On the permutability condition of quantum mechanics

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1. Introduction

A problem of importance for quantum mechanics is to find all couples of symmetric operators P, Q on Hilbert space $\mathfrak{H}$, satisfying the permutability condition

$$\text{(c)} \qquad PQ - QP = -iI,$$

I denoting the identity operator. There arise mathematical difficulties in the very formulation of this problem. Indeed, there exists no couple of bounded operators satisfying this condition[1]) and so it is necessary to consider non bounded and hence non everywhere defined operators too, and then one may require that (c) *holds only on some linear subset* $\mathfrak{D}$ of $\mathfrak{H}$, i. e.

$$(PQ - QP)f = -if \quad \text{for} \quad f \in \mathfrak{D};$$

in order to avoid trivial cases (such as the case $\mathfrak{D} = (0)$) it is however reasonable to require that this subset be not too sparse in $\mathfrak{H}$.

The Schrödinger functional operators

$$p\colon\ f(x) \to \frac{1}{i}\frac{df(x)}{dx}, \qquad q\colon\ f(x) \to xf(x)$$

on $L_2(-\infty, \infty)$ yield in this sense a particular solution of (c). $\mathfrak{D}_p$ [2]) consists of those $f(x) \in L_2$ which are absolutely continuous and such that $f'(x) \in L_2$; $\mathfrak{D}_q$ consists of those $f(x) \in L_2$ for which also $xf(x) \in L_2$; both p and q are selfadjoint. The condition (c) holds on the whole set $\mathfrak{D}_{pq-qp} = \mathfrak{D}_{pq} \cap \mathfrak{D}_{qp}$, consisting of those $f(x) \in L_2$ which are absolutely continuous and such that $xf(x)$, $f'(x)$, $xf'(x)$ also belong to L_2. This set is dense in L_2; moreover, the restrictions of p and q to $\mathfrak{D}_{pq-qp}$ are essentially selfadjoint[3]). Further, there

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1) See WIELANDT [9].

2) The domain of definition of an operator T will be denoted by $\mathfrak{D}_T$.

3) An operator T is essentially selfadjoint if its closure is selfadjoint.

exists a linear subset of $\mathfrak{D}_{pq-qp}$, which is invariant with respect to p and q,[4]) and nevertheless large enough that the restrictions of p, q, and p^2+q^2 to this set be all essentially selfadjoint: such is the set $\mathfrak{D}^\infty$ of all infinitely differentiable functions $f(x)$ for which $\lim_{|x|\to\infty} x^n f^{(m)}(x)=0$ $(m, n=0, 1, 2, \ldots)$. Both $\mathfrak{D}_{pq-qp}$ and $\mathfrak{D}^\infty$ are mapped onto themselves by the operators $p\pm iI, q\pm iI$.

These statements may be proved by more or less straightforward reasonings, using incidentally the fact that the Fourier transformation F maps both $\mathfrak{D}_{pq-qp}$ and $\mathfrak{D}^\infty$ onto themselves, and carries p in q over: $p=F^{-1}qF$. (Concerning the operator p^2+q^2 see RELLICH [5].)

Any couple of operators $\{P, Q\}$ on a Hilbert space of dimension $\aleph_0$, which is unitarily equivalent to the couple $\{p, q\}$, will be called a *Schrödinger couple.* Any Schrödinger couple, and also the *direct sums* $\{P, Q\}$ *of Schrödinger couples* $\{P_\alpha, Q_\alpha\}$[5]) are then equally solutions of the permutability relation (c) and inherit also the other properties mentioned above, of the couple $\{p, q\}$.

The problem is if we have obtained thus *all* the solutions of (c), at least if we suppose some suitable additional conditions implying in particular that the set $\mathfrak{D}$ on which (c) holds is not to sparse. This is the *unicity problem for the permutability condition* (c).

If we calculate with P and Q formally as if they were everywhere defined and bounded operators, we get from (c) the relations $PQ^n-Q^nP=-inQ^{n-1}$ $(n=0, 1, 2, \ldots)$ and hence

$$P\varphi(Q)-\varphi(Q)P=-i\varphi'(Q)$$

for any entire function $\varphi(\lambda)=a_0+a_1\lambda+a_2\lambda^2+\cdots$, in particular

$$Pe^{isQ}-e^{isQ}P=se^{isQ}, \quad \text{thus} \quad e^{-isQ}Pe^{isQ}=P+sI.$$

Calculating further we get $e^{-isQ}P^n e^{isQ}=(P+sI)^n$ $(n=0, 1, 2, \ldots)$ and hence

$$e^{-isQ}\psi(P)e^{isQ}=\psi(P+sI)$$

for any entire function $\psi(\lambda)=b_0+b_1\lambda+b_2\lambda^2+\cdots$, in particular

$$e^{-isQ}e^{itP}e^{isQ}=e^{it(P+sI)}=e^{its}e^{itP}.$$

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[4]) A set $\mathfrak{D}$ contained in the domain of definition of an operator T will be called invariant with respect to T if $T\mathfrak{D}\subseteqq\mathfrak{D}$.

[5]) If P_α and Q_α act on the Hilbert space $\mathfrak{H}_\alpha$ (of dimension $\aleph_0$) then P and Q act on $\mathfrak{H}=\sum_\alpha\oplus\mathfrak{H}_\alpha$ and we have, by definition, $P=\sum_\alpha\oplus P_\alpha$, $Q=\sum_\alpha\oplus Q_\alpha$.

Thus we are lead from (c), at least by a formal calculus which was indicated first by H. WEYL [8], to the permutability condition

$$\text{(C)} \qquad e^{itP} e^{isQ} = e^{its} e^{isQ} e^{itP} \qquad (-\infty < s, t < \infty).$$

Now, by the spectral theory, the exponential functions $U_t = e^{itP}$, $V_s = e^{isQ}$ $(-\infty < t, s < \infty)$ have a well defined meaning for any selfadjoint operators P, Q; $\{U_t\}$ and $\{V_s\}$ are namely the (uniquely determined) strongly continuous one parameter groups of unitary operators whose infinitesimal generators are iP and iQ, respectively, i. e.

$$iPf = \lim_{t \to 0} \frac{1}{t}(U_t - I)f, \qquad iQg = \lim_{s \to 0} \frac{1}{s}(V_s - I)g,$$

the domains of P and Q consisting of those elements f and g for which the respective limit exists.

In particular, the Schrödinger functional operators p, q (on L_2) generate in this sense the following operators $u_t = e^{itp}$, $v_s = e^{isq}$:

$$u_t f(x) = f(x+t), \qquad v_s f(x) = e^{isx} f(x),$$

and these operators satisfy obviously the permutability condition (C). This implies that (C) is satisfied by all one parameter groups $\{U_t = e^{itP}\}$, $\{V_s = e^{isQ}\}$, which are generated by Schrödinger couples $\{P, Q\}$ or by the direct sums $\{P, Q\}$ of Schrödinger couples.

J. VON NEUMANN [4] proved in 1931 that these are the only solutions of (C):[6])

Theorem of von Neumann. *In order that a couple $\{P, Q\}$ of selfadjoint operators on Hilbert space $\mathfrak{H}$ be a Schrödinger couple or the direct sum of Schrödinger couples it is necessary and sufficient that the one parameter unitary groups $\{U_t = e^{itP}\}$, $\{V_s = e^{isQ}\}$ $(-\infty < s, t < \infty)$ satisfy the permutability condition* (C).

The problem has been left open under which circumstances the formal equivalence of the two permutability conditions becomes an exact equivalence. Thus the solution of the unicity problem for (C), by VON NEUMANN, did not yield automatically a solution of the unicity problem for (c).

The unicity problem for (c) was attacked later, in 1946, directly, by F. RELLICH [5]; his results were recently improved by J. DIXMIER [1].

Theorem of Rellich—Dixmier. *In order that a couple of closed symmetric operators P, Q on Hilbert space $\mathfrak{H}$ be a Schrödinger couple or the*

[6]) See also MACKEY [3], [4].

direct sum of Schrödinger couples it is necessary and sufficient that there exists in $\mathfrak{D}_P \cap \mathfrak{D}_Q$ *a dense linear set* $\mathfrak{D}$, *such that*

(i) $\mathfrak{D}$ *is invariant with respect to* P *and* Q,

(ii) *the restriction of* $P^2 + Q^2$ *to* $\mathfrak{D}$ *is essentially selfadjoint,*[7])

(iii) *the permutability condition* (c) *holds on* $\mathfrak{D}$.

These conditions imply also that the restrictions of P *and* Q *to* $\mathfrak{D}$ *are essentially selfadjoint.*

2. The theorems of this paper

The aim of the present paper is to study in a direct way the exact connections between the permutability conditions (c) and (C). In fact, we shall consider the permutability condition (C) more generally, for any two one parameter semi-groups of contraction operators $\{S_s\}_{s \geqq 0}$, $\{T_t\}_{t \geqq 0}$, each depending (strongly) continuously on its parameter[8]); for sake of brevity, we shall call them simply *contraction semi-groups.* Our result is the following

Theorem I. *Let* $\{S_s\}_{s \geqq 0}$, $\{T_t\}_{t \geqq 0}$ *be two contraction semi-groups on Hilbert space* $\mathfrak{H}$, *and let* A *and* B *be their infinitesimal generators:*

$$A = \lim_{s \to +0} \frac{1}{s}(S_s - I), \qquad B = \lim_{t \to +0} \frac{1}{t}(T_t - I).^{9)}$$

In order that the permutability condition

$$T_t S_s = e^{its} S_s T_t \qquad (s, t \geqq 0) \tag{C'}$$

hold it is necessary that $\mathfrak{D}_{AB-BA}$ *be dense in* $\mathfrak{H}$, *invariant with respect to*

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[7]) In Rellich's version it was supposed that $P^2 + Q^2$ is "decomposable in $\mathfrak{D}$", i. e. that it has a selfadjoint closure $\int \lambda \, dE_\lambda$ such that, for any finite interval (a, b), the subspace $(E_b - E_a)\mathfrak{H}$ is contained in $\mathfrak{D}$.

[8]) I. e. we suppose that $S_{s_1} S_{s_2} = S_{s_1+s_2}$, $T_{t_1} T_{t_2} = T_{t_1+t_2}$ $(s_1, s_2, t_1, t_2 \geqq 0$, $S_0 = T_0 = I$, $\|S_s\| \leqq 1$, $\|T_t\| \leqq 1$, $\lim_{s \to 0} S_s = I$, $\lim_{t \to 0} T_t = I$ (strongly).

[9]) It is known that, for any contraction semi-group $\{W_t\}_{t \geqq 0}$ on Hilbert space $\mathfrak{H}$, the *infinitesimal generator* $A = \lim_{t \to +0} \frac{1}{t}(W_t - I)$ is a closed and densely defined operator. The operators $(A - I)^{-1}$ and $W = (A + I)(A - I)^{-1}$ are everywhere defined and bounded, W is moreover a contraction operator having not the eigenvalue 1; W is called the *infinitesimal cogenerator* of the semi-group $\{W_t\}$. Conversely, any contraction operator W which has not the eigenvalue $\lambda = 1$ is the infinitesimal cogenerator of exactly one contraction semi-group $\{W_t\}$. (See Sz.-Nagy—Foiaş [7].)

$(A-I)^{-1}$ and $(B-I)^{-1}$,[10]*) and the permutability condition*

$$(c') \qquad AB-BA=-iI$$

hold on $\mathfrak{D}_{AB-BA}$.

Conversely, in order that (C′) *hold it is sufficient that* (c′) *hold on some linear subset $\mathfrak{D}$ of $\mathfrak{D}_{AB-BA}$, for which $(B-I)(A-I)\mathfrak{D}$ or $(A-I)(B-I)\mathfrak{D}$ is dense in $\mathfrak{H}$.*

From Theorem I and from VON NEUMANN's theorem follows readily:

Theorem II. *In order that a couple of selfadjoint operators P, Q on Hilbert space $\mathfrak{H}$ be a Schrödinger couple or the direct sum of Schrödinger couples it is necessary and sufficient that there exist a linear set $\mathfrak{D}$, contained in $\mathfrak{D}_{PQ-QP}$, such that*

(i) *$(P+iI)(Q+iI)\mathfrak{D}$ or $(Q+iI)(P+iI)\mathfrak{D}$ be dense in $\mathfrak{H}$,*

(ii) *the permutability condition* (c) *hold on $\mathfrak{D}$.*

These conditions are namely, by Theorem I, necessary and sufficient that the unitary operators $U_t=e^{itP}$, $V_s=e^{isQ}$ satisfy (C) for $s, t\geqq 0$ and hence (as a consequence of the relation $U_{-t}=U_t^{-1}$, $V_{-s}=V_s^{-1}$) for all real s, t, and so we have only to apply the theorem of VON NEUMANN.

Corollary. *In order that a couple of closed symmetric operators P, Q on Hilbert space $\mathfrak{H}$ be a Schrödinger couple or the direct sum of Schrödinger couples it is necessary and sufficient that there exist in $\mathfrak{D}_{PQ-QP}$ a dense linear set $\mathfrak{D}$, such that*

(i) $(P\pm iI)\mathfrak{D}\supseteqq\mathfrak{D}$, $(Q\pm iI)\mathfrak{D}\supseteqq\mathfrak{D}$,

(ii) *the permutability condition* (c) *holds on $\mathfrak{D}$.*

Indeed, (i) and the fact that $\mathfrak{D}$ is dense in $\mathfrak{H}$ imply that the closure $\tilde{P}$ of the restriction of P to $\mathfrak{D}$ has the deficiency indices (0, 0), thus $\tilde{P}$ is selfadjoint, and since P is a symmetric extension of the selfadjoint $\tilde{P}$, so is $\tilde{P}=P$, i. e. P is itself selfadjoint; analogously for Q. Further, (i) implies that $(P+iI)(Q+iI)\mathfrak{D}\supseteqq\mathfrak{D}$ and since $\mathfrak{D}$ is dense, we have only to apply Theorem II to get the "sufficiency part" of the Corollary. The "necessity part" follows from what has been said above on the Schrödinger functional operators.

It is interesting to compare these results with those of RELLICH and DIXMIER. Though each set of conditions stated characterizes Schrödinger

[10]) This implies that $\mathfrak{D}_{AB-BA}\subseteqq(A-I)\mathfrak{D}_{AB-BA}$, $\mathfrak{D}_{AB-BA}\subseteqq(B-I)\mathfrak{D}_{AB-BA}\subseteqq(B-I)(A-I)\mathfrak{D}_{AB-BA}$, thus $(B-I)(A-I)\mathfrak{D}_{AB-BA}$ and analogously $(A-I)(B-I)\mathfrak{D}_{AB-BA}$ are dense in $\mathfrak{H}$.

couples and their direct sums (and thus they are equivalent), they do not seem to imply simply each other.

The theorems of RELLICH, DIXMIER, and VON NEUMANN have been stated by their authors also in the more general case where operators $P_1, Q_1, P_2, Q_2, \ldots, P_n, Q_n$ are concerned with the permutability relations $P_k Q_k - Q_k P_k = -iI$ $(k=1, \ldots, n)$, all operators with different subscripts being permutable. It is possible to generalize our results in the same direction too; but since this generalization is straightforward we treat the case $n=1$ only.

The rest of this paper deals with the proof of Theorem I.

3. Summary of the functional calculus for contractions

We shall make use of the functional calculus for contraction operators, developed by SZ.-NAGY and FOIAŞ [7], in the particular case of contraction operators which have not the eigenvalue 1, i. e. which are the infinitesimal cogenerators of some contraction semi-group (cf. [9]).

Let W be a contraction operator having not the eigenvalue 1. Correspondingly, we consider the class $\mathcal{C}$ of those complex-valued functions $\varphi(\lambda)$ which are defined and continuous on the set

$$K = \{\lambda: \ |\lambda| \leqq 1, \ \lambda \neq 1\}$$

in the complex plane, and holomorphic and bounded in its interior

$$K^0 = \{\lambda: \ |\lambda| < 1\}.$$

Then, by this functional calculus, to each $\varphi(\lambda) \in \mathcal{C}$ there corresponds a bounded linear operator $\varphi(W)$ such that

(α) $\varphi(\lambda) = \sum_0^\infty c_n \lambda^n \to \varphi(W) = \sum_0^\infty c_n W^n$ if $\sum_0^\infty |c_n| < \infty$;

(β) $c_1\varphi_1(\lambda) + c_2\varphi_2(\lambda) \to c_1\varphi_1(W) + c_2\varphi_2(W)$;

(γ) $\varphi_1(\lambda)\varphi_2(\lambda) \to \varphi_1(W)\varphi_2(W)$;

(δ) $\|\varphi(W)\| \leqq \sup_{\lambda\in K} |\varphi(\lambda)|$;

(ε) if a sequence $\varphi_n(\lambda) \in \mathcal{C}$ converges on K boundedly to a limit $\varphi(\lambda) \in \mathcal{C}$, then $\varphi_n(W) \to \varphi(W)$ strongly.

In particular, the functions

(1) $$e_t(\lambda) = \exp\left(t\frac{\lambda+1}{\lambda-1}\right) \qquad (t \geqq 0)$$

belong to $\mathcal{C}$ (we have $|e_t(\lambda)| \leqq 1$ on K), and the corresponding operators

$$W_t = e_t(W) \tag{2}$$

form precisely the contraction semi-group $\{W_t\}_{t \geqq 0}$ whose infinitesimal cogenerator is equal to the given W.

Conversely, W may be derived from W_t by the direct formula

$$W = \lim_{t \to +0} [W_t - (1-t) I] [W_t - (1+t) I]^{-1} \quad \text{(strong limit)}. \tag{3}$$

Cf., for all these statements, the paper [7].[11])

4. Proof of the "necessity" part of Theorem I

Let $\{S_s\}_{s \geqq 0}$, $\{T_t\}_{t \geqq 0}$ be two contraction semi-groups satisfying the permutability condition (C'). Let A, B be their infinitesimal generators and S, T their infinitesimal cogenerators, respectively;

$$S = (A+I)(A-I)^{-1}, \qquad T = (B+I)(B-I)^{-1}.$$

Differentiating both sides of (C') with respect to t at $t=0$ (from the right) we obtain that $S_s \mathfrak{D}_B \subseteqq \mathfrak{D}_B$ $(s \geqq 0)$ and

$$BS_s f = S_s B f + is S_s f \quad \text{for} \quad f \in \mathfrak{D}_B. \tag{4}$$

If we differentiate now with respect to s at $s=0$ (from the right) we obtain, using also the fact that B is closed, the relations

$$\mathfrak{D}_A \cap \mathfrak{D}_{AB} \subseteqq \mathfrak{D}_{BA} \tag{5}$$

and

$$BAf = ABf + if \quad \text{for} \quad f \in \mathfrak{D}_A \cap \mathfrak{D}_{AB}. \tag{6}$$

Since $\mathfrak{D}_{AB} = \mathfrak{D}_{AB} \cap \mathfrak{D}_B$ it results from (5):

$$\mathfrak{D}_A \cap \mathfrak{D}_{AB} = \mathfrak{D}_A \cap (\mathfrak{D}_{AB} \cap \mathfrak{D}_B) = \mathfrak{D}_B \cap (\mathfrak{D}_A \cap \mathfrak{D}_{AB}) \subseteqq \mathfrak{D}_B \cap \mathfrak{D}_{BA}.$$

If we carry out the differentiations in the reverse order we obtain analogously

$$\mathfrak{D}_B \cap \mathfrak{D}_{BA} \subseteqq \mathfrak{D}_A \cap \mathfrak{D}_{AB}.$$

Consequently,

$$\mathfrak{D}_A \cap \mathfrak{D}_{AB} = \mathfrak{D}_B \cap \mathfrak{D}_{BA} = \mathfrak{D}^*. \tag{7}$$

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[11]) The functional calculus, as developed in the cited paper, applies to a somewhat larger class of functions, but for our present needs it is sufficient to consider only the above class $\mathcal{C}$.

Since obviously

$$\mathfrak{D}^* \subseteqq \mathfrak{D}_{AB} \cap \mathfrak{D}_{BA} \subseteqq \mathfrak{D}_B \cap \mathfrak{D}_{BA} = \mathfrak{D}^*,$$

we have also

$$\mathfrak{D}^* = \mathfrak{D}_{AB} \cap \mathfrak{D}_{BA} = \mathfrak{D}_{AB-BA}. \tag{8}$$

Thus, by (6), the permutability condition (c′) holds on $\mathfrak{D}^* = \mathfrak{D}_{AB-BA}$.

We have still to prove that $\mathfrak{D}^*$ is dense in $\mathfrak{H}$ and invariant with respect to $(A-I)^{-1}$ and $(B-I)^{-1}$.

We need to this aim some relations connecting the infinitesimal generator of one semi-group and the infinitesimal cogenerator of the other.

First we observe that the operators $S_s^{(t)} = e^{its} S_s$ $(s \geqq 0)$ form, for any fixed $t \geqq 0$, a contraction semi-group whose infinitesimal generator is $A^{(t)} = A + itI$; denote its infinitesimal cogenerator by $S^{(t)}$. The permutability condition (C′) implies

$$T_t S_s = S_s^{(t)} T_t,$$

hence

$$(S_s^{(t)} - aI)\, T_t(S_s - bI) = (S_s^{(t)} - bI)\, T_t(S_s - aI)$$

for any scalar a, b, thus

$$T_t(S_s - bI)(S_s - aI)^{-1} = (S_s^{(t)} - aI)^{-1}(S_s^{(t)} - bI)\, T_t$$

if $|a| > 1$, for then the inverse operators indicated exist, are bounded and everywhere defined. Put in particular $a = 1+s$, $b = 1-s$ $(s > 0)$; letting $s \to +0$ apply the limit formula (3). It results

$$T_t S = S^{(t)} T_t \qquad (t \geqq 0). \tag{10}$$

We wish to differentiate both sides of (10) with respect to t at $t = 0$, from the right. To this effect we write $S^{(t)}$, for $t > 0$, in the following form:

$$S^{(t)} = (A^{(t)} + I)(A^{(t)} - I)^{-1} = \left[\left(1 + \frac{it}{2}\right)(A+I) - \frac{it}{2}(A-I)\right]\left[\frac{it}{2}(A+I) + \right.$$

$$\left. + \left(1 - \frac{it}{2}\right)(A-I)\right]^{-1} = \left[\left(1+\frac{it}{2}\right)S - \frac{it}{2} I\right]\left[\frac{it}{2} S + \left(1 - \frac{it}{2}\right) I\right]^{-1} = h_t(S)$$

where $h_t(S)$ is the operator corresponding to the function

$$h_t(\lambda) = \frac{\left(1 + \frac{it}{2}\right)\lambda - \frac{it}{2}}{\frac{it}{2}\lambda + \left(1 - \frac{it}{2}\right)} \in \mathfrak{C}$$

The function

$$\frac{1}{t}\,[h_t(\lambda)-\lambda]=-\frac{i}{2}\,\frac{(\lambda-1)^2}{1+\frac{it}{2}(\lambda-1)}$$

also belongs to $\mathfrak{C}$ and when $t\to+0$ it converges on K boundedly to the function $-\frac{i}{2}(\lambda-1)^2\in\mathfrak{C}$. By the functional calculus we have therefore

$$\frac{1}{t}(S^{(t)}-S)=\frac{1}{t}[h_t(S)-S]\to-\frac{i}{2}(S-I)^2.$$

Using this result and the fact that B is closed we obtain from (10) by differentiating with respect to t at $t=+0$ that

$$S\mathfrak{D}_B\subseteq\mathfrak{D}_B \tag{11}$$

and

$$BSf=SBf-\frac{i}{2}(S-I)^2f\quad\text{for}\quad f\in\mathfrak{D}_B. \tag{12}$$

These are the relations we need, between the infinitesimal generator B of $\{T_t\}$ and the infinitesimal cogenerator S of $\{S_s\}$.

Let $f\in\mathfrak{D}_B$, $g=(S-I)f$. From (11) and (12) we deduce

$$Bg=B(S-I)f=SBf-Bf-\frac{i}{2}(S-I)^2f=(S-I)\left[Bf-\frac{i}{2}(S-I)f\right].$$

Since, by the definition of S,

$$S-I=2(A-I)^{-1}, \tag{13}$$

we see that both g and Bg belong to $\mathfrak{D}_{A-I}=\mathfrak{D}_A$, i.e. $g\in\mathfrak{D}_A\cap\mathfrak{D}_{AB}$. This proves that

$$(A-I)^{-1}\mathfrak{D}_B\subseteq\mathfrak{D}_A\cap\mathfrak{D}_{AB}. \tag{14}$$

On the other hand we have

$$\mathfrak{D}_B\cap\mathfrak{D}_{BA}\subseteq(A-I)^{-1}\mathfrak{D}_B \tag{15}$$

since $f\in\mathfrak{D}_B\cap\mathfrak{D}_{BA}$ implies $(A-I)f=Af-f\in\mathfrak{D}_B$.

Comparing (14), (15), and (7), we obtain

$$\mathfrak{D}^*=(A-I)^{-1}\mathfrak{D}_B. \tag{16}$$

If we interchange the rôle of the two semi-groups in the above reasoning we obtain the analogous relation

$$\mathfrak{D}^*=(B-I)^{-1}\mathfrak{D}_A. \tag{17}$$

Since $\mathfrak{D}^* \subseteqq \mathfrak{D}_B$ and $\mathfrak{D}^* \subseteqq \mathfrak{D}_A$, (16) and (17) imply that $\mathfrak{D}^*$ is invariant with respect to $(A-I)^{-1}$ and $(B-I)^{-1}$.

Finally, $\mathfrak{D}^*$ is dense in $\mathfrak{H}$. In the contrary case there would exist $g \neq 0$ in $\mathfrak{H}$, orthogonal to $\mathfrak{D}^* = (A-I)^{-1}\mathfrak{D}_B = (S-I)\mathfrak{D}_B$. Then $(S^*-I)g$ would be orthogonal to $\mathfrak{D}_B$, thus $(S^*-I)g=0$ since $\mathfrak{D}_B$ is dense in $\mathfrak{H}$. But $S^*g=g$ implies $Sg=g$,[12]) in contradiction to the fact that S has not the eigenvalue 1.

Thus the "necessity" part of Theorem I is proved.

5. Proof of the "sufficiency" part of Theorem I

We suppose now that the infinitesimal generators A, B of the contraction semi-groups $\{S_s\}$, $\{T_t\}$ satisfy the permutability condition (c′) on a linear subset $\mathfrak{D}$ of $\mathfrak{D}_{AB-BA}$, such that $(B-I)(A-I)\mathfrak{D}$ is dense in $\mathfrak{H}$. The case when $(A-I)(B-I)\mathfrak{D}$ is dense in $\mathfrak{H}$ may be treated in an analogous way. Since $A-I$ and $B-I$ have continuous inverses, both conditions imply that $\mathfrak{D}$ itself is also dense in $\mathfrak{H}$.

By (c′) we have for any $f \in \mathfrak{D}$

$$(A \mp I)Bf = B(A \mp I)f - if.$$

Putting $g=(A-I)f$ and remembering of the definition of the infinitesimal cogenerator S we obtain hence that

$$S(Bg-if) = BSg - if, \qquad SBg - BSg = i(S-I)f,$$

i. e.

$$(SB-BS)g = \frac{i}{2}(S-I)^2 g \tag{18}$$

since, by (13), $f=(A-I)^{-1}g = \frac{1}{2}(S-I)g$. Thus (18) has been proved for the elements g of $(A-I)\mathfrak{D}$.

We shall now prove that (18) holds for *all* elements of $\mathfrak{D}_B$. Let g be such an element. Since $(B-I)(A-I)\mathfrak{D}$ is dense in $\mathfrak{H}$, there exists a sequence $\{f_n\} \in \mathfrak{D}$ such that, putting $g_n=(A-I)f_n$, we have

$$(B-I)g_n \to (B-I)g.$$

Since $B-I$ has a continuous inverse this implies $g_n \to g$ and consequently $Bg_n = (B-I)g_n + g_n \to (B-I)g + g = Bg$. Now (18) holds for $g_n \in (A-I)\mathfrak{D}$, thus, by the fact that S is continuous and B is closed, it results that $Sg \in \mathfrak{D}_B$ and (18) holds also for $g \in \mathfrak{D}_B$.

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[12]) A contraction operator and its adjoint have the same invariant elements, cf. [6] p. 402.

We have thus proved that $\mathfrak{D}_B$ is invariant with respect to S and (18) holds for the elements g of $\mathfrak{D}_B$. This implies by induction

$$(S^n B - B S^n) g = \frac{i}{2} n S^{n-1} (S-I)^2 g \qquad (g \in \mathfrak{D}_B;\ n = 0, 1, \ldots).$$

Using again the fact that B is closed we obtain hence

$$(19) \qquad [\varphi(S) B - B \varphi(S)] g = \frac{i}{2} (S-I)^2 \varphi'(S) g \qquad (g \in \mathfrak{D}_B)$$

for any function $\varphi(\lambda) = \sum_0^\infty c_n \lambda^n$ and its derivative $\varphi'(\lambda) = \sum_1^\infty c_n n \lambda^{n-1}$ if the convergence radius of these power series is greater than 1.

This is the case in particular for the function $e_{s,r}(\lambda) = e_s(r\lambda)$ $(s \geqq 0,$ $0 < r < 1)$, where $e_s(\lambda) = \exp\left(s \dfrac{\lambda+1}{\lambda-1}\right)$ (see (1)). Thus if we introduce also the function

$$\tilde{e}_{s,r}(\lambda) = \frac{1}{2} (\lambda - 1)^2 e'_{s,r}(\lambda) = -\frac{s}{r} \left(\frac{\lambda - 1}{\lambda - \frac{1}{r}} \right)^2 e_{s,r}(\lambda) \in \mathfrak{C}$$

we have

$$(20) \qquad [e_{s,r}(S) B - B e_{s,r}(S)] g = i \tilde{e}_{s,r}(S) g \quad \text{for} \quad g \in \mathfrak{D}_B.$$

On K we have $|e_{s,r}(\lambda)| \leqq 1$ and $|\tilde{e}_{s,r}(\lambda)| \leqq \dfrac{s}{r}$, and when $r \to 1-0$ $e_{s,r}(\lambda)$ converges on K to $e_s(\lambda)$ and $\tilde{e}_{s,r}(\lambda)$ converges to $-s e_s(\lambda)$. By the functional calculus, $e_{s,r}(S)$ converges therefore to $e_s(S) = S_s$, and $\tilde{e}_{s,r}(S)$ to $-s S_s$. So it results from (20), again by the fact that B is closed, that

$$(21) \qquad S_s \mathfrak{D}_B \subseteqq \mathfrak{D}_B$$

and

$$(S_s B - B S_s) g = -is S_s g$$

i. e.

$$(22) \qquad S_s B g = (B - isI) S_s g \quad \text{for} \quad g \in \mathfrak{D}_B.$$

For any fixed value of s, $B^{(s)} = B - isI$ is evidently the infinitesimal generator of the contraction semi-group $\{T_t^{(s)} = e^{-its} T_t\}_{t \geqq 0}$. Let $T^{(s)}$ be the corresponding infinitesimal cogenerator, i. e. $T^{(s)} = (B^{(s)} + I)(B^{(s)} - I)^{-1}$. From (22) we get

$$S_s (B \mp I) g = (B^{(s)} \mp I) S_s g,$$

whence we see that

$$T^{(s)}[S_s(B-I)g] = S_s(B+I)g \quad \text{for any} \quad g \in \mathfrak{D}_B$$

i. e.

$$T^{(s)}S_s h = S_s T h$$

for any

$$h = (B-I)g \in (B-I)\mathfrak{D}_B = \mathfrak{H}.$$

Thus

$$T^{(s)}S_s = S_s T.$$

From this equation we deduce

$$\varphi(T^{(s)})S_s = S_s \varphi(T)$$

first for the functions $\varphi(\lambda) = \lambda^n$ $(n = 0, 1, 2, \ldots)$, then for any function $\varphi(\lambda)$ which is holomorphic in a domain containing the closed unit disc in its interior, finally, reasoning again through the auxiliary functions $\varphi_r(\lambda) = \varphi(r\lambda)$ $(0 < r < 1)$, for any function $\varphi(\lambda) \in \mathfrak{C}$. If we take in particular $\varphi(\lambda) = e_t(\lambda)$ $(t \geqq 0)$ we obtain

$$T_t^{(s)} S_s = S_s T_t,$$

and this proves (C′).

Thus Theorem I is fully proved.

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