On the ordering of quotient rings and quotient semigroups

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Dedicated to L. Rédei on his 60th birthday

It is a rather familiar problem how to extend the ordering relation of a fully ordered ring R to a full order of a larger ring S, in particular, those cases are of great interest in which such an extension is uniquely possible. The most important special case of this problem is when R is a domain of integrity and S is its quotient field; this case is dealt with in most text-books on algebra. Other important special cases have been considered by several authors. ALBERT [1] and NEUMANN [6] discussed the case in which S was the Ore quotient skewfield of R. Recently, Prof. RÉDEI has been interested in this problem [7] when R was supposed to be a ring without divisors of zero and S its ring extension with identity containing no divisors of zero. GRÄTZER and SCHMIDT [4] considered the more general problem when R was an ideal of S and S was supposed to be free of divisors of zero. Here we wish to give a common generalization of these results. Our starting point is a rather general definition of quotient ring, one which includes the relationship of R and S in all above-mentioned cases.

Our main result can be carried over to semigroups. Because of the failure of a positive cone in semigroups by means of which it is easy to handle the ordering relation in rings, a certain amount of complication arises, but this is merely of technical character. Its effect appears in that the proof is somewhat longer. As corollaries we obtain well-known results of TAMARI—ALIMOV—NAKADA and CONRAD, respectively.

By a *fully ordered ring* R we mean an associative (but not necessarily commutative) ring which is at the same time a fully ordered set satisfying: $a \leqq b$ implies $a \pm c \leqq b \pm c$, $ac \leqq bc$ and $ca \leqq cb$ for all $c > 0$ in the ring. By the *positive cone* P of R we mean the set of all $x \in R$ with $x \geqq 0$. P has the characterizing properties: it is closed with respect to addition and multiplication, it contains 0 and for any $x \in R$, $x \neq 0$, exactly one of $x, -x$.

The positive cone P uniquely determines the ordering relation of R, since $a \leqq b$ if and only if $b-a \in P$.

Let R be an arbitrary ring and S an overring of R. Assume that to each $\alpha \in S \setminus R$ there exist elements $a, b \in R$ such that (i) a is not a left divisor of zero in S, (ii) b is not a right divisor of zero in S, (iii) $a\alpha = c$ and $\alpha b = d$ belong to R. In this case S will be called *a quotient ring of R*.

Our main result is the following theorem.

Theorem 1. *A full order of an (associative) ring R can be uniquely extended to a full order of an arbitrary quotient ring S of R.*

Let R be a fully ordered ring with the positive cone P and S a quotient ring of R. There is no loss of generality in assuming that in (i) and (ii) the elements a, b are >0. Then in (iii) the elements c and d have the same sign, for $cb = a\alpha b = ad$, and so the sign of $a\alpha$ and αb does not depend on the special choice of a or b. We define the positive cone Q of S to consist of P and of all $\alpha \in S \setminus R$ such that $a\alpha$ (and so αb) lies in P. Then we see immediately that for any $\alpha \neq 0$ in S, either α or $-\alpha$ belongs to Q, but not both. If $\alpha, \beta \in Q$, then there exist elements $a, b \in P$ with (i) and (ii) or $=1$ such that $a\alpha, \beta b \in P$. Hence $a(\alpha+\beta)b = (a\alpha)b + a(\beta b) \in P$, and so[1]) $\alpha+\beta \in Q$. Again, $a(\alpha\beta)b = (a\alpha)(\beta b) \in P$ whence $\alpha\beta \in Q$. Consequently, Q defines a full order in S. The uniqueness is evident. Q. e. d.

We mention the following consequences of our theorem.[2])

1. If R is a fully ordered ring having an Ore left quotient skewfield S, then S can be fully ordered uniquely so as to continue the ordering of R. (ALBERT [1], NEUMANN [6].)

2. Any full order of a domain of integrity can be uniquely extended to a full order of its quotient field.

3. Let R be an ideal of a ring S containing at least one element which is not a divisor of zero. Then any full order of R can be extended in a unique way to a full order of S. (Cp. GRÄTZER—SCHMIDT [4].)

4. Let R be a ring containing a non-divisor of zero and S a minimal ring with identity[3]) containing R. Every full order of R can be extended uniquely to a full order of S. (Cp. RÉDEI [7].)

Let us turn to semigroups. By a *fully ordered semigroup S* is meant a

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[1]) Here we make use of the fact that if $a, b > 0$ and $c = a\alpha b \in R$, then α and c have the same sign. In fact, if $d > 0$ is chosen so that $d(a\alpha) = f \in R$, then the elements α, f, $fb = dc$, c are simultaneously positive or negative.

[2]) The proofs are immediate and therefore may be left to the reader.

[3]) If the ring contains elements which are not divisors of zero, then it has a ring extension with identity in which they remain non-divisors of zero. We understand this by "minimal".

semigroup which is at the same time a fully ordered set satisfying: $a \leqq b$ implies $ac \leqq bc$ and $ca \leqq cb$ for all $c \in S$.

Let T be a semigroup containing the semigroup S. Assume that to any $\alpha \in T \setminus S$ there exist elements $a, b \in S$ such that (i) a is left-cancellable[4]) in T, (ii) b is right-cancellable in T and (iii) $a\alpha$, αb belong to S. Then we shall say that T is *a quotient semigroup of* S.

We have the following analogue of Theorem 1:

Theorem 2. *A full order of a semigroup S can be extended, in one and only one way, to a full order of an arbitrary quotient semigroup T of S.*

If T properly contains S, then there exist surely elements $a, b \in S$ which are left- resp. right-cancellable in T, and therefore in the above definition the case $\alpha \in S$ need not be excluded. Now if $\alpha, \beta \in T$ $(\alpha \neq \beta)$ and if a, b are left- resp. right-cancellable elements such that $a\alpha$, $\beta b \in S$, then $a\alpha b$ and $a\beta b$ are different elements of S, and we define $\alpha \gtrless \beta$ according as $a\alpha b \gtrless a\beta b$ in S. It is a trivial fact that for the elements of S this definition coincides with that originally given in S. The definition does not depend on the special choice of a, b. For, if a', b' are again left- resp. right-cancellable elements with $a'\alpha$, $\beta b' \in S$, then — taking a left- and a right-cancellable element $a'', b'' \in S$ such that $a''(a'\beta), (\alpha b)b'' \in S$ — we obtain e. g. from $a\alpha b < a\beta b$ in turn $a\alpha bb'' < a\beta bb''$, $\alpha bb'' < \beta bb''$, $a''a'\alpha bb'' < a''a'\beta bb''$, $a''a'\alpha < a''a'\beta$, $a''a'\alpha b' < a''a'\beta b'$, $a'\alpha b' < a'\beta b'$. A similar reasoning applies if a', b' are determined so as to have $a'\beta, \alpha b' \in S$. The transitivity of $<$ follows by a straightforward computation of similar kind. Finally, we show that $\alpha \leqq \beta$ implies $\gamma\alpha \leqq \gamma\beta$ for all $\gamma \in T$. If a, b, b'' are defined as before and $c \in S$ is left-cancellable such that $c\gamma \in S$, then we get successively $a\alpha b \leqq a\beta b$, $a\alpha bb'' \leqq a\beta bb''$, $\alpha bb'' \leqq \beta bb''$, $c\gamma\alpha bb'' \leqq c\gamma\beta bb''$. Hence, by multiplying by suitable elements we arrive at $\gamma\alpha \leqq \gamma\beta$. The uniqueness of the extension is evident.

We obtain the following corollaries.

1. Let S be a cancellative fully ordered semigroup satisfying:[5]) for each pair $a, b \in S$ there is a pair $x, y \in S$ such that $ax = by$. Then there exists a fully ordered group G containing S in such a way that every $g \in G$ has the form $g = ab^{-1}(a, b \in S)$ and $g > e$ if and only if $a > b$ in S. This G is unique within to order-isomorphism. (Conrad [3].)

2. A cancellative commutative fully ordered semigroup S has a quotient group which can be fully ordered in a unique manner. (Tamari [8], Alimov [2], Nakada [5].)

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[4]) An element a is left-cancellable if $ax = ay$ implies $x = y$.

[5]) This is just the Ore condition and ensures the existence of a quotient group of S.

References

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