Some sets of integers related to the k-free integers

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1. Introduction

In this paper e, n, r and k will denote natural numbers, *with k assumed* >1 *throughout.* Suppose that $p_1, \ldots, p_t$ are the distinct prime divisors of n and write

$$(1.1) \qquad n = p_1^{e_1} \ldots p_t^{e_t},$$

with the convention that $t=0$ in case $n=1$. We shall say that n is *unitarily k-free,* or simply *k-skew,* if $e_i \not\equiv 0 \pmod{k}$ for each exponent e_i $(1 \leqq i \leqq t)$ appearing in the factorization (1.1). Further, we shall say that n is *e-skew of rank r* if $e_i \neq je$ for all i $(1 \leqq i \leqq t)$ and all j $(1 \leqq j \leqq r)$.

Before proceeding further, it is convenient to introduce the following terminology. The *characteristic function* $\chi_S(n)$ of a set S is defined by $\chi_S(n) = 1$ or 0 according as $n \in S$ or $n \notin S$. If x is real and $\geqq 1$, the *enumerative function* of S is defined to be the number $S(x)$ of integers $\leqq x$ contained in S. The *asymptotic density* $\delta(S)$ of S is the limit, $\lim_{x \to \infty} S(x)/x$, whenever this limit exists. Finally, the generating function $f_S(s)$ is defined by the Dirichlet series,

$$f_S(s) = \sum_{n=1}^{\infty} \frac{\chi_S(n)}{n^s}.$$

Let now Q_k, Q_k^*, and $Q_{e,r}^*$ denote respectively the sets of the k-free integers, the k-skew integers, and the e-skew integers of rank r. As to the relation between these sets, it is evident that $Q_k \subset Q_{k,r}^* \subset Q_k^*$ for all r. Moreover, the set Q_k^* is the limiting case, as $r \to \infty$, of the sets $Q_{k,r}^*$ $(r = 1, 2, \ldots)$. Finally, we note a striking structural analogy between Q_k and Q_k^*. In particular, define d to be a *unitary* divisor of n if $d > 0, d\delta = n,$ and $(d, \delta) = 1$; a k-skew integer may be defined than as *an integer whose largest unitary k-th power divisor is* 1.

The principal aim of this paper is to determine the simplest asymptotic properties of the k-skew integers. In place of considering Q_k^* directly, we investigate the sets $Q^*_{k,r}$, and from the properties obtained, we deduce, in the limiting case of r, corresponding properties for Q^*_k. One will note in particular the following corollary of Theorem 3. 2.

Corollary. *The asymptotic density $\delta(Q^*_k)$ of Q^*_k satisfies*

$$\frac{1}{\zeta(k)} < \delta(Q^*_k) = \alpha_k < 1, \tag{1.2}$$

where $\zeta(s)$ denotes the Riemann zeta function.

Denote by $Q_k(x)$, $Q^*_k(x)$, and $Q^*_{e,r}(x)$, $x \geqq 1$, the enumerative functions of Q_k, Q^*_k, and $Q^*_{e,r}$, respectively. We shall use two different methods in treating $Q^*_{k,r}(x)$. In the first method (§ 3) we proceed in a manner parallel to the classical treatment of $Q_k(x)$. Sums over ordinary divisors are now replaced, however, by unitary divisors. This method is elementary to the extent that it is not even necessary to introduce generating functions in the argument. By introducing such functions in § 4, we are able to refine the estimates proved in § 3. In the method of § 4, however, in place of proceeding by analogy with the k-free integers, we express $Q^*_{k,r}(x)$ *directly* in terms of $Q_k(x)$, effectively reducing the problem under consideration to one whose solution is well known (cf. § 2).

The final results obtained by the second method are contained in Theorems 4. 1 and 4. 2. In particular, the remainder terms in the estimates for $Q^*_{k,r}(x)$ and $Q^*_k(x)$ proved in § 3 are diminished by a logarithmic factor.

Regarding previous work, we mention that the case $k=2$ of the estimate for $Q^*_k(x)$ proved in Theorem 3. 2 was obtained in [3, § 6] by the same method used to treat $Q^*_{k,r}(x)$ in § 3 of the present paper. As for $Q^*_{e,r}(x)$, the case $e=1$ (excluded in this paper) was treated by Erdős and Szekeres [4] by an elementary method, and recently, using more advanced methods, by Bateman and Grosswald [1].

2. Preliminaries concerning Q_k

The material of this section is classical and is included for purposes of comparison and reference. Let $q_k(n)$ denote the characteristic function of Q_k. The generating function of Q_k is $\zeta(s)/\zeta(ks)$; that is [6, Theorem 303, p. 255],

$$\sum_{n=1}^{\infty} \frac{q_k(n)}{n^s} = \frac{\zeta(s)}{\zeta(ks)}, \quad s>1. \tag{2.1}$$

From (2. 1) we obtain the following representation of $q_k(n)$ as an arithmetical integral,

$$q_k(n)=\sum_{d\delta=n}\mu^{(k)}(d),\quad \mu^{(1)}(n)=\mu(n), \tag{2.2}$$

where $\mu(n)$ is the Möbius function, and

$$\mu^{(k)}(n)=\begin{cases}\mu(m) & \text{if } \ n=m^k,\\ 0 & \text{otherwise.}\end{cases} \tag{2.3}$$

We recall that $1/\zeta(s)$ is the generating function of $\mu(n)$.

The principal elementary result for $Q_k(x)$ is contained in the following

Lemma. 2.1 ([5, p. 47], also cf. [6, § 18.6, $k=2$]). *If* $x\geqq 1$, *then*

$$Q_k(x)=\frac{x}{\zeta(k)}+O(\sqrt[k]{x}). \tag{2.4}$$

Proof. By (2.2) one obtains

$$Q_k(x)=\sum_{d\delta\leqq x}\mu^{(k)}(d)=\sum_{n^k\delta\leqq x}\mu(n)=\sum_{n\leqq\sqrt[k]{x}}\mu(n)\left[\frac{x}{n^k}\right]. \tag{2.5}$$

Hence by the boundedness of $\mu(n)$,

$$\begin{aligned}Q_k(x)&=\sum_{n\leqq\sqrt[k]{x}}\mu(n)\left(\frac{x}{n^k}+O(1)\right)=x\sum_{n\leqq\sqrt[k]{x}}\frac{\mu(n)}{n^k}+O(\sqrt[k]{x})\\ &=\frac{x}{\zeta(k)}+O\left(x\sum_{n>x^{1/k}}\frac{1}{n^k}\right)+O(\sqrt[k]{x}),\end{aligned}$$

and (2.4) results, because the O-sum is $O\left(\dfrac{1}{x^{1-\frac{1}{k}}}\right)$.

Corollary 2.1. *The asymptotic density of* Q_k *is* $\delta(Q_k)=1/\zeta(k)$.

3. Initial estimates

We first introduce some notation. Let $\mu_r(n)$ denote the unique multiplicative function of n defined as follows for $n=p_e$, p prime, $e>0$, $(\mu_r(1)=1)$,

$$\mu_r(p^e)=\begin{cases}-1 & (1\leqq e\leqq r)\\ 0 & (e>r).\end{cases} \tag{3.1}$$

Note that $\mu_1(n)=\mu(n)$. The function defined by $\mu_r(n)$ as $r\to\infty$ will be denoted $\mu^*(n)$; that is, $\mu^*(n)=(-1)^{\omega(n)}$, where $\omega(n)$ denotes the number of (distinct) prime divisors of n. Generalizing (2. 3), we define

$$\mu_r^{(k)}(n) = \begin{cases} \mu_r(m) & \text{if} \quad n = m^k \\ 0 & \text{otherwise.} \end{cases} \tag{3. 2}$$

The Legendre totient, defined to be the number of positive integers $\leqq x$ that are prime to n, will be denoted $\varphi(x, n)$. We also write $\varphi(n) = \varphi(n, n)$, and define $\theta(n)$ to be the number of unitary divisors of n.

As in [3], the *unitary product* (or convolution) of two arithmetical functions $a_1(n), a_2(n)$, is the function $a_3(n)$ defined by

$$a_3(n) = \sum_{\substack{d\delta = n \\ (d,\delta)=1}} a_1(d) a_2(\delta), \tag{3. 3}$$

where the summation is over all unitary divisors d of n. We recall three lemmas proved in [3].

Lemma 3. 1 ([2, Lemma 6. 1]). *If $a_1(n)$ and $a_2(n)$ are multiplicative, then the unitary product* (3. 3) *of $a_1(n)$ and $a_2(n)$ is also multiplicative.*

The next lemma is the "unitary" analogue of the Möbius inversion formula.

Lemma. 3. 2 ([3, Theorem 2. 3]).

$$a_1(n) = \sum_{\substack{d\delta = n \\ (d,\delta)=1}} a_2(d) \rightleftarrows a_2(n) = \sum_{\substack{d\delta = n \\ (d,\delta)=1}} \mu^*(d) a_1(\delta). \tag{3. 4}$$

Lemma 3. 3 ([2, (1)], [3, Lemma 3. 4]).

$$\varphi(x, n) = \frac{x\varphi(n)}{n} + O(\theta(n)), \tag{3. 5}$$

uniformly in x.

Finally, we mention the following simple property of $\theta(n)$:

$$R(x) \equiv \sum_{n \leqq x} \theta(n) = O(x \log x), \qquad x \geqq 2. \tag{3. 6}$$

Let $q_{k,r}^*(n)$ and $q_k^*(n)$ represent the characteristic functions of $Q_{k,r}^*$ and Q_k^*, respectively. We prove first the following analogue of (2. 2).

Lemma 3. 4.

$$\sum_{\substack{d\delta = n \\ (d,\delta)=1}} \mu_r^{(k)}(d) = q_{k,r}^*(n). \tag{3. 7}$$

Proof. By Lemma 3. 2 there exists a uniquely defined function $b(n)$ such that

$$q_{k,r}^*(n) = \sum_{\substack{d\delta = n \\ (d,\delta)=1}} b(d), \tag{3. 8}$$

and in fact,

$$b(n)=\sum_{\substack{d\delta=n\\(d,\delta)=1}}\mu^*(d)q^*_{k,r}(\delta). \tag{3.9}$$

Evidently, $\mu^*(n)$ and $q^*_{k,r}(n)$ are multiplicative in n; hence by (3.9) and Lemma 3.1, $b(n)$ is also multiplicative. It therefore suffices to evaluate $b(n)$ in case $n=p^e$. By (3.9) it is easily verified that $b(p^e)=-1$ when $e=k$, $2k,\ldots,rk$, otherwise $b(p^e)=0$; that is, $b(p^e)=\mu_r^{(k)}(p^e)$, which completes the proof.

Define

$$S_{k,r}\equiv\sum_{n=1}^{\infty}\frac{\mu_r(n)\varphi(n)}{n^{k+1}}; \tag{3.10}$$

that the series is absolutely convergent is a consequence of the boundedness of $\mu_r(n)$ and the fact that $\varphi(n)\leqq n$.

Lemma 3.5.

$$S_{k,r}=\alpha_{k,r}\equiv\zeta(k)\prod_p\left(1-\frac{2}{p^k}+\frac{1}{p^{k+1}}+\frac{1}{p^{kr+k}}-\frac{1}{p^{kr+k+1}}\right), \tag{3.11}$$

where the (absolutely convergent) product extends over the primes p.

Proof. By (3.10), the multiplicativity of $\varphi(n)$ and $\mu_r(n)$, and the familiar fact, $\varphi(p^e)=p^{e-1}(p-1)$, one obtains (cf. [6, § 17.4])

$$S_{k,r}=\prod_p\left(\sum_{j=0}^{\infty}\frac{\mu_r(p^j)\varphi(p^j)}{p^{j(k+1)}}\right)=\prod_p\left\{1-\left(\frac{p-1}{p}\right)\sum_{j=1}^{r}\frac{1}{p^{jk}}\right\},$$

so that on summing a progression,

$$S_{k,r}=\prod_p\left\{2-\frac{1}{p}+\left(\frac{1}{p}-1\right)\left(\frac{1-\dfrac{1}{p^{k(r+1)}}}{1-\dfrac{1}{p^k}}\right)\right\}.$$

Factoring out $\zeta(k)=\prod(1-p^{-k})^{-1}$ yields (3.11).

Remark 3.1. In case $r=1$, (3.11) simplifies to give

$$\alpha_k^*\equiv\alpha_{k,1}=\prod_p\left(1-\frac{1}{p^k}+\frac{1}{p^{k+1}}\right). \tag{3.12}$$

We now prove the following estimate for $Q^*_{k,r}(x)$.

Theorem 3.1. *If $x\geqq 2$, then*

$$Q^*_{k,r}(x)=\alpha_{k,r}x+O(\sqrt[k]{x}\log x) \tag{3.13}$$

uniformly in r, $\alpha_{k,r}$ being defined by (3.11).

Proof. The proof is analogous to that of Lemma 2. 1 and [3, Theorem 6. 1]. By (3. 7) and (3. 1),

$$Q^*_{k,r}(x) = \sum_{\substack{d^k\delta\leq x\\(\delta,d)=1}} \mu_r(d) = \sum_{n\leq\sqrt[k]{x}} \mu_r(n)\varphi\left(\frac{x}{n^k}, n\right). \tag{3. 14}$$

Application of (3. 5) and (3. 6) yields, since $|\mu_r(n)|\leqq 1$ for all r,

$$Q^*_{k,r}(x) = x \sum_{n\leq\sqrt[k]{x}} \frac{\mu_r(n)\varphi(n)}{n^{k+1}} + O(R(\sqrt[k]{x}))$$

$$= S_{k,r}x + O\left(x \sum_{n>x^{1/k}} \frac{1}{n^k}\right) + O(\sqrt[k]{x}\log x),$$

uniformly in r. The theorem results by Lemma 3. 5.

Corollary 3. 1. 1. *The asymptotic density of* $Q^*_{k,r}$ *is*

$$\delta(Q^*_{k,r}) = \alpha_{k,r}; \tag{3. 15}$$

in particular, $\delta(Q^*_{k,1}) = \alpha^*_k$.

An estimate for $Q^*_k(x)$ can now be deduced on the basis of Theorem 3. 1 and the observation

$$Q^*_k(x) = \lim_{r\to\infty} Q^*_{k,r}(x). \tag{3. 16}$$

Theorem 3.2. *If* $x\geqq 2$, *then*

$$Q^*_k(x) = \alpha_k x + O(\sqrt[k]{x}\log x), \tag{3. 17}$$

where

$$\alpha_k = \zeta(k)\prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right). \tag{3. 18}$$

Proof. Denote the general factor of the product in (3. 11) by $1 + L_p(k, r)$. We have for all r,

$$|L_p(k, r)| \leqq \frac{2}{p^k} + \frac{1}{p^{k+1}} + \frac{1}{p^{kr+k}} + \frac{1}{p^{kr+k+1}} < \frac{5}{p^k}.$$

Therefore, the series $\Sigma_p L_p(k, r)$ and hence the product in (3. 11) converge uniformly with respect to r. It follows then that

$$\lim_{r\to\infty} \alpha_{k,r} = \zeta(k)\prod_p \left\{\lim_{r\to\infty}(1 + L_p(k, r))\right\} = \alpha_k. \tag{3. 19}$$

The theorem now results by (3. 16) and the fact that the remainder term in (3. 13) is uniform in r.

Remark 3.2. We observe finally that the corollary stated in the Introduction follows from Theorem 3.2 by a simple computation similar to that in the case $k=2$ [3, Lemma 3.6].

4. Improved estimates

First we shall express the generating function $f_{k,r}(s)$ of $Q^*_{k,r}$ in terms of the generating function $\zeta(s)/\zeta(ks)$ of Q_k. To this purpose, put

$$(4.1)\qquad f_{k,r}(s)=\sum_{n=1}^{\infty}\frac{q^*_{k,r}(n)}{n^s},\quad h_{k,r}(s)=\sum_{n=1}^{\infty}\frac{g_{k,r}(n)}{n^s},$$

where

$$(4.2)\qquad h_{k,r}(s)\equiv f_{k,r}(s)\left(\frac{\zeta(ks)}{\zeta(s)}\right),\qquad s>1.$$

Remark 4.1. In the interest of clarity, we note that $h_{k,r}(s)$ is defined for $s>1$ by (4.2) and for possibly smaller values of s by the sum of the Dirichlet series for $h_{k,r}(s)$, whose coefficients, denoted $g_{k,r}(n)$, are obtained from (4.2) by Dirichlet multiplication.

Remark 4.2. On the basis of (4.1) and (4.2) and the multiplicativity of $q^*_{k,r}(n)$, it follows that $g_{k,r}(n)$ is a multiplicative function of n.

Lemma 4.1. (a) *The series expansion* (4.1) *of $h_{k,r}(s)$ converges absolutely for $s>1/(k+1)$;* (b) *for $s>1/k$,*

$$(4.3)\qquad h_{k,r}(s)=\zeta^2(ks)\prod_p\left(1-\frac{2}{p^{ks}}+\frac{1}{p^{s(k+1)}}+\frac{1}{p^{ks(r+1)}}-\frac{1}{p^{s(kr+k+1)}}\right);$$

(c) *for $s>1/(k+1)$, $h_{k,r}(s)$ is represented in the form,*

$$(4.4)\qquad h_{k,r}(s)=\zeta(2ks)\eta_{k,r}(s),\quad \eta_{k,r}(s)=\sum_{n=1}^{\infty}\frac{a_{k,r}(n)}{n^s},$$

the series being absolutely convergent for $s>1/(k+1)$. The function $\eta_{k,r}(s)$ is determined by (4.6).

Proof. For $s>1$,

$$f_{k,r}(s)=\prod_p\left(\sum_{j=0}^{\infty}\frac{1}{p^{js}}-\sum_{j=1}^{r}\frac{1}{p^{jks}}\right)$$

$$=\prod_p\left(\frac{1}{1-\frac{1}{p^s}}-\frac{1}{p^{ks}}-\frac{1}{p^{2ks}}-\cdots-\frac{1}{p^{rks}}\right),$$

so that, on dividing out $\zeta(s)=\Pi(1-p^{-s})^{-1}$, one obtains

$$h_{k,r}(s)=\zeta(ks)\prod_p\left\{1-\left(1-\frac{1}{p^s}\right)\left(\frac{1}{p^{ks}}+\cdots+\frac{1}{p^{rks}}\right)\right\}. \tag{4.5}$$

By an easy computation, (4. 3) follows from (4. 5).

Now the product on the right of (4. 3) converges absolutely for $s>1/k$. Hence the Dirichlet series with product representation (4. 3) must also converge absolutely for $s>1/k$. By the uniqueness theorem for Dirichlet series [6, p. 245], the coefficients are therefore furnished by $g_{k,r}(n)$, which proves Part (b).

Another simple computation based on (4. 5) yields the relation on the left of (4. 4), where

$$\begin{aligned}\eta_{k,r}(s)\equiv\prod_p\Big\{1+\frac{1}{p^{s(k+1)}}-\Big(\frac{2}{p^{2ks}}-\frac{2}{p^{(2k+1)s}}+\cdots\\ \cdots+\frac{2}{p^{rks}}-\frac{2}{p^{(rk+1)s}}\Big)-\frac{1}{p^{(r+1)ks}}+\frac{1}{p^{(rk+k+1)s}}\Big\}.\end{aligned} \tag{4.6}$$

Parts (c) and (a) result by the same type of argument used in the proof of Part (b).

By Lemma 4. 1 (b) and (3. 11), it follows that

Lemma 4. 2.

$$h_{k,r}(1)=\sum_{n=1}^{\infty}\frac{g_{k,r}(n)}{n}=\alpha_{k,r}\zeta(k). \tag{4.7}$$

Lemma 4. 3.

$$q^*_{k,r}(n)=\sum_{d\delta=n}q_k(d)g_{k,r}(\delta), \tag{4.8}$$

$$g_{k,r}(n)=\sum_{d\delta^{2k}=n}a_{k,r}(d). \tag{4.9}$$

Proof. By virtue of (2. 1) and the rule for Dirichlet multiplication, (4. 8) is a consequence of (4. 1) and (4. 2); (4. 9) follows from (4. 4).

Lemma 4. 4. *If $s>1/(k+1)$, then there exists a quantity $A_k(s)$, independent of r, such that*

$$T_{k,r}(s)\equiv\sum_{n=1}^{\infty}\frac{|g_{k,r}(n)|}{n^s}\leqq A_k(s),\qquad r\geqq 1. \tag{4.10}$$

Proof. By Lemma 4. 1 (a), the series in (4. 10) converges for $s>1/(k+1)$. By (4. 6), the factors in the infinite product representation of $\zeta(2ks)\eta_{k,r}(s)$ will contain, except for an initial 1, only terms of the form b_jp^{-js} where $j\geqq k+1$ and $|b_j|\leqq 2(j+1)$ for all occurring j. Hence, by the multiplicativity of $|g_{k,r}(n)|$, (Remark 4. 2),

$$T_{k,r}(s)=\prod_p\left(1+\sum_{j=k+1}^{\infty}\frac{|b_j|}{p^{js}}\right)\leqq\prod_p\left(1+\frac{2}{p^{s(k+1)}}\sum_{i=0}^{\infty}\frac{i+k+1}{p^{is}}\right)$$

$$\leqq\prod_p\left(1+\frac{2}{p^{s(k+1)}}\sum_{i=0}^{\infty}\frac{i+k+1}{2^{is}}\right)=\prod_p\left(1+\frac{2c_k}{p^{s(k+1)}}\right),$$

where c_k is independent of p and r. The convergence of the final product proves the theorem.

Lemma 4. 5. *If* $s>1/(k+1)$, *then for all* r

$$T^*_{k,r}(s)\equiv\sum_{n=1}^{\infty}\frac{|a_{k,r}(n)|}{n^s}\leqq A^*_k(s),$$

where $A^*_k(s)$ *is independent of* r.

The proof is similar to that of the preceding lemma except that the details are simpler.

Lemma 4. 6. *If* $\beta>1/(k+1)$, *then*

$$G^*_{k,r}(x)\equiv\sum_{n\leqq x}|g_{k,r}(n)|\leqq A^*_k(\beta)x^\beta, \tag{4.11}$$

where $A^*_k(\beta)$ *is independent of* r *and* x.

Proof. By (4. 9) and Lemma 4. 5,

$$G^*_{k,r}(x)\leqq\sum_{d\delta^{2k}\leqq x}|a_{k,r}(d)|=\sum_{n\leqq x}|a_{k,r}(n)|\left[\left(\frac{x}{n}\right)^{\frac{1}{2k}}\right]$$

$$\leqq\sum_{n\leqq x}|a_{k,r}(n)|\left[\left(\frac{x}{n}\right)^{\beta}\right]\leqq x^\beta\sum_{n\leqq x}\frac{|a_{k,r}(n)|}{n^\beta}\leqq A^*_k(\beta)x^\beta.$$

The proof is complete.

It is convenient to define

$$\Delta_k(x)=Q_k(x)-\frac{x}{\zeta(k)}. \tag{4.12}$$

We are now in a position to prove

Theorem 4. 1.

$$Q^*_{k,r}(x)=\alpha_{k,r}x+O(\sqrt[k]{x}) \tag{4.13}$$

uniformly with respect to r.

Proof. By (4. 8),

$$Q^*_{k,r}(x) = \sum_{d\delta \leqq x} q_k(d) g_{k,r}(\delta) = \sum_{n \leqq x} g_{k,r}(n) Q_k\left(\frac{x}{n}\right),$$

and hence by (4. 12),

$$Q^*_{k,r}(x) = \frac{x}{\zeta(k)} \sum_{n \leqq x} \frac{g_{k,r}(n)}{n} + V_{k,r}(x, n), \tag{4. 14}$$

where

$$V_{k,r}(x, n) = \sum_{n \leqq x} g_{k,r}(n) \Delta_k\left(\frac{x}{n}\right). \tag{4. 15}$$

Application of Lemma 4. 2 gives

$$\sum_{n \leqq x} \frac{g_{k,r}(n)}{n} = \alpha_{k,r} \zeta(k) + \sum_{n > x} \frac{g_{k,r}(n)}{n}. \tag{4. 16}$$

Let $G_{k,r}(x)$ denote the summatory function of $g_{k,r}(n)$; that is, $G_{k,r}(x) = \Sigma g_{k,r}(n)$ summed over $n \leqq x$. Then, by partial summation and Lemma 4. 6 with $\beta = 1/k$, one deduces

$$\left| \sum_{n > x} \frac{g_{k,r}(n)}{n} \right| = \left| \sum_{n > x} \frac{G_{k,r}(n)}{n(n+1)} - \frac{G_{k,r}(x)}{[x]+1} \right| \leqq \sum_{n > x} \frac{G^*_{k,r}(n)}{n^2} + \frac{G^*_{k,r}(x)}{x}$$

$$\leqq A^*_k(1/k) \left(\sum_{n > x} \frac{1}{n^{2-1/k}} + \frac{1}{x^{1-1/k}} \right).$$

Hence

$$\sum_{n > x} \frac{g_{k,r}(n)}{n} = O\left(\frac{1}{x^{1-1/k}}\right) \quad \text{uniformly in } r. \tag{4. 17}$$

We turn now to $V_{k,r}(x, n)$ in (4. 15). By Lemma 2. 1, there exists a constant $b_k > 0$, depending at most upon k, such that $|\Delta_k(x)| \leqq b_k \sqrt[k]{x}$. Hence by Lemma 4. 4,

$$|V_{k,r}(x, n)| \leqq b_k \sqrt[k]{x} \sum_{n \leqq x} \frac{|g_{k,r}(n)|}{n^{1/k}} \leqq b_k A_k(1/k) \sqrt[k]{x}.$$

That is,

$$V_{k,r}(x, n) = O(\sqrt[k]{x}) \quad \text{uniformly in } r. \tag{4. 18}$$

The theorem results on collecting (4. 14), (4. 16), (4. 17) and (4. 18).

Theorem 4. 2.

$$Q^*_k(x) = \alpha_k x + O(\sqrt[k]{x}). \tag{4. 19}$$

Proof. The theorem results from the application of (3. 16) and (3. 19) to (4. 13), in conjunction with the fact that the latter estimate is uniform in r.

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