On unitary dilations of bounded operators

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SZ.-NAGY's theorem on the unitary dilation of a linear operator T on Hilbert space with $\|T\| \leqq 1$ may be generalized so as that it applies simultaneously to any product of such operators. This was observed by SZ.-NAGY himself, as a consequence of the matrix construction, due to SCHÄFFER, of the unitary dilations[1]). We state this theorem in an equivalent form and give an alternative proof which does not make use of square roots of positive operators.

Theorem. *Let H be a Hilbert space of infinite dimension and P a projection operator on H such that the dimension of $(1-P)H$ is not less than that of PH. Corresponding to every bounded linear operator T on PH with $\|T\| \leqq 1$, we can find a unitary operator U_T on H such that*

$$T_1 T_2 \dots T_n P = P U_{T_1} U_{T_2} \dots U_{T_n} P$$

for every finite number of operators $T_1, T_2, \dots, T_n$ on PH with $\|T_\nu\| \leqq 1$ $(\nu = 1, 2, \dots, n)$.

Proof. We can find easily projection operators P_1, P_2, P_3 on H such that $P + P_1 + P_2 + P_3 = 1$, the dimension of $P_1 H$ is the same as that of PH, and $P_2 H$ and $P_3 H$ have each the dimension infinite and not less than that of PH. Let T be a linear operator on PH with $\|T\| \leqq 1$, and z_λ $(\lambda \in \Lambda)$ a complete orthonormal system in PH. Then we can find a system of elements $y_\lambda \in P_1 H$ $(\lambda \in \Lambda)$ such that $Tz_\lambda + y_\lambda$ $(\lambda \in \Lambda)$ constitutes an orthonormal system, i. e.

$$(Tz_\lambda + y_\lambda,\ Tz_\varrho + y_\varrho) = \delta_{\lambda,\varrho} \qquad (\lambda, \varrho \in \Lambda)$$

with the Kronecker $\delta_{\lambda,\varrho}$. Because, putting

$$\alpha_{\lambda,\varrho} = \delta_{\lambda,\varrho} - (Tz_\lambda, Tz_\varrho) \qquad (\lambda, \varrho \in \Lambda),$$

[1]) See F. RIESZ—B. SZ.-NAGY, *Vorlesungen über Funktionalanalysis* (Berlin, 1956), Nachtrag (p. 460).

we see easily that $\sum_{\lambda,\varrho}\xi_\lambda\bar{\xi}_\varrho\alpha_{\lambda,\varrho}\geqq 0$ for every finite number of complex numbers ξ_λ, and hence putting

$$((\xi_\lambda),\ (\eta_\lambda))=\sum_{\lambda,\varrho}\xi_\lambda\bar{\eta}_\varrho\,\alpha_{\lambda,\varrho},$$

we can introduce an inner product, not always proper, into the linear space of vectors $(\xi_\lambda)_{\lambda\in\Lambda}$: $\xi_\lambda=0$ except for a finite number of λ. As the dimension of P_1H is not less than the cardinal number of Λ we can find $y_\lambda\in P_1H$ such that $(y_\lambda, y_\varrho)=\alpha_{\lambda,\varrho}$ $(\lambda,\varrho\in\Lambda)$ and hence, $Tz_\lambda+y_\lambda$ $(\lambda\in\Lambda)$ is an orthonormal system. Then, putting $U_Tz_\lambda=Tz_\lambda+y_\lambda$, we obtain an isometric operator U_T from PH into $(P+P_1)H$, and we have obviously $TP=PU_TP$. Now we extend U_T as follows. As the dimension of $(P_1+P_2)H$ coincides with that of P_2H, we can extend U_T such that U_T is an isometric operator from $(P_1+P_2)H$ onto P_2H. Denoting by Q the projection operator of U_TPH, the dimension of P_3H coincides with that of $(P+P_1-Q+P_3)H$, and hence we can extend U_T such that U_T is an isometric operator from P_3H onto $(P+P_1-Q+P_3)H$. Then U_T becomes a unitary operator on H and we have obviously

$$U_T(P+P_1+P_2)=(P+P_1+P_2)U_T(P+P_1+P_2),$$
$$PU_T(P_1+P_2)=0,$$
$$TP=PU_TP.$$

For every finite number of linear operators T_ν with $\|T_\nu\|\leqq 1$ $(\nu=1,2,\dots,n)$, we have then

$$A=PU_{T_1}, U_{T_2}\dots U_{T_n}P=PU_{T_1}(P+P_1+P_2)U_{T_2}(P+P_1+P_2)\dots$$
$$\dots(P+P_1+P_2)U_{T_n}P$$

because $(P+P_1+P_2)P=P$,

$$A=PU_{T_1}PU_{T_2}P\cdots PU_{T_n}P$$

because $PU_T(P+P_1+P_2)=PU_TP+PU_T(P_1+P_2)=PU_TP$, and, finally

$$A=T_1T_2\dots T_nP$$

because $TP=PU_TP$.

Remark. When the dimension of H is finite and not less than n times that of PH, then we can find projection operators P_ν $(\nu=1,2,\dots,n)$ such that

$$P+\sum_{\nu=1}^{n}P_\nu=1$$

and the dimension of each $P_\nu H$ for $\nu \leqq n-1$ coincides with that of PH. Then, for any linear operator T on PH with $\|T\| \leqq 1$, we can find an isometric U_T from PH into $(P+P_1)H$ by the same way as above. If we extend U_T so that U_T is an isometric operator from $P_\nu H$ onto $P_{\nu+1}H$ for $\nu \leqq n-2$, from $P_{n-1}H$ onto $(P+P_1-Q)H$, and from $P_n H$ onto $P_n H$, then we see easily that

$$T_1 T_2 \dots T_\nu P = P U_{T_1} U_{T_2} \dots U_{T_\nu} P \text{ for } \nu \leqq n-1.$$

(Received November 29, 1960)