Integration with respect to operator-valued functions

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1. Introduction

1. 1. Our basic problem is to integrate scalar-valued functions with respect to operator-valued functions that are not of bounded variation.

Given a fixed measure space (R, μ), let $\mathfrak{E}_r$ denote the Banach space of endomorphisms of $L_r(R, \mu)$ (see [4, p. 51][1])). Let J be a fixed compact subinterval of $]-\infty, \infty[$. Suppose that E_r is a function on J which assumes its values in $\mathfrak{E}_r$; this article is chiefly concerned with the convergence in $\mathfrak{E}_r$ of the integral

$$(1) \qquad \int_J f(\lambda)\cdot dE_r(\lambda),$$

where f belongs to the class $\mathfrak{D}(J)$ of all simply-discontinuous, complex-valued functions. The integrator E_r need not be of bounded variation in the sense of HILLE [4, p. 59]; see for example 10. 7. Part III of this article deals with applications to the theory of multipliers of Fourier series.

Suppose for a moment that E_r is a resolution of the identity in $L_2(R, \mu)$ (in the sense of [17, p. 174]). The integral (1) need not converge in $\mathfrak{E}_2$ for all f in $\mathfrak{D}(J)$. This situation is remedied by interpreting (1) in the *modified* Pollard—Moore—Stieltjes sense [3, p. 273]; the integral shall then be symbolized by either of the following two notations:

$$(2) \qquad \mathbf{E}_r(f) = (\mathfrak{E}_r) \oint f\cdot dE_r.$$

1. 2. The Wiener—Young class $\mathfrak{W}_p(J)$ consists of all complex-valued functions f such that $V_p(f) \neq \infty$, where

$$V_p(f) = \sup \Big(\sum_k |f(\beta_k) - f(\alpha_k)|^p\Big)^{1/p}$$

over every choice of a finite number of non-overlapping subintervals $[\alpha_k, \beta_k]$

1) The topology of $\mathfrak{E}_r$ is also called the uniform operator-topology.

of J (WIENER [18], YOUNG [19], and 4.2). Let $L^0 = L^0(R, \mu)$ be the class of all (R, μ)-simple functions. If $T \in \mathfrak{E}_2$, then we define:

$$|T|_r = \sup\{\|Tx\|_r : x \in L^0 \text{ and } \|x\|_r \leqq 1\}; \tag{3}$$

it is clear that the eventuality $|T|_r \neq \infty$ implies that $T_r \in \mathfrak{E}_r$, where T_r denotes the continuous extension of T from L^0 to $L_r(R, \mu)$.

1.3. Suppose that E is a resolution of the identity in $L_2(R, \mu)$ such that

$$\infty \neq \sup_{\lambda \in J} |E(\lambda)|_s \text{ whenever } 1 < s < \infty. \tag{v}$$

The integrator E_r is now defined for all λ in J by the relation $E_r(\lambda) = E(\lambda)_r$ (as in 1.2, $E(\lambda)_r$ is the continuous extension to $L_r(R, \mu)$). It will be proved that, if $1 \leqq p < \infty$, then there exists an interval $I(p)$ such that the integral (2) converges in $\mathfrak{E}_r$ for each r in $I(p)$ whenever $f \in \mathfrak{W}_p(J)$. It turns out that the mapping $f \to \mathbf{E}_r(f)$ is a continuous linear transformation of the Banach space $\mathfrak{W}_p(J)$ into $\mathfrak{E}_r$. If $p > q$ then:

$$\mathfrak{D}(J) \supset \mathfrak{W}_p(J) \supset \mathfrak{W}_q \supset \mathfrak{W}_1(J) = \{\text{bounded variation}\}$$
$$2 \in I(p) \subset I(q) \subset I(1) =]1, \infty[.$$

1.4. Motivation. Suppose $1 < r < \infty$, $r \neq 2$. Let f be the function defined by $f(\lambda) = \lambda$ for each λ in J. There are some integrators E_r with the following property: there exists no spectral measure M such that

$$\int \lambda \cdot M(d\lambda) = (\mathfrak{E}_r) \oint f \cdot dE_r,$$

although the integral on the right-hand side converges. More details are given in 6.1 and 10.7.

2. Two applications to the theory of multiplier transformations

2.1. Consider a complete orthonormal system $\{\Phi_n : n \in R\} \subset \mathfrak{D}(J)$, where R is now a subset of the integers. In this case, the measure space (R, μ) is so chosen that $L_r(R, \mu)$ becomes the sequence space usually denoted l_r. If $x \in l_1$, let $\hat{x}$ be the function defined as follows:

$$\hat{x}(\lambda) = \sum_{r \in R} x_r \cdot \Phi_r(\lambda) \qquad (\lambda \in J).$$

If $f \in \mathfrak{D}(J)$, then $f \# x$ is defined as the sequence of Fourier coefficients of the function $\lambda \to f(\lambda) \cdot \hat{x}(\lambda)$. Let $f_\#$ denote the mapping $x \to f \# x$ defined on l_1; HIRSCHMAN[2]) calls $f_\#$ a "*multiplier transformation*".

[2]) The writer is indebted to Professors HIRSCHMAN, GOFFMAN, and HENRIKSEN for valuable suggestions.

An important problem in the theory of multiplier transformations is to find conditions on f which will insure that $|f_{\#}|_r \neq \infty$; this in turn implies that the continuous extension $f_{\# r}$ belongs to $\mathfrak{E}_r$. In § 10 we examine two systems $\{\Phi_n : n \in R\}$ that give rise to a resolution of the identity E; condition 1.3 (v) is satisfied in both cases, and it results from our theory that

(i) *if* $f \in \mathfrak{W}_p(J)$ *and* $r \in I(p)$, *then* $|f_{\#}|_r \neq \infty$.

In fact, it will be proved that

(ii) *if* $f \in \mathfrak{W}_p(J)$ *and* $r \in I(p)$, *then* $f_{\# r} = \mathbf{E}_r(f) \in \mathfrak{E}_r$.

The first system $\{\Phi_n : n \in R\}$ is the system of normalized Legendre polynomials (see 10.5); the proof of (i)—(ii) depends in this case on an article by HIRSCHMAN [7]. Property (i) was discovered by HIRSCHMAN [5] in the case where $\{\Phi_n : n \in R\}$ is the trigonometric system; for this second system we derive (i)—(ii) directly from two properties of the Hilbert transformation on l_p (see 10.6).

3. Hölder-type inequalities and the variation-norm

3.1. We now return to the general setting described in 1.3. Suppose that $(x, y) \in L^0 \times L^0$; the relation

$$E_{x,y}(\lambda) = \int_R y \cdot E(\lambda) x \cdot d\mu \qquad (\lambda \in J)$$

defines a complex-valued function $E_{x,y}$. We write

(1) $$U_r = \{(x, y) \in L^0 \times L^0 : \|x\|_r \leqq 1 \text{ and } \|y\|_{r'} \leqq 1\},$$

where $r' = r/(r-1)$. The variation-norm is defined as follows:

$$V_q(E)_r = \sup \{V_q(E_{x,y}) : (x, y) \in U_r\}.$$

When $f \in \mathfrak{D}(J)$ it is easy to verify the familiar inequalities

(iii) $$\left|(\mathfrak{E}_2) \oint f \cdot dE_2\right|_2 \leqq V_1(E)_2 \cdot \mathfrak{N}(f; J)_\infty < \infty,$$

where $\mathfrak{N}(f; J)_\infty = \sup \{|f(\lambda)| : \lambda \in J\}$. The norm $W(f)_p = \mathfrak{N}(f; J)_\infty + V_p(f)$ makes $\mathfrak{W}_p(J)$ into a Banach space.

Suppose $1 < p < \infty$ and $r \in I(p)$. Our approach involves establishing the existence of a number $q > 1$ such that $q^{-1} + p^{-1} > 1$ and

(iii*) $$\left|(\mathfrak{E}_r) \oint f \cdot dE_r\right|_r \leqq B(r, p) \cdot V_q(E)_r \cdot W(f)_p < \infty$$

(where $B(r, p)$ is independent of f and E), for all f in $\mathfrak{W}_p(J)$. This is closely related to a theorem obtained for scalar-valued integrators by LOVE and YOUNG [15]; in fact, their results originate from the same inequality[3]) that we use to prove (iii*). A suitable definition of $V_\infty(E)_r$ conserves the inequality (iii*) in the case $p=1$ (see 9.7).

PART I

4. Preliminaries

4.1. A closed interval J is kept fixed throughout. Let $\mathfrak{J}$ be the family of all half-open intervals $]\alpha, \beta]$ with end-points α, β in J. Let Π be the class of all finite families of disjoint members of $\mathfrak{J}$. In other words, if $\pi \in \Pi$, then π is a disjoint family of intervals $i =]\alpha, \beta] \subset J$.

4.2. Suppose that F is a vector-valued function on J. In case $i \in \mathfrak{J}$ we write

(1) $$\Delta F(i) = F(\beta) - F(\alpha) \text{ whenever } i =]\alpha, \beta].$$

The relation (1) defines on $\mathfrak{J}$ the function ΔF. If a is a subset of $]-\infty, \infty[$, then $\mathfrak{F}(a)$ will denote the class of all complex-valued functions on a. If $\varphi \in \mathfrak{F}(a)$, we shall write

$$\mathfrak{N}(\varphi; a)_p = \left(\sum_{i \in a} |\varphi(i)|^p\right)^{1/p},$$

and

$$\mathfrak{N}(\varphi; a)_\infty = \sup\{|\varphi(i)| : i \in a\}.$$

In case $F \in \mathfrak{F}(J)$ and $1 \leqq p \leqq \infty$, then we write

(2) $$V_p(F) = \sup_{\pi \in \Pi} \mathfrak{N}(\Delta F; \pi)_p.$$

5. The variation-norm

5.1. Besides the interval J, we hold fixed a measure space (R, μ). Let $L^0 = L^0(R, \mu)$ be the corresponding class of (R, μ)-simple functions. The spaces $L_r = L_r(R, \mu)$ are subjected to the usual norm $\|x\|_r$. Let $\mathfrak{L}(L_r, L_r)$ denote the class of all linear mappings of L_r into itself.

If $T \in \mathfrak{L}(L_2, L_2)$ and $(x, y) \in L^0 \times L^0$, then we may write

$$T_{x,y} = \int_R y \cdot Tx \cdot d\mu.$$

[3]) Due to L. C. YOUNG [19].

Note that

(1) $$|T|_r = \sup\{|T_{x,y}| : (x, y) \in U_r\};$$

$|T|_r$ was defined in 1.2 (3) and U_r in 3.1 (1).

Unless otherwise specified, E will consistently denote a function on J which assumes its values in $\mathfrak{L}(L_2, L_2)$. If $(x, y) \in L^0 \times L^0$ and $\lambda \in J$, then $E(\lambda)$ is some member T of $\mathfrak{L}(L_2, L_2)$, so that $E(\lambda)_{x,y} = T_{x,y}$ is a scalar; the function $E_{x,y}$ is defined by the relation

(2) $$E_{x,y}(\lambda) = E(\lambda)_{x,y} \qquad (\lambda \in J).$$

In case $1 \leqq r,\ q \leqq \infty$ we define

(3) $$V_q(E)_r = \sup\{V_q(E_{x,y}) : (x, y) \in U_r\}.$$

5.2. Theorem. *If* $M(\alpha, \beta) = V_{1/\beta}(E)_{1/\alpha}$, *then* $\log M(\alpha, \beta)$ *is a convex function of* (α, β) *in the rectangle* $0 \leqq \alpha, \beta \leqq 1$.

5.3. Remark. Let $P_0 = (\alpha_0, \beta_0)$ and $P_1 = (\alpha_1, \beta_1)$ be any two points in the rectangle $[0, 1] \times [0, 1]$. Theorem 5.2 is clearly equivalent to the following assertion: if $0 \leqq t \leqq 1$ and

(4) $$(\alpha, \beta) = tP_0 + (1-t)P_1,$$

then

(5) $$M(\alpha, \beta) \leqq M(P_0)^t \cdot M(P_1)^{1-t}.$$

Proof of 5.2. Take $(x, y) \in L^0 \times L^0$ and $\pi \in \Pi$; set $T(x, y) = \Delta E_{x,y}$ (see 4.2 (1)). Note that $T(x, y) \in \mathfrak{F}(\pi)$. In view of (5), 5.1 (3), and 4.2 (2), it will clearly suffice to show that

(6) $$\mathfrak{N}(T(x, y); \pi)_{1/\beta} \leqq M(P_0)^t \cdot M(P_1)^{1-t} \cdot \|x\|_r \cdot \|y\|_{r'},$$

where $r' = r/(r-1)$ and $r = 1/\alpha$. Counting-measure μ_0 makes (π, μ_0) into a measure space such that the norm of $L_r(\pi, \mu_0)$ coincides with the norm $\{\varphi \to \mathfrak{N}(\varphi; \pi)_r\}$ (see 4.2). It is easily checked that $\{(x, y) \to T(x, y)\}$ is a multilinear mapping into the class of μ_0-measurable functions on the set π; the conclusion (6) is now a direct consequence of the Riesz—Thorin theorem [20, p. 106].

6. The type of integrator that will be used

6.1. Let M be a spectral measure which assumes its values in the space $\mathfrak{E}_r$ (see 1.1). By definition, M is weakly countably-additive and satisfies the relation

(v*) $$\infty \neq \sup_{\sigma \in \mathfrak{B}} |M(\sigma)|_r,$$

where $\mathfrak{B}$ is the ring of Borel subsets of J (see [1, p. 324]). Consequently, the integral

$$\int_J f(\lambda)\cdot M(d\lambda)$$

can be defined when $f\in\mathfrak{D}(J)$ (see [1, p. 340]). It will be shown in 10.7 that this type of integration is too restrictive for our purposes. Let $\mathfrak{C}(J)$ be the class of all complex-valued, continuous functions on J, and suppose that E_r assumes its values in $\mathfrak{C}_r$. If $\infty\neq V_1(E_r)_r$, then the Stieltjes integral

$$\text{(1)}\qquad \int_J f(\lambda)\cdot dE_r(\lambda) \qquad (\text{where } f\in\mathfrak{C}(J)),$$

is easily seen to converge in $\mathfrak{C}_r$. General results concerning the case $\infty=V_1(E_r)_r$ have apparently not yet been published. In 10.7 will be displayed an integrator E_r with $\infty=V_1(E_r)_r$ when $r\neq 2$, $1<r<\infty$, although the integral (1) converges in $\mathfrak{C}_r$. Let M be the extension to the Borel ring $\mathfrak{B}$ of the set-function ΔE_r (defined by 4.2 (1)); in view of 5.1 (1) it is easily verified that

$$\text{(2)}\qquad \sup_{\pi\in\Pi}\Big|\sum_{i\in\pi}\Delta E_r(i)\Big|_r\leqq V_1(E_r)_r\leqq 4\sup_{\sigma\in\mathfrak{B}}|M(\sigma)|_r,$$

where the second inequality comes from [2, p. 97], 5.1 (2) and 5.1 (1). Consequently, from (2) and [4, p. 60] it follows that $\infty\neq V_1(E_r)_r$ iff E_r is of "*bounded variation*" as defined in [4, p. 59].[5])

6.2. Remark. If M is a spectral resolution, or if $E=E_r$ is a resolution of the identity in $L_2(R,\mu)$ (cf. [17, p. 174]), then the relation

$$\text{(iv)}\qquad \infty\neq V_1(E)_2$$

follows easily from 6.1 (2).

6.3. Definitions. Set $L_2=L_2(R,\mu)$. We will say that E is a $V(R,\mu)$-type integrator iff E is a function on J that assumes its values in $\mathfrak{L}(L_2,L_2)$, and which simultaneously satisfies (iv) and

$$\text{(v)}\qquad \infty\neq\sup_{\lambda\in J}|E(\lambda)|_s \quad \textit{whenever} \quad 1<s<\infty.$$

If $1\leqq p<\infty$, then

$$I(p)=\left\{\lambda:\frac{2p}{p+1}<\lambda<\frac{2p}{p-1}\right\},$$

and $I(\infty)$ will denote the limit (as $p\to\infty$) of the closure of $I(p)$.

[5]) 'iff' stands for 'if and only if'.

6.4. Remark. Note that $I(\infty)$ is the set whose only element is 2. let I be the region obtained by adding the vertex $P_0=(1/2, 1)$ and the open base segment $OB=]0, 1[$ to the open triangle OBP_0 (see Fig. 1 below). If $0\leqq\beta<1$ it is immediately verified that

$$(3) \qquad (\alpha, \beta)\in I \quad \text{iff} \quad \frac{1}{2}\beta<\alpha<1-\frac{1}{2}\beta.$$

Consequently, if $1\leqq p<\infty$, then

$$(4) \qquad r\in I(p) \quad \text{iff} \quad (1/r, 1/p')\in I,$$

where $p'=p/(p-1)$.

6.5. Lemma. *Suppose $1<p<\infty$ and $r\in I(p)$. There exists a number q such that $(r^{-1}, q^{-1})\in I$ and $1<q^{-1}+p^{-1}$.*

Proof. From (4) and (3) it follows readily that

$$m'=\frac{1}{2}\frac{1}{p'}<m''=$$
$$=\min\left\{\frac{1}{r}, 1-\frac{1}{r}\right\}.$$

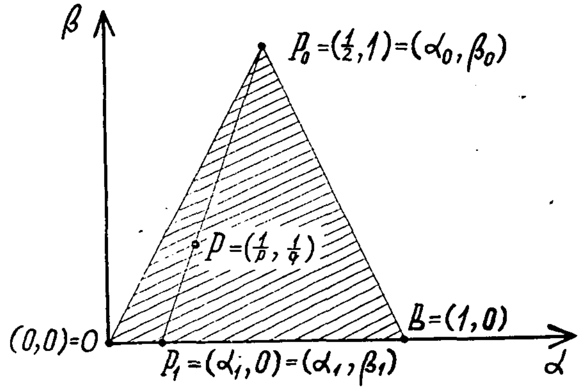

Fig. 1

Choose q such that $m'<(2q)^{-1}<m''$. From $(2q)^{-1}>m'$ follows $q^{-1}+p^{-1}>1$, and the relation $(r^{-1}, q^{-1})\in I$ comes from $(2q)^{-1}<m''$ and 6.4 (3).

6.6. In case $1<p<\infty$ and $r\in I(p)$, we define

$$E(r, p)=V_q(E)_r,$$

where q is the number that was mentioned in 6.5. It will be convenient to write $E(2, \infty)=V_1(E)_2$ and $E(r, 1)=2\sup\{|E(\lambda)|_r : \lambda\in J\}$.

6.7. Theorem. *Let E be a $V(R, \mu)$-type integrator. If $1\leqq p\leqq\infty$ and $r\in I(p)$, there exists a number t in $[0, 1]$ and a number s such that $1<s<\infty$ and*

$$(5) \qquad E(r, p)\leqq E(2, \infty)^t\cdot E(s, 1)^{1-t}<\infty.$$

Proof. Note first that $t=1$, 0 when $p=\infty$, 1, respectively. Next, suppose $1<p<\infty$. From 6.5 we see that the point $P=(r^{-1}, q^{-1})$ lies in the open triangle I; since $P_0=(\alpha_0, \beta_0)$ is the vertex of I, it follows that the line from P_0 to the point P meets the open basis-segment $OB=]0, 1[$ at a point $P_1=(\alpha_1, \beta_1)=(\alpha_1, 0)$ (see Fig. 1). Therefore $P=(\alpha, \beta)=(r^{-1}, q^{-1})$ lies on the open segment $]P_0, P_1[$, and there exists a number $t \in]0, 1[$ satisfying 5.3 (4). But the convexity theorem 5.2—5.3 shows that 5.3 (4) implies 5.3 (5), and $M(\alpha, \beta) = V_{1/\beta}(E)_{1/\alpha} = E(r, p)$; a similar reformulation of $M(P_0)$ and $M(P_1)$ yields the conclusion.

7. Inequalities for Riemann sums

7.1. Until further notice, E will be a $V(R, \mu)$-type integrator (see Definition 6.3). The variation $V(f)_p$ was defined by 4.2 (2) in the case $f \in \mathfrak{F}(J)$; now set

$$\mathfrak{W}_p(J) = \{f \in \mathfrak{F}(J);\ \infty \neq V_p(f)\},$$

and

$$W(f)_p = V_p(f) + \sup\{|f(\lambda)| : \lambda \in J\}.$$

Let λ' and λ'' be the end-points of the interval J; we have $J = [\lambda', \lambda'']$ and $-\infty < \lambda' < \lambda'' < \infty$. A member π of the class Π (defined in 4.1) will be called a *partition for J* if π is a cover[4]) of $]\lambda', \lambda'']$. In other words: π is a partition for J iff π is a finite, disjoint family of intervals $]\alpha, \beta] \subset J$ such that

$$]\lambda', \lambda''] = \bigcup \{i : i \in \pi\}.$$

7.2. Definition. The domain of a function z will be denoted $[z]$. The class $\mathfrak{Z}$ will consist of all functions z such that $[z]$ is a partition for J, while

$$z_i \in \operatorname{int}(i) \textit{ for each } i \textit{ in } [z];$$

$(\operatorname{int}(i) =$ the interior of $i)$.

7.3. Remarks. Suppose $z \in \mathfrak{Z}$. The members of the partition $[z]$ can be arranged in such a way that we can write $[z] = \{i(k) : 1 \leq k \leq n\}$, where $i(k) =]d_{k-1}, d_k]$ and $\lambda' = d_0 < d_1 < \ldots < d_n = \lambda''$ (recall that $[\lambda', \lambda''] = J$); if $c_k = z_{i(k)}$, then $d_{k-1} < c_k < d_k$. We write

$$S(f;\, z) = \sum_{i \in [z]} f(z_i) \cdot \Delta E(i) = \sum_{k=1}^{n} f(c_k)\bigl(E(d_k) - E(d_{k-1})\bigr). \tag{1}$$

7.4. Lemma. *If $f \in \mathfrak{W}_1(J)$ and $z \in \mathfrak{Z}$, then*

$$|S(f;\, z)|_r \leq E(r, 1) \cdot W(f)_1.$$

Proof. An application to 7.3 (1) of Abel's partial summation formula shows that

$$S(f;\, z) = \sum_{k=2}^{n} \bigl(f(c_{k-1}) - f(c_k)\bigr) E(d_{k-1}) - f(c_1) E(d_0) + f(c_n) E(d_n);$$

the conclusion now comes from the definition (given in 6.6) of $E(r, 1)$.

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[4]) In the sense of [10, p. 49].

7.5. Lemma. *Suppose* $p^{-1}+q^{-1}>1$ *and* $z\in\mathfrak{Z}$. *If* $f\in\mathfrak{F}(J)$ *and* $g\in\mathfrak{F}(J)$, *then*

$$\Big|\sum_{i\in[z]} f(z_i)\cdot\Delta g(i)\Big|\leqq A(p,q)\cdot V_q(g)\cdot W(f)_p,$$

where $A(p,q)$ *is the number defined as follows:*

$$A(p,q)=1+\sum_{n=1}^{\infty} n^{-t} \qquad (t=p^{-1}+q^{-1}).$$

This lemma is due to L. C. YOUNG; its proof is sketched in the appendix.

7.6. Theorem. *Suppose* $1\leqq p\leqq\infty$ *and* $r\in I(p)$. *There exists a number* $B(r,p)$ *with the following property: if* $f\in\mathfrak{W}_p(J)$ *and* $z\in\mathfrak{Z}$, *then*

$$|S(f;z)|_r\leqq B(r,p)\cdot E(r,p)\cdot W(f)_p. \tag{2}$$

Proof. We define B as follows. If p is an end-point of the interval $[1,\infty]$ and $r\in I(p)$, then $B(r,p)=1$; if $p\in]1,\infty[$ and $r\in I(p)$, then $B(r,p)=A(p,q)$, where q is the number that was introduced in 6.5. If $p=1$, then (2) is a re-statement of 7.4. Now for the case $1<p<\infty$. Suppose that $(x,y)\in U_r$ (as in 3.1 (1)), and let g be the function $E_{x,y}$ that was defined in 5.1 (2): from 7.5 we therefore see that

$$\Big|\sum_{i\in[z]} f(z_i)\cdot\Delta E_{x,y}(i)\Big|\leqq B(r,p)\cdot V_q(E_{x,y})\cdot W(f)_p. \tag{3}$$

From 5.1 (3) and 6.6 follows that $V_q(E_{x,y})\leqq V_q(E)_r=E(r,p)$. Set $T=S(f;z)$ and observe that

$$T_{x,y}=\sum_{i\in[z]} f(z_i)\cdot\Delta E_{x,y}(i).$$

From (3) we accordingly obtain that

$$|T_{x,y}|\leqq B(r,p)\cdot E(r,p)\cdot W(f)_p,$$

and a glance at 5.1 (1) now yields the conclusion (2). In the remaining case $p=\infty$ then $r=2$, and the proof of (2) is exactly the same as above, except that $q=1$; in this case, relation (3) is easily obtained directly.

8. Simply-discontinuous functions

8.1. As before, $\mathfrak{F}(J)$ is the class of all complex-valued functions on J. Let $\mathfrak{D}(J)$ be the class of all f in $\mathfrak{F}(J)$ such that the limits $f(\lambda\pm 0)$ exist for each λ in the interior of J. N. WIENER has proved that $\mathfrak{D}(J)\supset\mathfrak{W}_p(J)$ whenever $1\leqq p\leqq\infty$ (see [18] and [19, p. 261]).

8.2. Lemma. *Suppose $f \in \mathfrak{D}(J)$ and $\varepsilon > 0$. The discontinuities of f on J form a denumerable set N, and there exists a member z^ε of $\mathfrak{Z}$ such that*

(1) $$\Omega(f; \operatorname{int}(i)) \leqq \varepsilon \text{ for each } i \text{ in } [z^\varepsilon],$$

where $\operatorname{int}(i) =$ *interior of* i, *and*

(2) $$\Omega(f; A) = \sup\{|f(\theta) - f(\lambda)| : (\theta, \lambda) \in A \times A\}.$$

This important lemma was first proved by LEBESGUE [13]; see also [9, top of p. 705].

8.3. Remark. If $A \subset J$, let χA denote the characteristic function of the set A. If $\lambda \in J$ we write $e(\lambda) = \chi[\lambda', \lambda]$. In other words:

(3) $$e(\lambda)(\theta) = \begin{cases} 1 & \text{if } \theta \leqq \lambda \\ 0 & \text{if } \theta > \lambda \end{cases} \qquad (\theta \in J).$$

Note that, if $i =]\alpha, \beta]$, then $\Delta e(i) = e(\beta) - e(\alpha) = \chi i$. Consider the function defined by the equation

(4) $$f^\varepsilon = \sum_i f(z_i^\varepsilon) \cdot \Delta e(i) \qquad (i \in [z^\varepsilon]);$$

this step-function plays an important role in the articles [14, 9]. It is easily seen that, if $f \in \mathfrak{D}(J)$, then

(5) $$0 = \lim_{\varepsilon \to 0+} \sup\{|f(\lambda) - f^\varepsilon(\lambda)| : \lambda \notin N \text{ and } \lambda \in J\};$$

in other words: $f = \lim f^\varepsilon$ (as $\varepsilon \to 0+$) uniformly in the complement of the denumerable set N that was introduced in 8.2.

8.4. Definition. The set $\mathfrak{Z}$ can be partially ordered as follows: $z \geqq s$ iff z is a *refinement of* s, in the sense that every member of $[z]$ is a subset of some member of $[s]$.

8.5. Remark. Suppose that z and z' belong to $\mathfrak{Z}$, and let $[z \vee z']$ denote the set $\{(i, i') \in [z] \times [z'] : \emptyset \neq i \cap i'\}$. Let s be a member of $\mathfrak{Z}$ such that $[s]$ is the set $\{i \cap i' : (i, i') \in [z \vee z']\}$. Clearly $s \geqq z$ and $s \geqq z'$. The relation '$\geqq$' directs the set $\mathfrak{Z}$ (see [10, p. 65 and p. 79]).

8.6. Theorem. *Suppose $f \in \mathfrak{D}(J)$ and $\varepsilon > 0$. Let z^ε be as in* 8.2. *If z and z' are refinements of z^ε which belong to $\mathfrak{Z}$, then*

$$|S(f; z) - S(f; z')|_2 < \varepsilon^* = \varepsilon \cdot E(2, \infty).$$

Proof. Note that

$$S(f; z) = \sum \{f(z_i) \cdot \Delta E(i \cap i') : (i, i') \in [z \vee z']\}.$$

Consequently, if $T=S(f;z)-S(f;z')$ and $(x,y)\in U_2$ (as in 3.1 (1)), then

$$T_{x,y}=\sum\{(f(z_i)-f(z'_{i'}))\cdot(\Delta E(i\cap i'))_{x,y}:(i,i')\in[z\vee z']\}. \tag{6}$$

Let s be as in 8.5; by transitivity it follows that $s\geqq z^\varepsilon$. This says that $i\cap i'\subset j$ for some j in $[z^\varepsilon]$. On the other hand, $\emptyset\neq i\cap i'\subset j$, whence both i and i' are included in j. By hypothesis, $z_i\in\operatorname{int}(i)$ and $z'_{i'}\in\operatorname{int}(i')$, so that z_i and $z'_{i'}$ both belong to $\operatorname{int}(j)$. Thus, by 8.2 (1), $|f(z_i)-f(z'_{i'})|\leqq\varepsilon$. Note that $(\Delta E(s_n))_{x,y}=\Delta E_{x,y}(s_n)$. Accordingly, from (6) it can be inferred that

$$|T_{x,y}|\leqq\varepsilon\cdot\sum_{n\in[s]}|\Delta E_{x,y}(s_n)|\leqq\varepsilon\cdot V_1(E)_2.$$

The conclusion now comes from 5.1 (1) and 6.6.

9. The modified Stieltjes integral

9.1. Let $\mathfrak{E}_r$ be the Banach algebra of all bounded linear transformations of $L_r(R,\mu)$ into itself; the topology of $\mathfrak{E}_r$ is the norm-topology (the norm $\{T\to|T|_r\}$ is defined by 1.2 (3)).

Suppose that $f\in\mathfrak{D}(J)$, and let E_r be a function on J which assumes its values in a Banach space $\mathfrak{B}$. We write

$$S(f;z)_r=\sum_{i\in[z]}f(z_i)\cdot\Delta E_r(i). \tag{1}$$

It is easily seen that the partial ordering '$\geqq$' (defined in 8.4) directs the set $\mathfrak{Z}$; consequently $\{S(f;z)_r, z\in\mathfrak{Z},\geqq\}$ forms a net $(S,\geqq)$ (see [10. p. 65]). If $(S,\geqq)$ converges in the topology of $\mathfrak{B}$, then we will say that f *is* $\mathfrak{B}$-*integrable with respect to* E_r, and denote by

$$(\mathfrak{B})\oint f\cdot dE_r$$

the limit in $\mathfrak{B}$ of the net $(S,\geqq)$.

9.2. Remark. This is a straightforward generalization of what T. H. HILDEBRANDT calls the "modified Stieltjes σ-integral" (see [3, p. 273] and [9]). Our main theorem (given in 9.5 below) involves the norm-topology of $\mathfrak{E}_r$; the choice of this topology has motivated our choice of the type of integral described in 9.1 (see, however, 9.8). For the sake of brevity, $\mathfrak{E}_r$ will not be subjected to other topologies in this article. Nevertheless, it may be of interest to mention that our main results apply equally well to the ordinary Riemann—Stieltjes integral 1.1 (1) when the latter is interpreted in the strong operator-topology of $\mathfrak{E}_r$.

9.3. The class of step-functions is the linear span of the set $\{e(\lambda):\lambda\in J\}$ (see 8.3 (3)). It is easily checked that, if $z\in\mathfrak{Z}$ and if g is the step-function $\sum f(z_i)\cdot\Delta e(i)$ (where $i\in[z]$), then

$$(1)\qquad (\mathfrak{B})\oint g\cdot dE_r=\sum_{i\in[z]} f(z_i)\cdot\Delta E_r(i)=S(f;z)_r.$$

9.4. Theorem. *Suppose that E_r is a function on J which is of bounded variation in $\mathfrak{E}_r$; then all members of $\mathfrak{D}(J)$ are $\mathfrak{E}_r$-integrable with respect to E_r.*

Proof. Bounded variation is equivalent to the property $\infty\neq V_1(E_r)_r$. Observe that the conclusion of 8.6 is not restricted to the case $r=2$.

9.5. Main theorem. *Let E be a $V(R,\mu)$-type integrator, as defined in* 6.3. *Set* $1\leqq p\leqq\infty$ *and* $r\in I(p)$. *If* $f\in\mathfrak{W}_p(J)$ *then f is $\mathfrak{E}_r$-integrable with respect to E_r. Moreover*

$$(\text{vi})\qquad (\mathfrak{E}_r)\oint f\cdot dE_r=\lim_{\varepsilon\to 0+}(\mathfrak{E}_r)\oint f^{\varepsilon}\cdot dE_r,$$

where f^{ε} is the step-function that was introduced in 8.3 (4).

9.6. We here recall some of the notation that was defined in 1.2. Suppose $\lambda\in J$, and let $E(\lambda)_r^0$ denote the restriction of $E(\lambda)$ to the set $L^0(R,\mu)$ of simple functions. Let $E(\lambda)_r$ be the continuous extension of $E(\lambda)_r^0$ to $L_r(R,\mu)$; note that $|E(\lambda)_r|_r=|E(\lambda)|_r\neq\infty$ (see 1.2 (3) and 6.3 (v)). The integrator E_r is defined by the equality: $E_r(\lambda)=E(\lambda)_r$. Note that $|S(f;z)|_r=|S(f;z)_r|_r$, where $S(f;z)$ and $S(f;z)_r$ are the expressions defined in 7.3 (1) and 9.1 (1), respectively; this type of property justifies our using $S(f;z)$ instead of $S(f;z)_r$ in the following proof.

Proof of 9.5. In the case $r=2$, the conclusion follows from 9.4, the hypothesis 6.2 (iv), and from the fact (mentioned in 8.1) that $\mathfrak{W}_p(J)\subset\subset\mathfrak{D}(J)$. Now for the case $r\neq 2$. Note that $r\neq 2$ implies the inequalities $1\leqq p<\infty$, whence $I(p)$ is an open interval containing the point 2. Consequently, $r\in I(p)$ implies the existence of a number u in $I(p)$ such that r lies between u and 2; this in turn implies the existence of a number m such that

$$(2)\qquad \frac{1}{r}=\frac{1}{2}m+\frac{1}{u}(1-m)\quad\text{and}\quad 0<m<1.$$

Set $M=B(u,p)\cdot E(u,p)\cdot W(f)_p$ ($<\infty$ since 6.7). From 7.6 we see that

$$(3)\qquad |S(f;z)-S(f;z')|_u\leqq 2M\qquad (z,z'\in\mathfrak{Z}).$$

Take any $\varepsilon>0$ and let z^{ε} be as in 8.2. Take any two refinements z, z' of z^{ε}.

In order to prove the $\mathfrak{C}_r$-integrability of f, it will suffice to show that $T = S(f; z) - S(f; z')$ satisfies the relation

$$|T|_r \leqq \varepsilon^m \cdot E(2, \infty)^m \cdot (2M)^{1-m}. \tag{4}$$

But from (2) and the Riesz—Thorin convexity theorem [20, p. 95] we have

$$|T|_r \leqq |T|_2^m |T|_u^{1-m} \leqq |S(f; z) - S(f; z')|_2^m \cdot (2M)^{1-m}; \tag{5}$$

where the second inequality comes from (3). The conclusion (4) now comes from (5) and 8.6. The $\mathfrak{C}_r$-integrability of f having now been established, we turn to the proof of (vi). From 8.3 (4) and 9.3 (1) we see that

$$(\mathfrak{C}_r) \oint f^\varepsilon \cdot dE_r = S(f; z^\varepsilon)_r. \tag{6}$$

On the other hand, by replacing z' by z^ε in the preceding part of this proof, we obtain that $T = S(f; z) - S(f; z^\varepsilon)$ satisfies (4), so that (6) gives the conclusion (vi).

PART II

9.7. Suppose that E is a $V(R, \mu)$-type integrator (see 6.3). Set $1 \leqq p \leqq \infty$ and $r \in I(p)$. Note that

$$\left|(\mathfrak{C}_r) \oint f \cdot dE_r\right|_r \leqq B(r, p) \cdot E(r, p) \cdot W(f)_p; \tag{iii*}$$

the existence of the integral was proved in 9.5, and the three numbers on the right-hand side were defined in 7.6, 6.6, 7.1, respectively. The norm $\{f \to W(f)_p\}$ makes $\mathfrak{W}_p(J)$ into a Banach space, and from (iii*) it follows that the transformation $\mathbf{E}_r$ that is defined for each f in $\mathfrak{W}_p(J)$ by the equation

$$\mathbf{E}_r(f) = (\mathfrak{C}_r) \oint f \cdot dE_r$$

is a continuous mapping of $\mathfrak{W}_p(J)$ into $\mathfrak{C}_r$. Recall that the eventuality $p = \infty$ corresponds to the Hilbert space case $r = 2$. As p decreases to 1, the space $\mathfrak{W}_p(J) = \mathfrak{W}_p$ contracts while the range of r expands:

$$\mathfrak{W}_\infty \supset \mathfrak{W}_p \supset \mathfrak{W}_1 = \{\text{all functions of bounded variation}\},$$
$$\{2\} = I(\infty) \subset I(p) \subset I(1) =]1, \infty[.$$

9.8. In case the integrand f is continuous, then the conclusions of our Main Theorem (9.5) apply to the *ordinary* Riemann—Stieltjes integral. This can be seen as follows. Let us suppose for a moment that the letter f stands for a continuous function throughout this article; moreover, let us change the meaning of the symbolism 'int(i)' (that occurs only in 7.2 and 8.2) to mean 'the closure of i'. Under these circumstances, it is easily seen that the results in § 7—§ 9 remain unchanged, while the integral that was defined in 9.1 becomes the Pollard—Moore—Stieltjes integral [3, p. 269]; the latter is in turn easily shown to coincide with the ordinary Riemann—Stieltjes integral [3, p. 269].

PART III

10. Two applications to the theory of multipliers

10. 1. The forthcoming applications involve two orthonormal systems; in either case, the system is a denumerable family $\{\Phi_n : n \in a\}$ of complex-valued, continuous functions on a compact interval J. In order to apply the results of Part I, we specialize the measure space (R, μ_0) by taking $R = a$ and $\mu_0 =$ counting measure; $L^0 = L^0(a, \mu_0)$ is henceforth to be interpreted as the class of all functions in $\mathfrak{F}(a)$ that vanish off finite subsets of a (the notation $\mathfrak{F}(a)$ is defined in 4. 2). Note that $L_r(a, \mu_0)$ is now the space usually denoted l_r.

10. 2. Definitions. If $f \in \mathfrak{D}(J)$ and $x \in L^0$, then $f \# x$ is the sequence y defined by

(1) $$(f \# x)_n = y_n = \sum_{r \in a} x_r \cdot \int_J f \cdot \Phi_r \cdot \Phi_n^- \qquad (n \in a),$$

where Φ_n^- is the function whose value at λ is the complex conjugate of $\Phi_n(\lambda)$. Let $f_\#$ denote the mapping $\{x \to f \# x\}$ defined on L^0. We write

$$\mathfrak{M}(r) = \{f \in \mathfrak{D}(J) : \infty \neq |f_\#|_r\}.$$

10. 3. Remarks. Hirschman [5, 6] calls $f_\#$ a *"multiplier transformation"*. If $f \in \mathfrak{M}(r)$, we denote by $f_{\#r}$ the continuous extension of $f_\#$ to l_r (this is consistent with our previous notation, since the domain L^0 of $f_\#$ is dense in l_r).

Each of the forthcoming applications involve a $V(a, \mu_0)$-type integrator E such that

(vii) $$E(\lambda) = e(\lambda)_{\#2} \qquad \textit{for each } \lambda \textit{ in } J,$$

where $e(\lambda)$ is the step-function defined by 8. 3 (3).

10. 4. Theorem. *Set* $1 \leqq p \leqq \infty$. *If* E *is a* $V(a, \mu_0)$*-type integrator that satisfies* (vii), *then*

(i*) $$\mathfrak{W}_p(J) \subset \bigcap \{\mathfrak{M}(r) : r \in I(p)\};$$

more precisely:

(ii) $$\textit{if } f \in \mathfrak{W}_p(J) \textit{ and } r \in I(p), \textit{ then } f_{\#r} = \mathbf{E}_r(f) \in \mathfrak{E}_r.$$

Proof. Take $z \in \mathfrak{Z}$; from (vii) follows that $\Delta E(i) = (\Delta e(i))_{\#}$ (see 4.2 (1)), whence

(2) $$\Big(\sum_{i\in[z]} f(z_i)\cdot \Delta e(i)\Big)_{\#r} = \sum_{i\in[z]} f(z_i)\cdot \Delta E_r(i) = S(f;\, z)_r\,.$$

Consequently, if $z = z^{\varepsilon}$ (as in 8.2), then 8.3 (4) and (2) show that $f^{\varepsilon}_{\#r} = S(f;\, z^{\varepsilon})_r$. But, from 9.3 (1) and 8.3 (4) we see that $S(f;\, z^{\varepsilon})_r = \mathbf{E}_r(f^{\varepsilon})$, so that $f^{\varepsilon}_{\#r} = \mathbf{E}_r(f^{\varepsilon})$. In view of 9.5 (vi) therefore:

(3) $$\mathbf{E}_r(f) = \lim_{\varepsilon\to 0+} f^{\varepsilon}_{\#r} \qquad \text{(convergence in } \mathfrak{E}_r).$$

Take $x \in L^0$ and $n \in a$. Consider the relation

(4) $$(f \# x)_n = \lim_{\varepsilon\to 0+} (f^{\varepsilon} \# x)_n = (\mathbf{E}_r(f)x)_n\,;$$

the first equality is an easy consequence of 8.3 (5) and of the definition 10.2 (1), while the second equality comes from (3) by observing that convergence in $\mathfrak{E}_r$ implies pointwise convergence (which in turn comes from the fact that the norm $\|x\|_r$ coincides with the norm $\mathfrak{N}(x;\, a)_r \geqq x_n$ that was defined in 4.2). Both conclusions (i*)—(ii) follow immediately from (4) and 9.5.

10.5. *First application.* Set $J = [-1, 1]$ and let $\{\Phi_n : n \in a\}$ be the system of normalized Legendre polynomials; $a = \{0, 1, 2, \dots\}$. In this setting, HIRSCHMAN defines an operator Γ_λ by means of the equation $\Gamma_\lambda = e(\lambda)_{\#}$ (compare formula (5) in [7] with 10.2 (1)). Thus, if $E(\lambda)$ is another notation for Γ_λ, then E satisfies 10.3 (vii) by definition. HIRSCHMAN points out that E is a resolution of the identity; the rest of the article [7] is devoted to the task of proving that E satisfies 6.3 (v).

Consequently, the hypotheses of 10.4 follow from 6.2—6.3: this establishes 10.4 (i*)—2.1 (i) and 10.4 (ii).

10.6. *Second application.* Here $\Phi_n(\lambda) = \exp(2\pi i n\lambda)$ for each λ in $J = [0, 1]$ and $n \in a = \{0, \pm 1, \pm 2, \pm 3, \dots\}$. If $f \in \mathfrak{D}(J)$, let φ denote the sequence of Fourier coefficients of f. If $x \in L^0$ it is easy to show that

$$(f \# x)_n = (\varphi * x)_n = \sum_{\nu\in a} \varphi_{n-\nu}\cdot x_\nu \qquad \text{(all } n \text{ in } a).$$

The article [8] contains an interesting study of the operator $f_{\#}$ in the present setting. If $x \in l_2$, then $H^{(\lambda)}x$ will denote the function y defined for all n in a by the following equality:

$$(H^{(\lambda)}x)_n = y_n = e^{-2\pi i\lambda n} \sum_{\nu\in a} e^{2\pi i\lambda\nu} x_\nu \frac{i}{2\pi\cdot(n-\nu)}$$

where $\nu \neq n$. Let E be defined by the relation 10.3 (vii). It is easily seen that E is the resolution of the identity pertaining to the self-adjoint operator $H^{(0)}=\{x\to H^{(0)}x\}$. Take $x\in L^0$ and $\lambda\in J$. From 10.3 (vii) and 10.2 (1) it immediately follows that $E(\lambda)x=H^{(\lambda)}x-H^{(0)}x+\lambda x$. Consequently, $|E(\lambda)|_r\leqq\leqq|\lambda|+|H^{(\lambda)}|_r+|H^{(0)}|_r\leqq 1+2\cdot|H^{(0)}|_r<\infty$ (this last inequality has been proved by M. RIESZ [16]). Accordingly, E satisfies 6.3 (v), and from 6.2 we conclude that E is a $V(a,\mu_0)$-type integrator that satisfies 10.3 (vii). The properties 10.4 (i*)—2.1 (i) and 10.4 (ii) are now a consequence of 10.4.

Property 10.4 (i*)—2.1 (i) has been proved by STEČKIN in the case $p=1$; HIRSCHMAN [5] discovered it in its present generality, and based his proof on STEČKIN's result.

10.7. *Counter-examples.* Let E and l_r be as in 10.6; as usual, E_r denotes the extension to l_r of the restriction of E to L^0. Suppose $1\leqq p\leqq\infty$, $r\in I(p)$, and $r\neq 2$. In [12, p. 461] it has been established that E_r is not of bounded variation: from 6.1 (2) it therefore follows that $\infty=V_1(E_r)_r$. On the other hand, 10.6 and 10.4 (ii) imply the convergence of the integral

$$f_{\#r}=(\mathfrak{C}_r)\oint f\cdot dE_r \qquad (\text{where } f\in\mathfrak{W}_p(J)),$$

and from 9.8 we see that the ordinary Riemann—Stieltjes integral

$$f_{\#r}=\int_0^1 f(\lambda)\cdot dE_r(\lambda)$$

converges in $\mathfrak{C}_r$ whenever f is a continuous member of $\mathfrak{W}_p(J)$. In particular,

$$T_r=\int_0^1 e^{-2\pi i\lambda}\cdot dE_r(\lambda),$$

where T_r is the unitary shift operator defined by the relation $T_r x=\{n\to x_{n+1}\}$ for all x in l_r (see [12, p. 461]).

It has been shown in [11] that there exists no spectral measure M (see 6.1) such that

$$\int_0^1 f(\lambda)\cdot dE_r(\lambda)=\int_J f(\lambda)\cdot M(d\lambda),$$

where $f(\lambda)=\lambda$ for $\lambda\in J=[0,1]$ and $r\neq 2$.

APPENDIX

10.8. Lemma 7.5 can be inferred indirectly by observing that the inequality (6.2) of [19, p. 256] is based on calculations which remain valid in our slightly more general setting. The purpose of this appendix is to sketch a direct verification of 7.5 based on the pivotal lemma of [19]. We suppose

$1<p, q<\infty$ and $t=p^{-1}+q^{-1}>1$ throughout. The letters k, m, n, ν consistently shall stand for non-negative integers. The letters a and b are subsequently reserved for functions whose domain are denoted $[a]$ and $[b]$, respectively.

10.9. Lemma. *If* $[a]=[b]=\{\nu: 0<\nu\leqq n+1\}$, *then there exists a number* $k\leqq n$ *such that* $k>0$ *and*

$$|a_{k+1}b_k|\leqq n^{-t}\cdot\mathfrak{R}(a;[a])_p\cdot\mathfrak{R}(b;[b])_q.$$

Proof. See [19, p. 251] and the notation in 4.2.

10.10. If $[a]=\{\nu: 0<\nu\leqq n+1\}$ and $k>0$, then we define $T_k a$ as follows:

$$(T_k a)_m=\begin{cases} a_m & \text{if } 0<m\leqq k\\ a_m+a_{m+1} & \text{if } m=k\\ a_{m+1} & \text{if } k<m\leqq n.\end{cases}$$

10.11. If $[a]=[b]=\{\nu: 0<\nu\leqq n+1\}$, then

$$\sum_{\nu=1}^{n+1} b_\nu \sum_{m=1}^{\nu} a_m-\sum_{\nu=1}^{n}(T_k b)_\nu \sum_{m=1}^{\nu}(T_k a)_m=-a_{k+1}b_k;$$

this routine calculation is performed in [19, p. 255].

10.12. Let $\mathfrak{S}(n)$ be the class of all sequences c with range $\{c_0, c_1, c_2, \ldots, c_{n+1}\}\subset J$ such that $c_0<c_1<c_2<\ldots<c_{n+1}$. We define $(I_0c)_m=]c_{m-1}, c_m]$ and

$$(I_k c)_m=\begin{cases} (I_0 c)_m & \text{if } 0<m<k\\ (I_0 c)_m\cup(I_0 c)_{m+1} & \text{if } m=k\\ (I_0 c)_{m+1} & \text{if } k<m\leqq n.\end{cases}$$

10.13. If $F\in\mathfrak{F}(J)$ (as in 4.2), then clearly

$$\mathfrak{R}(\varDelta F\circ I_0 c;[c])_p\leqq V_p(F), \tag{5}$$

where $[c]=\{1,2,3,\ldots,n+1\}$ and $(\varDelta F\circ I_0 c)_m=\varDelta F((I_0 c)_m)=F(c_m)-F(c_{m-1})$ (see 4.2). Note further that $I_k c\in\mathfrak{S}(n-1)$ and $T_k(\varDelta F\circ I_0 c)=(\varDelta F\circ I_k c)$.

10.14. If $c, d\in\mathfrak{S}(n)$ we set

$$Q(c,d)=\sum_{\nu=1}^{n+1}(\varDelta g\circ I_0 d)_\nu\sum_{m=1}^{\nu}(\varDelta f\circ I_0 c)_m.$$

Here f and g are as in 7.5. Note that

$$\sum_{\nu=1}^{n+1} f(c_\nu)\cdot(\varDelta g\circ I_0 d)_\nu=Q(c,d)+f(c_0)\big(g(d_{n+1})-g(d_0)\big). \tag{6}$$

10.15. If $c, d\in\mathfrak{S}(n)$, then there exist c^* and d^* in $\mathfrak{S}(n-1)$ such that

$$|Q(c,d)|\leqq n^{-t}\cdot V_p(f)\cdot V_q(g)+|Q(c^*,d^*)|.$$

Proof. Set $b = \Delta g \circ I_0 d$ and $a = \Delta f \circ I_0 c$; from 10.9 and 10.13 (5) there exists therefore a number $k \leqq n$ such that

$$|a_{k+1} b_k| \leqq n^{-t} \cdot V_p(f) \cdot V_q(g),$$

and the conclusion now follows from 10.11 and the remarks made in 10.13.

10.16. Lemma. Let $\psi(0) = 1$ and $\psi(\nu) = \nu$. If $c, d \in \mathfrak{S}(n)$, then

$$|Q(c,d)| \leqq V_p(f) \cdot V_q(g) \cdot \sum_{\nu=0}^{n} (\psi(\nu))^{-t};$$

the proof is a simple induction argument based on 10.15.

10.17. Let $A(p,q)$ be the number that was introduced in 7.5. It follows from 10.16 that $|Q(c,d)| \leqq V_p(f) \cdot V_q(g) \cdot A(p,q)$. The conclusion 7.5 is now an easy consequence of 10.14 (6). Compare with the notation of 7.3 (1).

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Errata

G. ZAPPA, *Sull'esistenza di sottogruppi normali di Hall in un gruppo finito* (*Acta Sci. Math.*, **21** (1960), 223—227).

	Errata	*Corrige*
Da pag. 227, riga 36, a pag. 228, riga 3, dappertutto	C	LC
pag. 228, riga 8	$(C \cap H)$	$(LC \cap H)$
pag. 228, riga 8	$(r^{-1}C.r)$	$(r^{-1}LCr)$
pag. 228, riga 8	$C \cap H^*$	$LC \cap H^*$
pag. 228, riga 9	C	LC
pag. 228, riga 11	$C=BD$	$LC=BD$

Questi errori sono stati segnalati all'autore da Z. Janko (Lištica, Iugoslavia).
