Von Neumann's Arithmetics of Continuous Rings

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This paper is a revision of an unpublished manuscript *Arithmetics of Regular Rings Derived From Continuous Geometries* written by J. VON NEUMANN in 1937[1]) and summarized in §§ 6, 7 of his note [3]. I am grateful for permission to present this work of VON NEUMANN. The material has been freely re-arranged, the introduction, footnotes and Lemma 6. 1 have been added and I have strengthened Lemmas 2. 1, 2. 2, 3. 1, 6. 2 and 6. 3 by using different proofs. Any faults in the present exposition are, of course, mine.

1. Introduction

1. 1. We use terminology close to that of [4]. $\mathfrak{R}$ will usually denote a fixed regular ring with unit, $\overline{R}_{\mathfrak{R}}$ = lattice of all principal right ideals of $\mathfrak{R}$ and $\overline{L}_{\mathfrak{R}}$ = lattice of all principal left ideals of $\mathfrak{R}$. For any idempotent e in $\mathfrak{R}$, $\mathfrak{R}(e)$ will denote the ring of all eae, $a\in\mathfrak{R}$. We shall say "a has a reciprocal in $\mathfrak{R}(e)$" if $a\in\mathfrak{R}(e)$ and $ab=ba=e$ for some $b\in\mathfrak{R}(e)$.

If $\mathfrak{R}$ is a complete rank ring, $R(a)$ will denote the unique normalized rank, defined for all a in $\mathfrak{R}$ and related to the dimension functions D in $\overline{R}_{\mathfrak{R}}$ and D' in $\overline{L}_{\mathfrak{R}}$ by:

$$R(a)=D((a)_r)=D'((a)_l).$$

A complete rank ring $\mathfrak{R}$ is (cf. [4]) either a discrete or a continuous ring according as the range of R is $0, 1/n, \ldots, n/n$ for some integer $n\geqq 1$ or the set of all real numbers $0\leqq t\leqq 1$. If $e\neq 0$, $\mathfrak{R}(e)$ is discrete or continuous along with $\mathfrak{R}$ and the rank function of $\mathfrak{R}(e)$ coincides with $R(a)/R(e)$, a in $\mathfrak{R}(e)$.

Z will denote the center of $\mathfrak{R}$ and we let $P=P(\mathfrak{R})$ denote the set of all polynomials

$$p(t) = t^m + z_{m-1}t^{m-1} + \ldots + z_0 t^0$$

with all z_i in Z and $m\geqq 1$. When $p(a)$ is calculated, t^0 is ordinarily to be replaced by the unit of $\mathfrak{R}$. But whenever we write "$p(a)$ in $\mathfrak{R}(e)$" we shall mean that t^0 is to be replaced by e.

We shall show below (cf. Lemma 2. 1) that if $\mathfrak{R}$ satisfies a weak condition (in particular, if $\mathfrak{R}$ is discrete or continuous) then the center of $\mathfrak{R}(e)$ consists of

[1]) The original manuscript is kept in the VON NEUMANN file of the Institute for Advanced Study in Princeton and may be seen there.

all $ez, z \in Z$. Hence if $a \in \mathfrak{R}(e)$ and $q \in P(\mathfrak{R}(e))$ then $q(a)$ coincides with $(p(a)$ in $\mathfrak{R}(e))$ for some $p \in P(\mathfrak{R})$.

Suppose $\mathfrak{R}$ is a discrete or continuous ring. If $p \in P$ and ε is a real number >0, we shall say a is *ε-p-algebraic* or simply *ε-algebraic* if $R(p(a))<\varepsilon$. If a is *ε-algebraic* for every $\varepsilon>0$ we shall say that a is *almost-algebraic*. If $R(p^m(a)) \to 0$ as $m \to \infty$ we shall say a is *limiting p-algebraic* or simply *limiting algebraic*. If $R(p(a))=0$, equivalently $p(a)=0$, we shall say a is *p-algebraic* or simply *algebraic*.

On the other hand, if $R(p(a))=1$ for every $p \in P$ we shall say a is *purely transcendental*.[2])

1. 2. In this paper we shall prove the theorem:

(1. 2. 1) *In a continuous ring the algebraic elements are everywhere dense in the sense of rank-metric.*

(1. 2. 1) means: for every $a \in \mathfrak{R}$ and every real $\varepsilon>0$ there exists an algebraic $b \in \mathfrak{R}$ such that $R(a-b)<\varepsilon$. It is easy to see that (1. 2. 1) follows from (1. 2. 2)—(1. 2. 5) below:

(1. 2. 2) If a is ε-algebraic then $R(a-b)<\varepsilon$ for some algebraic b (cf. Lemma 3. 2).

(1. 2. 3) If a is limiting p-algebraic then for every real $\varepsilon>0$, a is ε-q-algebraic when $q=p^m$ for sufficiently large integer m (cf. § 4).

(1. 2. 4) If a is purely transcendental then for every real $\varepsilon>0$ and every $p \in P$ with p of degree greater than $1/\varepsilon$, $R(a-b)<\varepsilon$ for some ε-p-algebraic b (cf. Lemma 5. 3).

(1. 2. 5) If (1. 2. 3), (1. 2. 4) hold then for every a and every real $\varepsilon>0$, $R(a-b)<\varepsilon$ for some ε-algebraic b (cf. Lemma 7. 1 and Theorem 8. 1).

Before going into the detailed proof (cf. sections 2—8) we give a brief indication of its principal ideas.

1. 3. To prove (1. 2. 2) we suppose $R(p(a))<\varepsilon$ and let $(e)_r=(p(a))_r$. Set $b=(1-e)a+x$ with x arbitrary in $\mathfrak{R}(e)$. We show that $(1-e)a$ is in $\mathfrak{R}(1-e)$ and $p(b)=(p(x)$ in $\mathfrak{R}(e))$. Since $R(a-b)=R(ea-ex)\leqq R(e)<\varepsilon$, (1. 2. 2) will be verified if for given $p \in P$ and given e we can find $x \in \mathfrak{R}(e)$ such that $(p(x)$ in $\mathfrak{R}(e))=0$. In the Corollary to Lemma 3. 1 we show that this is possible.

1. 4. The statement (1. 2. 3) is almost trivial (cf. § 4).

1. 5. The proof of (1. 2. 4) is technically the most difficult part of the entire proof. We show first that if a is purely transcendental then for every integer $N \geqq 1$ there exists a decomposition into independent elements: $\mathfrak{R}=\Sigma((a^ie_j)_r;$ $i=1, \ldots, j; j \geqq N)$. Now choose $m>1/\varepsilon$, choose any integer $N>\dfrac{m}{\varepsilon}$ and define an idempotent g so that $(g)_r=\Sigma((a^ie_j)_r; j\geqq N;\ i=m, 2m, \ldots,$ but $i \leqq j)$, $(1-g)_r=$ $=\Sigma((a^ie_j)_r;\ j \geqq N,\ i \leqq j$ but $i \neq m, 2m, \ldots)$. We set[3]) $b \equiv a-a^{-m+1}p(a)g$ and show: $p(b)a^ie_j=0$ for all $j \geqq N$ and all i satisfying $i \leqq ms$ for some $ms \leqq j$. It follows that $R(p(b)) \leqq \dfrac{m}{N}<\varepsilon$ and $R(a-b) \leqq R(g) \leqq \dfrac{1}{m}<\varepsilon$, hence this b verifies (1. 2. 4).

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2) von Neumann discovered [3] that in every continuous ring there exist purely transcendental elements but the manuscript (see 1)) gives no details of his proof. See [2] for a proof.

3) The symbol $\equiv$ means "equality by definition".

1. 6. To verify (1. 2. 5) we define $P' = P'(a)$ to the set of irreducible polynomials p in P for which $R(p(a)) < 1$. We show that P' is finite or denumerable, $P' \equiv (p_1, p_2, \ldots)$. Then we determine a "resolution of the identity" for a, that is, a sequence of orthogonal idempotents $e_0, e_1, \ldots$, each of which commutes with a, such that $\mathfrak{R} = \Sigma(e_i)_r$ and:

(1. 6. 1) $ae_0 = e_0 a$ is purely transcendental in the ring $\mathfrak{R}(e_0)$,

(1. 6. 2) for $i \geqq 1$, $ae_i = e_i a$ is limiting p_i-algebric in the ring $\mathfrak{R}(e_i)$.

Suppose now (1. 2. 3) and (1. 2. 4) hold for every continuous ring. Then if n_i is sufficiently large, $R(p_i^{n_i}(ae_i)$ in $\mathfrak{R}(e_i)) < \frac{\varepsilon}{2^{i+2}}$. And if p_0 is any polynomial of sufficiently high degree there exists b_0 in $\mathfrak{R}(e_0)$ such that $R(ae_0 - b_0) < \frac{\varepsilon}{2}$ and $R(p_0(b_0)$ in $\mathfrak{R}(e_0)) < \frac{\varepsilon}{4}$. Now we choose j so large that $R(1-(e_0 + \ldots + e_j)) < \frac{\varepsilon}{2}$ and set

$$b = b_0 + ae_1 + \ldots + ae_j, \quad p = p_0 p_1^{n_1} \ldots p_j^{n_j}.$$

It follows that $R(a-b) \leqq R(ae_0 - b_0) + R(1-(e_0 + \ldots + e_j)) < \varepsilon$, and $R(p(b)) < \varepsilon$. This verifies (1. 2. 5).

1. 7. Some of our Lemmas are proved under hypotheses weaker than the requirement that $\mathfrak{R}$ be a continuous ring; in particular, irreducibility of $\mathfrak{R}$ is frequently not required. This will facilitate an extension (to the reducible case) of Theorem (1. 2. 1).

2. Preliminary lemmas

Lemma 2. 1. *Suppose e is an idempotent in an associative ring $\mathfrak{R}$ and that* (i): *$e = 0$ or* (ii): *$\mathfrak{R}$ possesses a set of matrix units s_{ij}, $i, j = 1, \ldots, k$ for some $k = 1, 2, \ldots$ with $es_{11} = s_{11}e = s_{11}$. Then the center of $\mathfrak{R}(e)$ is the set of all ze, $z \in Z$.*

Proof. The Lemma is trivially true if $e = 0$. Consider the case $e \neq 0$ and suppose a is any element in the center of $\mathfrak{R}(e)$. Then clearly $\bar{a} \equiv as_{11}$ is in the center of $\mathfrak{R}(s_{11})$. Let $z = \sum_{i=1}^{k} s_{i1}\bar{a}s_{1i}$. Then z is in Z: for if x is in $\mathfrak{R}$, $x = \sum_{i=1}^{k} xs_{ii} = \sum_{i=1}^{k} s_{ii}x$, so (since $s_{1j}xs_{i1}$ is in $\mathfrak{R}(s_{11})$ and $\bar{a}$ is in the center of $\mathfrak{R}(s_{11})$)

$$xz = \sum_{i=1}^{k} xs_{i1}\bar{a}s_{1i} = \sum_{j=1}^{k} s_{j1}\left(\sum_{i=1}^{k} s_{1j}xs_{i1}\bar{a}s_{1i}\right) = \sum_{j=1}^{k} s_{j1}\left(\sum_{i=1}^{k} \bar{a}s_{1j}xs_{i1}s_{1i}\right) =$$

$$= \left(\sum_{j=1}^{k} s_{j1}\bar{a}s_{1j}\right)x = zx.$$

We shall show that $y \equiv ze - a$ satisfies $y = 0$. Clearly, y is in the center of

$\mathfrak{R}(e)$ and $ys_{11}=\bar{a}s_{11}-as_{11}=0$. Hence for all u, v in $\mathfrak{R}$, $yus_{11}v=(yeues_{11})v=$ $=(eue)ys_{11}v=0$. Then for each $i=1, \ldots, k$, $ys_{ii}=ys_{i1}s_{11}s_{1i}=0$ so $y=\sum_{i=1}^{k} ys_{ii}=0$.

Thus, if a is in the center of $\mathfrak{R}(e)$ then $a=ze$ for some z in Z. Conversely if z is in Z and x is in $\mathfrak{R}(e)$ then $(ze)x=zx=xz=(xe)z=x(ze)$ so ze is in the centre of $\mathfrak{R}(e)$. This completes the proof of Lemma 2. 1.

Corollary. *If e is an idempotent in a continuous or discrete ring $\mathfrak{R}$, then the center of $\mathfrak{R}(e)$ consists of all ze, $z\in Z$. And if $q(t)=t^m+z_{m-1}t^{m-1}+\ldots+z_0t^0$ with all z_i in the center of $\mathfrak{R}(e)$ then for some p in P, for every a in $\mathfrak{R}(e)$, $(p(a)$ in $\mathfrak{R}(e))$ coincides with $ep(a)=q(a)$.*

Proof. If $e\neq 0$, $\mathfrak{R}$ does possess a set of matrix units s_{ij}, $i,j=1, \ldots, k$ for some $k=1, 2, \ldots$ with $es_{11}=s_{11}e=s_{11}$. Hence Lemma 2. 1 applies and the Corollary follows.

Lemma 2. 2. *Suppose $\mathfrak{R}$ is an associative ring possessing a set of matrix units, s_{ij}, $i,j=1, \ldots, k$ for some $k\geqq 2$. Then for any a in $\mathfrak{R}$, if $faf=af$ for every idempotent f in $\mathfrak{R}$, a is in the center of $\mathfrak{R}$.*

Proof. The condition $faf=af$ can be written: $(1-f)af=0$. With f in place of $1-f$ this gives: $fa(1-f)=0$, hence $fa=faf=af$.

Now for any x in $\mathfrak{R}$, $f+fx(1-f)$ is idempotent along with f, so $(f+fx(1-f))a=$ $=a(f+fx(1-f))$. But $fa=af$, and so we obtain

$$(fx(1-f))a=a(fx(1-f))$$

for every x in $\mathfrak{R}$ and every idempotent f in $\mathfrak{R}$.

Thus, $xa=ax$ whenever $x=fx(1-f)$ for some idempotent f, in particular whenever $x=s_{ii}x=xs_{jj}$ for some $i\neq j$ (use $f=s_{ii}$). Hence, for every x, if $i\neq j$ then $s_{ii}xs_{jj}$ commutes with a, and for any i, using some $j\neq i$ (here we use the hypothesis $k\geqq 2$),

$$a(s_{ii}xs_{ii})=a(s_{ii}xs_{ij})s_{ji}=(s_{ii}xs_{ij})as_{ji}=(s_{ii}xs_{ij})s_{ji}a=(s_{ii}xs_{ii})a,$$

so

$$xa=\sum_{i,j=1}^{k}(s_{ii}xs_{jj})a=\sum_{i,j=1}^{k}a(s_{ii}xs_{jj})=ax$$

showing that a is in the center of $\mathfrak{R}$, as stated.

Corollary. *Suppose a is an element in $\mathfrak{R}(e)$ with e idempotent in an associative ring $\mathfrak{R}$ and suppose $faf=af$ for every idempotent f in $\mathfrak{R}(e)$. Then $a=ze$ for some z in the center of $\mathfrak{R}$ if* (i): *$e=0$ or* (ii): *$\mathfrak{R}(e)$ possesses a set of matrix units s_{ij}, $i,j=1, \ldots, k$ for some $k\geqq 2$ and $\mathfrak{R}$ possesses some set of matrix units S_{ij}, $i,j=1, \ldots, K$, for some $K\geqq 1$ with $eS_{11}=S_{11}e=S_{11}$.*

Proof. The Corollary follows at once from Lemma 2. 2 and Lemma 2. 1.

Remark. The conditions of the Corollary to Lemma 2. 2 are always satisfied if $\mathfrak{R}$ is a continuous ring. But if $\mathfrak{R}$ is a discrete ring (then $\mathfrak{R}$ is the ring of $n\times n$ matrices over some division ring $\mathfrak{D}$) and $(e)_r$ is an atom in $\bar{R}_{\mathfrak{R}}$ (so $\mathfrak{R}(e)$ is ring isomorphic to $\mathfrak{D}$) the conditions fail to hold, and in fact if $\mathfrak{D}$ is not commutative, the Corollary actually fails to hold.

3. Proof of (1. 2. 1)

Lemma 3. 1. *Suppose $\mathfrak{R}$ is any associative ring with unit (not assumed regular) and let $\mathfrak{R}_N$ denote the ring of all $N\times N$ matrices over $\mathfrak{R}$. If $N\geqq 1$ and*

$$p(t) = t^N + z_{N-1}t^{N-1} + \dots + z_0$$

is any polynomial of degree N with all z_i in the centre of $\mathfrak{R}$ then there exists a matrix M in $\mathfrak{R}_N$ such that $p(M)=0$.

Proof. For $i, j=1, \dots, N$, let S_{ij} be the matrix with ij-th entry equal to 1 and all other entries 0. Then our requirements are satisfied by the matrix $M=\sum_{i=1}^{N-1} S_{i+1,1} - \sum_{i=0}^{N-1} z_i S_{i+1,N}$. We have

$$M=\begin{vmatrix} 0 & 0 & \dots & 0 & -z_0 \\ 1 & 0 & \dots & 0 & -z_1 \\ 0 & 1 & \dots & 0 & -z_2 \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 0 & -z_{N-2} \\ 0 & 0 & \dots & 1 & -z_{N-1} \end{vmatrix}.$$

In fact, $MS_{i1}=S_{i+1,1}$ for $i=1, \dots, N-1$, hence $M^h S_{11}=S_{h+1,1}$ for $h=0, \dots, N-1$. Now,

$$M^N S_{11}=MM^{N-1}S_{11}=MS_{N,1} = -\sum_{i=0}^{N-1} z_i S_{i+1,1} =$$

$$= -\sum_{i=0}^{N-1} z_i M^i S_{11} = -(z_{N-1}M^{N-1} + \dots + z_0)S_{11}.$$

Thus $p(M)\cdot S_{11}=0$ and for $i>1, p(M)S_{i1}=p(M)M^{i-1}S_{11}=M^{i-1}p(M)S_{11}=0$. Hence, $p(M)S_{ii}=p(M)S_{i1}S_{1i}=0$ for $i=1, \dots, N$ and summation over i gives $p(M)=0$. This completes the proof.

Corollary. *If e is an idempotent in a continuous ring $\mathfrak{R}$ and p is a polynomial in P then for some x in $\mathfrak{R}(e)$, $(p(x)$ in $\mathfrak{R}(e))=0$.*

Proof. The statement is obvious if $e=0$ (with $x=0$). So we may assume $e\neq 0$, and then by replacing $\mathfrak{R}$ by $\mathfrak{R}(e)$ we may suppose $e=1$. Since $\mathfrak{R}$ is a *continuous* ring it can be represented (cf. [4, page 99, Theorem 3. 3 and page 93, Definition 3. 2]) as a ring of $n\times n$ matrices over an associative ring $\mathfrak{S}$ with unit for *every n* (the center of $\mathfrak{R}$ can be identified with center of $\mathfrak{S}$). The Corollary now follows from Lemma 3. 1.

Remark. In the proof of this Corollary we made use of the fact that $\mathfrak{R}$ is continuous, not discrete.

Lemma 3. 2. *For arbitrary* $\varepsilon>0$, *if* a *is* ε*-algebraic in a continuous ring* $\mathfrak{R}$ *then* $R(a-b)<\varepsilon$ *for some algebraic* b *in* $\mathfrak{R}$.

Proof. Suppose $R(p(a))<\varepsilon$, choose an idempotent e with $(e)_r=(p(a))_r$ and let $b=(1-e)a+x$ with x arbitrary in $\mathfrak{R}(e)$. Then $e=p(a)\cdot u$ for some u in $\mathfrak{R}$ so $ae=ap(a)u=p(a)\cdot au=ep(a)\cdot au=eap(a)u=eae$; thus $(1-e)ae=0$, $(1-e)a=(1-e)a(1-e)\in\mathfrak{R}(1-e)$. Then $p(b)=p((1-e)a(1-e)+exe))=(1-e)\cdot$ $\cdot p(a)+(p(x)\text{ in }\mathfrak{R}(e))=(p(x)\text{ in }\mathfrak{R}(e))$, since $(1-e)p(a)=(1-e)ep(a)=0$.

Now $a-b=ea-x=e(a-x)$ so $R(a-b)\leqq R(e)=R(p(a))<\varepsilon$. And $p(b)=0$ if x is chosen in $\mathfrak{R}(e)$ so that $(p(x)\text{ in }\mathfrak{R}(e))=0$. Now x can be so chosen by the Corollary to Lemma 3. 1 and then the Lemma is proved.

4. Proof of (1. 2. 3)

If $R((p(a))^m)\to 0$ as $m\to\infty$ then the polynomial $q(t)=p^m(t)$ has the property $R(q(a))<\varepsilon$ if m is sufficiently large. This proves (1. 2. 3).

5. Proof of (1. 2. 4)

We shall use the symbol $E_n(e)$ to denote the expression $\Sigma(A, (a^ie)_r; i=1, \dots, n)$.

Lemma 5. 1. *Suppose* a *is purely transcendental in a continuous ring* $\mathfrak{R}$, *let* N *be any integer* $\geqq 1$, *and suppose* A *is a principal right ideal such that* $A\neq\mathfrak{R}$, $aA\leqq A$. *Then there exists an idempotent* $e\neq 0$ *which is a solution for*

(5. 1) $$(A, (ae)_r, \dots, (a^Ne)_r)\perp.$$ [4]

Proof. We note for future use that if d, c are in $\mathfrak{R}$ then

$$D(d(c)_r)=R(dc)\leqq R(c)=D((c)_r).$$

If $R(d)=1$ and $dA\leqq A$ (in particular if $d=p(a)$ for some p in P) then $dc\in A$ implies $d((c)_r+A)\leqq A$, hence $(c)_r+A\leqq d^{-1}A$ and so $D((c)_r+A)\leqq D(A)$, and so $(c)_r\leqq A$. Thus for such d, if $dc\in A$ then $c\in A$.

Now suppose $N=1$. Choose B to be a complement of A in $\mathfrak{R}$. Then $B=a(a^{-1}B)$ (a^{-1} exists since $R(a)=1$). Choose an idempotent e such that $(e)_r=a^{-1}B$. Then $e\neq 0$ since $A\neq\mathfrak{R}$, hence $B\neq 0$. Also (5. 1) holds since it asserts only that $A(ae)_r=0$ and this is true since $(ae)_r=B$. So the Lemma holds for $N=1$.

Next, suppose the Lemma is established and $f_0\neq 0$ is a solution for the case $N=n$ for some integer $n\geqq 1$. We shall prove below:

(5. 2) $(a^{n+1}f)_r\leqq E_n(f)$ is false for some idempotent $f\neq 0$, $f\in(f_0)_r$.

Assuming (5. 2) we have

(5. 3) $$(a^{n+1}f)_rE_n(f)\neq(a^{n+1}f)_r$$

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[4]) The symbol $\perp$ signifies independence of the lattice elements (cf. [4, page 8]).

so we can choose an idempotent $e'\neq 0$ with $(e')_r$ a relative complement of the left side of (5. 3) in the right side of (5. 3).

Now choose an idempotent e with $(e)_r=(a^{-(n+1)}e')_r$. We shall prove: *this e is a solution of* (5. 1) *for* $N=n+1$. In fact, $(e')_r\leqq(a^{n+1}f)_r$ so $(e)_r\leqq(f)_r$ and $(a^{n+1}e)_r=(e')_r$. Thus $e\neq 0$ since $e'\neq 0$, and $(A,(ae)_r,\dots,(a^ne)_r)\perp$ since $(e)_r\leqq\leqq(f)_r\leqq(f_0)_r$ and f_0 is a solution for (5. 1) with $N=n$. Furthermore, $(a^{n+1}e)_rE_n(e)\leqq\leqq(e')_rE_n(f)=0$. Thus (5. 1) holds with $N=n+1$, as required. Thus by induction on N the Lemma would be proved for all N if (5. 2) were verified.

Assume (5. 2) false, if possible. Then for every f in $(f_0)_r$, $(a^{n+1}f)_r\leqq E_n(f)$. This implies

$$a^{n+1}f=y+\sum_{i=1}^{n}a^ifv_i \tag{5.4}$$

for some y in A and some v_i in $\mathfrak{R}$. Using right multiplication by f we could suppose $y=yf$ and $v_i=fv_if$ for all i. Choose, in particular, $f=f_0$ and let the resulting y, v_i in (5. 4) be denoted by x, u_i respectively. Then (5. 4) becomes

$$a^{n+1}f_0=x+\sum_{i=1}^{n}a^iu_i. \tag{5.5}$$

Right multiplication of (5. 5) by f and subtraction from (5. 4) yields

$$0=(y-xf)+\sum_{i=1}^{n}a^i(v_i-u_if). \tag{5.6}$$

Since the addends in (5. 6) are in the principal right ideals A, $(a^if_0)_r$, $i=1,\dots,n$, respectively, and $(A,(af_0)_r,\dots,(a^nf_0)_r)\perp$, therefore all of $y-xf$, $a^i(v_i-u_if)$ must be 0. Then $v_i-u_if=(a^{-1})^ia^i(v_i-u_if)=0$, so $v_i=u_if$. But $fv_i=v_i$ so $fu_if=u_if$ for every idempotent f in $(f_0)_r$, in particular for every idempotent f in $\mathfrak{R}(f_0)$. Hence, by the Corollary to Lemma 2. 2, $u_i=z_if_0$ for some z_i in Z. Now (5. 5) becomes

$$a^{n+1}f_0=x+\sum_{i=1}^{n}a^iz_if_0,$$

$$\left(a^{n+1}-\sum_{i=1}^{n}z_ia^i\right)f_0=x.$$

Put $p(t)=t^{n+1}+\sum_{i=1}^{n}(-z_i)t^i$. Then p is in P and $p(a)f_0=x\in A$. Since $R(p(a))=1$, $f_0\in A, f_0\in(f_0)_rA=0$, $f_0=0$. This contradicts $f_0\neq 0$, so (5. 2) cannot be false and this completes the proof of the Lemma.

Corollary. *Under the hypotheses of Lemma* 5. 1 *there exists a maximal solution e. This means: if f is a solution and* $(e)_r\leqq(f)_r$ *then* $(e)_r=(f)_r$.

Proof. By transfinite induction there exists an ordinal number Ω (not necessarily a limit ordinal) and a set of solutions of (5. 1) e_α, $\alpha<\Omega$, such that $(e_\alpha)_r<(e_\beta)_r$ whenever $\alpha<\beta<\Omega$ and such that no solution f satisfies $(e_\alpha)_r<(f)_r$ for all $\alpha<\Omega$.

Let $(e)_r=\Sigma_\alpha(e_\alpha)_r$. Then by Axiom III of [4, page 2] (the continuity of lattice operations) assumed for $\bar{R}_{\mathfrak{R}}$, we have: for $n=1, 2, \ldots, N$

$$(a^n e)_r E_{n-1}(e)=\sum_\alpha\left((a^n e_\alpha)_r E_{n-1}(e)\right)=\sum_{\alpha,\beta}(a^n e_\alpha)_r E_{n-1}(e_\beta)=$$
$$=\sum_\alpha(a^n e_\alpha)_r E_{n-1}(e_\alpha)=\sum_\alpha 0=0.$$

This implies that e is a solution of (5. 1). Since $(e)_r\geqq(e_\alpha)_r$ for all $\alpha<\Omega$ it follows that no solution f can satisfy $(f)_r>(e)_r$.

Lemma 5. 2. *Let $\mathfrak{R}$, a, N, A be as in Lemma 5. 1. Then there exists a sequence of idempotents $e_N, e_{N+1}, \ldots$ with the properties:*

$$(5.7) \qquad \left(A, (a^i e_j)_r;\ i=1, \ldots, j;\ j\geqq N\right)\perp.$$

$$(5.8) \qquad \sum\left(A, (a^i e_j)_r;\ i=1, \ldots, j;\ j\geqq N\right)=\mathfrak{R}.$$

Proof. Let $\bar{e}_N$ be a maximal solution of (5. 1) (existing by the Corollary to Lemma 5. 1) and for each $j\geqq N$ define idempotents $e_j, \bar{e}_{j+1}$ by induction on j so that:

$$(e_j)_r=a^{-(j+1)}\sum_{n=N}^{j}E_n(\bar{e}_n)(\bar{e}_j)_r,$$
$$(\bar{e}_{j+1})_r=\text{relative complement of } (e_j)_r \text{ in } (\bar{e}_j)_r.$$

Then for all $k\geqq j\geqq N$ and all i,

$$(a^i e_j)_r+(a^i\bar{e}_{j+1})_r=(a^i\bar{e}_j)_r,$$
$$(a^{j+1}\bar{e}_{j+1})_r\sum_{n=N}^{j}E_n(\bar{e}_n)=0,$$
$$(a^{j+1}e_j)_r\leqq\sum_{n=N}^{j}E_n(\bar{e}_n),$$
$$E_j(\bar{e}_j)=\sum_{n=j}^{k}E_j(e_n)+\sum_{i=1}^{j}(a^i\bar{e}_{k+1})_r.$$

Since $R(\bar{e}_{k+1})\leqq\dfrac{1}{k+1}$, $D\left(\sum_{i=1}^{j}(a^i\bar{e}_{k+1})_r\right)\leqq\dfrac{j}{k+1}$ and, consequently, converges to 0 as $k\to\infty$ for fixed j; hence $E_j(\bar{e}_j)=\sum\left(E_j(e_n);\ n\geqq j\right)$.

It is now easily verified that (5. 7) holds. We need only show $\left(A, (a^i e_j)_r;\ i=1, \ldots, j;\ k\geqq j\geqq N\right)\perp$ for all $k\geqq N$, and it is therefore sufficient to prove

$$(5.9) \qquad \left(A, (a^i e_j)_r;\ i=1, \ldots, j;\ k-1\geqq j\geqq N;\ (a^i\bar{e}_k)_r;\ i=1, \ldots, k\right)\perp$$

for all $k\geqq N$. But (5. 9) holds for $k=N$, (by the definition of $\bar{e}_N$), and if (5.9) holds for some $k\geqq N$ then it holds also for $k+1$ since

(i): $(a^i\bar{e}_k)_r$ is the union of the independent elements $(a^i e_k)_r$, $(a^i\bar{e}_{k+1})_r$ for $i=1, \ldots, k$ and

(ii): $(a^{k+1}\bar{e}_{k+1})_r(A+\sum((a^i e_j)_r;\ i=1,\dots,j;\ k\geqq j\geqq N)+$
$+\sum((a^i\bar{e}_{k+1})_r;\ i=1,\dots,k))\leqq(a^{k+1}\bar{e}_{k+1})_r\sum_{j=N}^{k}E_j(\bar{e}_j)=0.$

Thus, by induction on k, (5. 9) holds for all $k\geqq N$ and so (5. 7) holds.

If the left side of (5. 8) is denoted by E then for each addend S on the left side of (5. 8) we have $aS\leqq E$. From this we deduce $aE\leqq E$ since the mapping: $x\to ax(x\in\bar{R}_{\mathfrak{R}})$ is an order-isomorphism of $\bar{R}_{\mathfrak{R}}$ onto itself with inverse mapping: $x\to a^{-1}x$(use: $x=(e)_r$ for some e, and then $ax=(ae)_r$).

Now if $E=\mathfrak{R}$ were false we could apply Lemma 5. 1 to obtain an $f\neq 0$ with $(E, (af)_r, \dots, (a_N f)_r)\perp$. Since $(af)_r(a\bar{e}_N)_r\leqq(af)_r E=0$, so $(f)_r(\bar{e}_N)_r=0$; choosing an idempotent e' with $(e')_r=(f)_r+(\bar{e}_N)_r$ we would have a solution e' of (5. 1) with $(e')_r>(\bar{e}_N)_r$, contradicting the choice of $\bar{e}_N$ as a maximal solution. Thus $E=\mathfrak{R}$ and (5. 8) holds. This completes the proof of the Lemma.

Lemma 5. 3. *Suppose a is purely transcendental in a continuous ring $\mathfrak{R}$. Then for every real $\varepsilon>0$ and for every $p(t)=t^m+z_{m-1}t^{m-1}+\dots+z_0t^0$ with all z_i in Z, and $m>1/\varepsilon$ there exists b in $\mathfrak{R}$ such that $R(p(b))<\varepsilon$ and $R(a-b)<\varepsilon$.*

Proof. Choose $N>\dfrac{m}{\varepsilon}$ and apply Lemma 5. 2 (with $A=0$) to obtain $\mathfrak{R}=$ $=\Sigma((a^i e_j)_r;\ i=1,\dots,j;\ j=N, N+1,\dots)$. For each j let t_j be the largest integer with $mt_j\leqq j$ and set

$$A=\sum((a^{ms}e_j)_r;\ j\geqq N;\ s=1,\dots,t_j),$$
$$B=\sum((a^i e_j)_r;\ j\geqq N;\ i\neq ms \text{ for all } s=1,\dots,t_j).$$

Then A and B are complementary in $\mathfrak{R}$ and

$$(g)_r=A,\quad (1-g)_r=B \text{ for some idempotent } g$$

and $R(g)=D(A)\leqq\dfrac{1}{m}$.

Choose $b=a-a^{-m+1}p(a)g$. Then

$$R(a-b)\leqq R(g)\leqq\frac{1}{m}<\varepsilon.$$

Now we shall prove below:

(5. 10) $\qquad p(b)a^i e_j=0$ for $j\geqq N$ and $i=1,\dots,mt_j$.

From (5. 10) it will follow at once that

$$R(p(b))=D'(p(b))_l=1-D(p(b))_l^r\leqq$$
$$\leqq D(\Sigma((a^i e_j)_r;\ j\geqq N;\ mt_j<i<m(t_j+1)))\leqq\frac{m}{N}<\varepsilon$$

as required. Thus we need only prove (5. 10).

Clearly $bx=ax$ whenever $gx=0$, that is, whenever $x\in B$. By repetition $b^h(a^i e_j)=a^h(a^i e_j)$ whenever $a^i e_j, a^{i+1}e_j, \dots, a^{i+h-1}e_j$ are all in B.

Hence, if $i=ms+1$ for some s with $0\leqq s<t_j$ we can deduce:

$$b^h(a^ie_j)=a^h(a^ie_j) \text{ for } h=0, 1, \ldots, m-1,$$

$$\begin{aligned} b^m(a^ie_j)&=bb^{m-1}(a^ie_j)=b(a^{m-1+i}e_j)=\\ &=\big(a-a^{-m+1}p(a)g\big)a^{m(s+1)}e_j=\\ &=a^{m(s+1)+1}e_j-p(a)a^{ms+1}e_j \quad (\text{use } ga^{m(s+1)}=a^{m(s+1)})=\\ &=a^m(a^ie_j)-p(a)a^ie_j;\end{aligned}$$

and so $p(b)a^ie_j=-p(a)a^ie_j+p(a)a^ie_j=0$. Therefore $p(b)a^ie_j=0$ for all $i=ms+1$, $0\leqq s<t_j$. But this implies $p(b)a^h(a^ie_j)=p(b)b^h(a^ie_j)=b^hp(b)a^ie_j=0$ for such i and all $h=0, 1, \ldots, m-1$. This proves (5. 10) and establishes the Lemma.

6. Preliminary Lemmas

In this section L will be assumed to be a relatively complemented modular lattice which is $\aleph_0$-complete and $\aleph_0$-continuous. This means: whenever I is countable,

(6. 1) $$\Sigma(a_\alpha;\, \alpha\in I) \quad\text{and}\quad \Pi(a_\alpha;\, \alpha\in I) \quad \text{exist},$$

(6. 2) $$b\Sigma(a_\alpha;\, \alpha\in I)=\Sigma_F\big(b\Sigma(a_\alpha;\, \alpha\in F)\big),$$

$$b+\Pi(a_\alpha;\, \alpha\in I)=\Pi_F\big(b+\Pi(a_\alpha;\, \alpha\in F)\big)$$

(where F varies over all finite subsets of I). $\mathfrak{R}$ will be assumed to be a regular ring not necessarily with unit such that

(6. 3) $$\bar{R}_{\mathfrak{R}} \text{ is } \aleph_0\text{-complete and } \aleph_0\text{-continuous.}$$

If $\mathfrak{R}$ has a unit $\bar{R}_{\mathfrak{R}}$ and $\bar{L}_{\mathfrak{R}}$ are anti-isomorphic under the mutually inverse mappings

(6. 4) $$(a)_r\to(a)_r^l\equiv(x;\, xa=0), \quad (a)_l\to(a)_l^r\equiv(x;\, ax=0)$$

(cf. [4, page 71, Corollary 2 to Lemma 2.2]). Since $\aleph_0$-completeness and $\aleph_0$-continuity are self-dual properties it follows that they are possessed by $\bar{L}_{\mathfrak{R}}$ if by $\bar{R}_{\mathfrak{R}}$.

Lemma 6. 1. *Let L be an $\aleph_0$-complete $\aleph_0$-continuous relatively complemented modular lattice. Suppose a, b are in L and suppose $Tx=y$ defines a $(1, 1)$ mapping of $L(0, a)$ onto $L(0, b)$ such that (i): $a_1\leqq a_2\leqq a$ holds if and only if $Ta_1\leqq Ta_2\leqq Ta(=b)$ and (ii): $a_1\sim Ta_1$ whenever $a_1(Ta_1)=0$. Then $a\sim b$.*

Proof. (i) If $Ta_1\leqq a_1$ let d be a relative complement of Ta_1 in a_1. Then $d, Td, \ldots, T^nd, \ldots$ are all defined, independent and mutually perspective. To prove this, note firstly that $Ta_1\leqq a_1$ implies that $T^2a_1=T(Ta_1)$ is defined and $T^2a_1\leqq Ta_1\leqq a_1$. By induction, T^na_1 is defined for all n and so T^nd is defined for all n since $d\leqq a_1$. Then for all $n\geqq 0, m\geqq 1$,

$$(T^nd)(T^{n+1}d+\ldots+T^{n+m}d)=T^n\big(d(Td+\ldots+T^md)\big)\leqq T^n\big(d(Ta_1)\big)=T^n(0)=0,$$

so d, Td,... are independent Finally, $T(T^nd)=T^{n+1}d$ and $(T^nd)\ (T^{n+1}d)=0$, so the hypotheses of Lemma 6. 1 yield $T^n d \sim T^{n+1} d$. From this it follows that $d, Td, \ldots$ are mutually perspective.

Now the argument given in [4, page 21, Theorem 3. 8] is valid with our present hypothesis on L, so $d=0$. Thus $Ta_1 \leqq a_1$ *implies* $Ta_1 = a_1$.

(ii) Set $a'_1 = a$, $b'_1 = b$. Define $a'_{n+1}, a_n, b'_{n+1}, b_n$ for all $n \geqq 1$ by induction on n so that:

$$a'_{n+1} = a'_n b'_n, \qquad a_n = \text{relative complement of } a'_{n+1} \text{ in } a'_n,$$

$$b'_{n+1} = Ta'_{n+1}, \qquad b_n = Ta_n.$$

Then (cf. [1, page 543, Lemma 2.11]),

$$a'_n = a'_{n+1} + a_n, \quad a'_0 \geqq a'_1 \geqq \ldots,$$

$$a = \Pi a'_n + \Sigma a_n, \quad (\Pi a'_n, a_1, a_2, \ldots) \perp,$$

$$b'_n = b'_{n+1} + b_n, \quad b'_0 \geqq b'_1 \geqq \ldots,$$

$$b = \Pi b'_n + \Sigma b_n, \quad (\Pi b'_n, b_1, b_2, \ldots) \perp,$$

$$T(\Pi a'_n) = \Pi b'_n \geqq \Pi a'_n.$$

Applying the preceding paragraph (with T^{-1} in place of T) it follows that $T(\Pi a'_n) = \Pi a'_n = \Pi b'_n$. Since $a_n b_n = 0$ for $n \geqq 1$, so $a_n \sim b_n$. Now under the present hypothesis on L, perspectivity is additive for countable independent families (cf. [1, page 561, Theorem 6. 2]). Then $a = \Pi a'_n + \Sigma a_n$ is perspective to $b = \Pi b'_n + \Sigma b_n$, proving the Lemma.

Lemma 6. 2.[5] *Let* $\mathfrak{R}$ *be a regular ring with unit such that* $\bar{R}_{\mathfrak{R}}$ *(hence also* $\bar{L}_{\mathfrak{R}}$*) is* $\aleph_0$*-complete and* $\aleph_0$*-continuous. Then for all* a, b *in* $\mathfrak{R}$:

(6. 5) $\qquad (a)_r \sim (b)_r$ *if and only if* $(a)_l \sim (b)_l$.

(6. 6) $\qquad (a)_r \precsim (b)_r$ *if and only if* $(a)_l \precsim (b)_l$.

(6. 7) $\qquad (a)_l^r \Pi((a^n)_r;\ n \geqq 1) = (a)_r^l \Pi((a^n)_l;\ n \geqq 1) = 0.$

(6. 8) *There exists a unique idempotent* e *such that*

$$(e)_r = \Pi((a^n)_r;\ n \geqq 1) \quad \text{and} \quad (e)_l = \Pi((a^n)_l;\ n \geqq 1).$$ [6]

Proof of (6. 5). Suppose $(a)_r$ and $(b)_r$ have a common complement in $\mathfrak{R}$. Then by [4, page 69, Theorem 2. 1] there exist idempotents f, g such that $(a)_r = (f)_r$, $(b)_r = (g)_r$ and the common complement is $(1-f)_r = (1-g)_r$. This implies $f = fg$, $g = gf$ so $(f)_l = (g)_l$. Now perspectivity is transitive in $\bar{L}_{\mathfrak{R}}$ under our present hypotheses on $\mathfrak{R}$ (see [1, page 550, Theorem 5. 1]), so if we can show $(a)_r = (f)_r$ implies $(a)_l \sim (f)_l$, the same result for $(b)_r = (g)_r$ will yield $(a)_l \sim (f)_l$, $(f)_l = (g)_l$, $(g)_l \sim (b)_l$,

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5) (6. 5) was proved in [4, page 223] for the special case of complete rank rings.

6) Although not required for this paper, the following remark may be of interest to the reader. Assume the hypotheses of Lemma 6. 2 and suppose $x \in \mathfrak{R}$. From (6. 6), (6. 7) and (6. 8) it follows that $(x)_r = (e)_r$, $(x)_l = (e)_l$ for some idempotent e (necessarily unique) if and only if $(x^2)_r = (x)_r$, equivalently: for $a \in \mathfrak{R}$, $x^2 a = 0$ implies $xa = 0$.

hence $(a)_l \sim (b)_l$. This will prove: $(a)_r \sim (b)_r$ implies $(a)_l \sim (b)_l$. We shall show now that $(a)_r = (f)_r$ implies $(a)_l \sim (f)_l$.

Since $(a)_r = (f)_r$, so $a = fa$, $f = ad$ for some d. Replacing d by df we can suppose $d = df$ so $(d)_l = (f)_l$. Now define the mappings T, T_1 by:

$$T(x)_l = (xd)_l \quad \text{for} \quad (x)_l \leqq (a)_l,$$

$$T_1(x)_l = (xa)_l \quad \text{for} \quad (x)_l \leqq (f)_l.$$

If $(x)_l \leqq (a)_l$ then $x = ua$, $xda = uada = ufa = ua = x$. It follows that T and T_1 are mutually inverse (1, 1) order preserving mappings between $L(0, (a)_l)$ and $L(0, (f)_l)$. Moreover, if $(x)_l \leqq (a)_l$ and $(x)_l\,(xd)_l = 0$ then $(x)_l \sim T(x)_l = (xd)_l$ with axis $(x + xd)_l$; for

(i) $(x)_l + (x+xd)_l = (x)_l + (xd)_l = (xd)_l + (x+xd)_l$;

(ii) $(x)_l (x+xd)_l = 0$ since $y = ux = v(x+xd)$ implies $(u-v)x = v(xd) \in \in (x)_l (xd)_l = 0$, $vxd = 0$, $vx = vxda = 0$, $y = vx + vxd = 0 + 0 = 0$.

(iii) $(xd)_l (x+xd)_l = 0$ since $y = uxd = v(x+xd)$ implies $vx = (u-v)xd \in \in (x)_l (xd)_l = 0$, $vx = 0$, $y = vx + vxd = 0$.

Now Lemma 6. 1 applies and shows that $(a)_l \sim (f)_l$.

Thus $(a)_r \sim (b)_r$ implies $(a)_l \sim (b)_l$. This result is equivalent to its dual: $(a)_l \sim (b)_l$ implies $(a)_r \sim (b)_r$. This proves (6. 5).

Proof of (6. 6). If $(a)_r \sim (c)_r \leqq (b)_r$ there exist orthogonal idempotents e, f such that $(e)_r = (c)_r$, $(f)_r =$ relative complement of $(c)_r$ in $(b)_r$. Then $(a)_r \sim (e)_r$, $ef = 0$, $(e+f)_r = (e)_r + (f)_r = (b)_r$.

By (6. 5) $(e+f)_l \sim (b)_l$. Since $e = e(e+f)$, so $(e)_l \leqq (e+f)_l$, $(e)_l \precsim (b)_l$. Using (6. 5) again and the transitivity of perspectivity [1, page 550, Theorem 5. 1] it follows that $(a)_l \sim (e)_l \precsim (b)_l$, and $(a)_l \precsim (b)_l$. So $(a)_r \precsim (b)_r$ implies $(a)_l \precsim (b)_l$. Combining this result with its dual we obtain (6. 6).

Proof of (6. 7). We need only prove $x \in ((a)_l^r \Pi((a^n)_r; n \geqq 1)$ implies $x = 0$ (together with its dual this yields (6. 7)). We have: $ax = 0$ but for every $n \geqq 1$, $x = a^n y_n$ for some y_n. For each $n \geqq 1$ let $(b_n)_l$ be a relative complement of $(a^{n+1})_l$ in $(a^n)_l$. Then $(a^n)_l = (a^{n+1})_l + (b_n)_l$, $a_n = ua^{n+1} + vb_n$ for some u, v;

$$x = a^n y_n = (ua^{n+1} + vb_n) y_n = uaa^n y_n + vb_n y_n = uax + vb_n y_n = 0 + vb_n y_n = vb_n y_n.$$

So $(x)_l \leqq (b_n y_n)_l$ and (6. 6) yields $(x)_r \precsim (b_n y_n)_r \leqq (b_n)_r$. Hence, again by (6. 6), $(x)_l \precsim \precsim (b_n)_l$, that is, $(x)_l \sim (x_n)_l \leqq (b_n)_l$ for some x_n. But the $(b_n)_l$ are independent, so the $(x_n)_l$ are independent. It follows that all $(x_n)_l = 0$ (cf. [4, page 21, Theorem 3. 8]) and so $(x)_l = 0$, $x = 0$ as required.[7]

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7) This type of argument shows: *if* $x \in \Pi_n (a^n)_r$ *and* $\Pi_n (a^n x)_l = 0$ *then* $x = 0$. Indeed, for fixed m, $a^m = ua^{m+n} + vb_n$ with $(b_n)_l =$ relative complement of $(a^{m+n})_l$ in $(a^m)_l$. So $x = a^m y = ua^{m+n} y + + vb_n y = ua^n x + vb_n y$; $(x)_l \leqq (a^n x)_l + (b_n y)_l$; $(x)_l \leqq \Pi_n ((a^n x)_l + \Sigma_m (b_m y)_l) = \Pi_n (a^n x)_l + \Sigma_n (b_n y)_l = = \Sigma_n (b_n y)_l$. But the b_n can be chosen so that $((b_n(_l;\ n \geqq 1) \perp$ and $\Sigma_n (b_n)_l =$ relative complement of $\Pi_n (a^n)_l$ in $(a^m)_l$ (use [1, Theorem 6. 2]). Then $(x)_l \leqq \Sigma_n (b_n y)_l \precsim$ relative complement of $\Pi_n (a^n)_l$ in $(a^m)_l$ and letting $m \to \infty$, we obtain $(x)_l = 0$ (use [1, Theorem 6. 1]). Hence, $x = 0$.

Proof of (6. 8). The existence of e as described is equivalent to

(6. 9) $$\Pi((a^n)_r;\ n\geqq 1)(\Pi((a^n)_l;\ n\geqq 1))^r=0,$$

and

(6. 10) $$\Pi((a^n)_r;\ n\geqq 1)+(\Pi((a^n)_l;\ n\geqq 1))^r=\mathfrak{R},$$

and then [4, page 69, Theorem 2. 1] shows that if e exists, it is uniquely determined.

The statement (6. 10) is equivalent to the left-right dual of (6. 9) (by (6. 3) and (6. 4)). So we need only prove (6. 9), equivalently:

(6. 11) $$\left(\sum_{n=1}^{\infty}(a^n)_l^r\right)\Pi((a^n)_r;\ n\geqq 1)=0.$$

Because of (6. 2) we need only prove (6. 11) with arbitrary finite m in place of ∞. Since $(a^1)_l^r\leqq(a^2)_l^r\leqq\ldots$ we need only prove: for each $m\geqq 1$,

(6. 12) $$(a^m)_l^r\Pi((a^n)_r;\ n\geqq 1)=0.$$

But from (6. 7), with a there replaced by a^m, the left side of (6. 22) is less than or equal to $(a^m)_l^r\Pi((a^{mn})_r;\ n\geqq 1)=0$. This proves (6. 8).

Lemma 6. 3. *Let $\mathfrak{R}$ be a regular ring with unit such that $\overline{R}_{\mathfrak{R}}$ and $\overline{L}_{\mathfrak{R}}$ satisfy the axioms I—V of discrete or continuous geometry* (cf. [4, pages 1, 2], *irreducibility is not assumed). Suppose S is a subset of $\mathfrak{R}$ such that for any a, b in S there is some c in S such that $(c)_r\leqq(ab)_r(ba)_r$ and some d in S such that $(d)_l\leqq(ab)_l(ba)_l$. Then there exists an idempotent $e=e(S)$ such that*

$$(e)_r=\Pi((a)_r;\ a\in S),\qquad (e)_l=\Pi((a)_l;\ a\in S)$$

(this e is unique) and e commutes with any u which commutes with every a in S.

Remark. The hypotheses on S are satisfied if $a, b\in S$ imply $ab=ba\in S$, in particular if S consists of all $p(a)$, $p\in P$ with a fixed, or if S consists of all $p^n(a)$, $n\geqq 1$ with $p\in P$, p fixed and a fixed.

Proof. Note that the hypotheses on S imply: for each d in S and each $n\geqq 1$, $(d^n)_r\geqq(a)_r$ for some a in S.

Now to prove e exists we need only prove (as in the proof of (6. 8)) that $(\Sigma((a)_l^r;\ a\in F))\,\Pi((a)_r;\ a\in S)=0$ for every finite subset F of S. But the hypotheses on S imply that for some d in S, $(d)_l\leqq(a)_l$ for all $a\in F$, hence $(d)_l^r\geqq(a)_l^r$ for all a in F. Thus we need only prove $(d)_l^r\Pi((a)_r;\ a\in S)=0$ for every d in S. But the hypotheses on S together with (6. 7) yield, $(d)_l^r\Pi((a)_r;\ a\in S)\leqq(d)_l^r\Pi((d^n)_r;\ n\geqq 1)=0$, so e does exist as described.

If $ua=au$ for all $a\in S$, then $(e)_r\leqq(a)_r$ yields $av=e$, $ue=uav=auv\in(a)_r$, so $ue\in\Pi((a)_r;\ a\in S)$, $ue=eue$. By the dual of this result, $eu=eue$ so $eu=ue$ as required.

Corollary. *If $ua=au$ for all a in S then there is a unique decomposition $u=u_1+u_2$ with u_1 in $\mathfrak{R}(e)$, u_2 in $\mathfrak{R}(1-e)$, namely $u_1=ue$, $u_2=u(1-e)$ and for every p in P*

(i) $p(u)=(p(ue)$ *in* $\mathfrak{R}(e))+(p(u(1-e))$ *in* $\mathfrak{R}(1-e))$,

(ii) $(p(ue)$ *in* $\mathfrak{R}(e))_r$ *and* $(p(u(1-e))$ *in* $\mathfrak{R}(1-e))_r$ *are independent and their lattice union is* $(p(u))_r$.

(iii) *If this* u *is also in* S *then* $(ue)_r=(e)_r$, *equivalently, if* $e\neq 0$, ue *has a reciprocal in* $\mathfrak{R}(e)$.

Proof. By Lemma 6. 3, $eu=ue$ and the unique decomposition of u follows immediately. Since

$$p(u) = p(u)e+p(u)\ (1-e) = \big(p(ue)\ \text{ in }\ \mathfrak{R}(e)\big)+\big(p(u(1-e))\ \text{ in }\ \mathfrak{R}(1-e)\big),$$

(i) holds. Since the two addends on the right side of (i) are orthogonal, (ii) follows.

Finally, if u *is also in* S *then* $(u)_r\geqq(e)_r$ so $e=uv$ for some v, $e=(eu)v$, hence $(e)_r\geqq(eu)_r$. But $(eu)_r\leqq(e)_r$, so (iii) holds.

7. Decomposition into limiting algebraic and transcendental parts

7. 1. Assume that $\mathfrak{R}$ is a regular ring with unit for which $\overline{R}_{\mathfrak{R}}$ and $\overline{L}_{\mathfrak{R}}$ satisfy the axioms I—V of discrete or continuous geometry (cf. [4, pages 1, 2]) (irreducibility is not assumed). Let a be a fixed element of $\mathfrak{R}$ and set $S_0=(p(a);p\in P)$ and for each p in P, $S_p=(p^n(a);n\geqq 1)$.

Clearly S_0 (and each S_p) satisfies the hypotheses of Lemma 6. 3. We may set $e_0=e(S_0)$, $f_0=1-e_0$, $f_p=e(S_p)$, $e_p=1-f_p$.

Since all members of S_0 commute, it follows from Lemma 6. 3 that they all commute with e_0 and with every f_p, and then, again from Lemma 6. 3, that e_0, all f_p commute. So e_0, all e_p commute. Moreover $S_0\supset S_p$ so $(e_0)_r\leqq(f_p)_r$, hence $f_pe_0=e_0$, $e_pe_0=0$.

Suppose p, q in P have the property:

(7. 1) $\quad p(t)h+q(t)k = 1$ for some h, k of the form $z_mt^m+\ldots+z_0$ with $m\geqq 0$ and all z_i in Z.

Then p^m, q^n have this property (7. 1) also (use: $1 = (ph+qk)^{m+n} = p^mh_1+q^nk_1$ for some h_1, k_1 of the form $z_jt^j+\ldots+z_0$ with $j\geqq 0$ and all z_i in Z). Hence $p^m(a)h_1(a)+q^n(a)k_1(a) = 1$,

$$\begin{aligned}\mathfrak{R} &= (p^m(a))_r+(q^n(a))_r = \Pi\big((p^m(a))_r+(q^n(a))_r;\ m\geqq 1,\ n\geqq 1\big) =\\ &= \Pi\big((p^m(a))_r+\Pi((q^n(a))_r;\ n\geqq 1);\ m\geqq 1\big) =\\ &= \Pi\big((p^m(a))_r;\ m\geqq 1\big)+\Pi\big((q^n(a))_r;\ n\geqq 1\big) =\\ &= (f_p)_r+(f_q)_r = \mathfrak{R} \text{ and, consequently, } 0 = \mathfrak{R}^l = (f_p)_r^l(f_q)_r^l = (e_p)_l(e_q)_l\end{aligned}$$

Thus (7. 1) implies $e_pe_q=0$, $(e_q)_r\leqq(f_p)_r$.

7. 2. From now on we assume also that $\overline{R}_{\mathfrak{R}}$ is irreducible so that $\mathfrak{R}$ is a complete rank ring, either discrete or continuous and so Z is a commutative division ring. Let $P'=P'(a)$ denote the set of $p\in P$ for which $(p(a))_r\neq\mathfrak{R}$ and p is irreducible (that is, $p\neq p_1p_2$ with $p_1, p_2\in P$). If p, q are in P' and $p\neq q$, then p, q are relatively prime with respect to coefficient domain Z so (7. 1) holds, hence $e_pe_q=0$. Moreover if $p \in P'$ then $(f_p)_r \leqq (p(a))_r \neq \Re$ so $f_p \neq 1$, $e_p \neq 0$. Since the e_p are mutually orthogonal, $1 \geqq R(e_p + e_q + \dots) = R(e_p) + R(e_q) + \dots$ with all $R(e_p) > 0$, so P' is finite or denumerable.

Let P' be enumerated: $p_1, p_2, \dots$ (say) and from now on write e_m, f_m in place of e_p, f_p (with $p = p_m$). Since each p in P can be expressed as a product of powers of irreducible polynomials $p = p_1^{n_1} \dots p_m^{n_m}$ for suitable $n_1, \dots, n_m \geqq 0$, it follows that

$$(e_o)_r = \Pi\big((p(a)_r;\ p \in P\big) =$$

$$= \Pi\big((p_m^n(a))_r;\ m \geqq 1,\ n \geqq 1\big) = \Pi\big((f_m)_r;\ m \geqq 1\big),$$

$$(e_0)_r^l = (f_0)_l = \Sigma\big((f_m)_r^l;\ m \geqq 1\big) = \Sigma\big((e_m)_l;\ m \geqq 1\big).$$

By the dual argument, $(f_0)_r = \Sigma\big((e_m)_r;\ m \geqq 1\big)$. Since the e_i, $i \geqq 0$ are orthogonal, for all $m \geqq 0$

$$(f_m)_r = \Sigma\big((e_i)_r;\ i \geqq 0,\ i \neq m\big),$$

$$(f_m)_l = \Sigma\big((e_i)_l;\ i \geqq 0,\ i \neq m\big).$$

Also

$$\Re = \Sigma\big((e_m)_l;\ m \geqq 0\big) = \Sigma\big((e_m)_r;\ m \geqq 0\big),$$

$$\Sigma\big(R(e_m);\ m \geqq 0\big) = 1,$$

$$R\big(1 - (e_0 + \dots + e_m)\big) \to 0 \text{ as } m \to \infty.$$

Lemma 7. 1. *With notation as in the preceding paragraphs,*

(7. 2) ae_0 *is purely transcendental in* $\Re(e_0)$,

(7. 3) *if* $i \geqq 1$, ae_i *is limiting* p_i*-algebraic in* $\Re(e_i)$,

(7. 4) *if* $p = p_1^{n_1} \dots p_m^{n_m}$ *then* $\big(1 - (e_1 + \dots + e_m)\big)_r$ *and the* $\big(p_i^{n_i}(ae_i) \text{ in } \Re(e_i)\big)_r$, $i = 1, \dots, m$ *are an independent set and their lattice union is* $\big(p(a)\big)_r$.

Proof. For every p in P, $\big(p(ae_0) \text{ in } \Re(e_0)\big) = e_0 p(ae_0)$ and, by (iii) of the Corollary to Lemma 6. 3, has a reciprocal in $\Re(e_0)$. The Corollary to Lemma 2. 1 now yields (7. 2).

For every $i \geqq 1$, a and e_i commute so $\big(p_i(ae_i) \text{ in } \Re(e_i)\big) = e_i p_i(a)$. Hence

$$\Pi\big((p_i^n(ae_i))_r;\ n \geqq 1\big) = (e_i)_r \Pi\big((p_i^n(a))_r;\ n \geqq 1\big) = (e_i)_r (f_i)_r = 0.$$

This proves (7. 3).

Now let $g_m = 1 - (e_1 + \dots + e_m) = f_1 \dots f_m$. Then $\big((g_m)_r, (e_1)_r, \dots, (e_m)_r\big) \perp$ (the g_m, e_i; $i = 1, \dots, m$ are even orthogonal) and $\big(p_i^{n_i}(ae_i) \text{ in } \Re(e_i)\big) = e_i p_i^{n_i}(ae_i) \in (e_i)_r$. It follows that $\big((g_m)_r, (p_i^{n_i}(ae_i))_r;\ i = 1, \dots, m\big) \perp$. Also,

$$p(a) = p(ae_1 + \dots + ae_m + ag_m) =$$

$$= e_1 p(ae_1) + \dots + e_m p(ae_m) + g_m p(ag_m) =$$

$$= p(ae_1) e_1 + \dots + p(ae_m) e_m + p(ag_m) g_m.$$

Since $g_m, e_1, \ldots, e_m$ are orthogonal,

$$(p(a))_r = (p(ae_1)e_1)_r + \ldots + (p(ae_m)e_m)_r + (p(ag_m)g_m)_r.$$

But $e_i p(ae_i) = e_i p(a) = e_i p_1^{n_1}(a) \ldots p_m^{n_m}(a)$. If for some $j = 1, \ldots, m$ we consider $S = S_{p_j} = (p_j^n(a);\ n \geqq 1)$ and apply (iii) of the Corollary to Lemma 6.3 with $u = p_j^{n_j}(a)$ and $e = f_j$ we see that $p_j^{n_j}(a) b_j = f_j$ for some b_j. If $j \neq i$, then $f_j e_i = e_i$ so $p_j^{n_j}(a)(b_j e_i) = f_j e_i = e_i$ which implies

$$(e_i p(ae_i))_r = (e_i p_i^{n_i}(a) \Pi (p_j^{n_j}(a);\ j \neq i,\ j = 1, \ldots, m))_r = (e_i p_i^{n_i}(a))_r$$

for $i = 1, \ldots, m$. Similarly since $f_j g_m = g_m$,

$$(g_m p(ag_m))_r = (g_m p(a))_r = (g_m)_r.$$

From this, (7.4) follows at once.

Remark. ae_0 will be called the *transcendental part* of a, $a(1-e_0)$ will be called the *almost algebraic part* of a, and for $i \geqq 1$, the ae_i will be called the p_i — *limiting algebraic parts* of a.

8. The main theorem

Theorem 8.1. *Let a be an element of a continuous ring $\mathfrak{R}$ and suppose given any real $\varepsilon > 0$. Then there exists $p \in P$ and $b \in \mathfrak{R}$ such that $p(b) = 0$ and $R(a-b) < \varepsilon$. Moreover p can be of the form $p = p_1^{n_1} \ldots p_m^{n_m}$ if $P'(a) = (p_1, p_2, \ldots)$ is not empty and p can be required to be any assigned polynomial in P of degree $> \frac{1}{\varepsilon}$ if $P'(a)$ is empty.*

Proof. We shall show first that with p as described, b' exists with $R(a-b') < \frac{\varepsilon}{2}$, $R(p(b')) < \frac{\varepsilon}{2}$. This follows from Lemma 5.3 if $P'(a)$ is empty (then a is purely transcendental) so we may suppose $P'(a)$ is not empty.

Set $g_m = 1 - (e_0 + \ldots + e_m)$. Choose m so large that $R(g_m) < \frac{\varepsilon}{4}$ and for $i = 1, \ldots, m$ choose n_i so large that $n = n_1 + \ldots + n_m$ and $p = p_1^{n_1} \ldots p_m^{n_m}$ satisfy; $n > 1$, $R(ae_0 - b_0) < \frac{\varepsilon}{4}$, $R(e_0 p(b_0)) < \frac{\varepsilon}{4(m+1)}$ for some b_0 in $\mathfrak{R}(e_0)$, and $R(p_i^{n_i}(ae_i)e_i) < \frac{\varepsilon}{4(m+1)}$ for $i = 1, \ldots, m$.

To see that such n_i exist, use the fact that for any c and any idempotent e, if $c \in e\mathfrak{R}e$ then: Rank of c in $\mathfrak{R}$ is equal to Rank of e in $\mathfrak{R}$ multiplied by (normalized) Rank of c in $e\mathfrak{R}e$. Now by (7.3), ae_i is limiting p_i-algebraic in $\mathfrak{R}(e_i)$, a fortiori, $R(e_i p_i^{n_i}(ae_i)) \to 0$ as $n_i \to \infty$; and by (7.2), ae_0 is purely transcendental in $\mathfrak{R}(e_0)$ which implies by Lemma 5.3, whenever $\eta > 0$ and p is given of degree n, $n > \frac{1}{\eta}$, there exists b_0 in $\mathfrak{R}(e_0)$ (and so in $\mathfrak{R}$), with the properties $R(e_0 p(b_0)) \leqq R(e_0)\eta$, $R(ae_0 - b_0) \leqq R(e_0)\eta$.

Set $b' = b_0 + ae_1 + \ldots + ae_m$. Then

$$R(a-b') = R(ae_0 - b_0) + R(ag_m) \leqq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},$$

$$R(p(b')) \leqq R(e_0 p(b_0)) + R(e_1 p(ae_1)) + \ldots + R(e_m p(ae_m)) + R(g_m) \leqq$$
$$\leqq R(e_0 p(b_0)) + R(p_1^{n_1}(ae_1)e_1) + \ldots + R(p_m^{n_m}(ae_m)e_m) + R(g_m) <$$
$$< (m+1)\frac{\varepsilon}{4(m+1)} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},$$

so b' exists as stated above.

Now with this b' let $(e)_r = (p(b'))_r$ so $e = p(b')u$ for some u. Then $b'e = b'p(b')u = p(b')b'u$ so $b'e = eb'e$, $(1-e)b'e = 0$, $(1-e)b'(1-e) = (1-e)b'$; so for all x in $\mathfrak{R}(e)$,

$$p((1-e)b' + x) = (1-e)p(b') + ep(x) =$$
$$= (1-e)ep(b') + ep(x) = ep(x).$$

By the Corollary to Lemma 3.1 we can choose x_0 in $\mathfrak{R}(e)$ so that $p(x_0) = 0$. Choose $b = (1-e)b' + x_0$. Then $p(b) = 0$ and $R(a-b) \leqq R(a-b') + R(eb' - ex_0) < < \frac{\varepsilon}{2} + R(p(b')) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus this b satisfies the requirements of the Theorem.

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