Slender Groups*)

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Let P be the direct product of countably many copies of the integers Z, i. e., the group of all sequences $x=(x_1, x_2, \ldots)$ of integers with termwise addition. For each natural number n let δ^n be the element of P whose n-th coordinate is 1 and whose other coordinates are 0. J. Łoś calls a torsion-free abelian group *slender* if every homomorphism of P into it sends all but a finite number of the δ^n into 0. The concept first appeared in [3]. E. SĄSIADA [8] has shown that all reduced countable groups are slender. The purpose of this paper is to give a new description of the slender groups and to apply it to show that certain classes of groups are slender. All groups in this paper are abelian.

Our starting point is the observation that a group is slender if and only if every homomorphic image of P in it is slender. Our first task will be to describe the homomorphic images of P (theorem 5). Once this is done it is easy to describe the slender groups (corollary 6). The proof of theorem 5 is preceded by four lemmas. The first two of these are extensions of two results contained in [6]. Some details of their proofs will therefore be omitted. Lemma 4 is a consequence of theorem 1 of BALCERZYK's paper [2].

Following J. J. ROTMAN we call a group G a *B-group* if $\operatorname{Ext}(G, T)=0$ for every torsion group T, and a *W-group* if $\operatorname{Ext}(G, Z)=0$. In [7] ROTMAN showed that every separable B-group and every W-group is slender. His proof however requires the continuum hypothesis. We shall prove ROTMAN's results assuming neither the separability of the B-groups nor the continuum hypothesis.

1. The dual of the group A is the group $A^* = \operatorname{Hom}(A, Z)$. There is a natural homomorphism $\sigma_A\colon A \to A^{**}$ defined by considering the elements of A as homomorphisms of A^* into Z. Using the fact that P^* is the free group generated by the coordinate projections, it is easy to show that σ_P is an isomorphism between P and P^{**} so that P may be identified with its double dual.

Suppose A is a subgroup of P. Taking duals gives an exact sequence

(1) $$0 \to (P/A)^* \to P^* \to A^*.$$

Let A' be the image of $(P/A)^*$ in P^*. It consists of all h in P^* such that $h(A)=0$,

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i. e., it is the annihilator of A in P^*. Let B be the image of P^* in A^* so that the sequence

(2) $$0 \to (P/A)^* \to P^* \to B \to 0$$

is exact. Taking duals again gives a commutative diagram

(3) $$\begin{array}{ccccccc} 0 \to & A & \to & P & \to & P/A & \to 0 \\ & \downarrow \lambda & & \downarrow \sigma_P & & \downarrow \sigma & \\ 0 \to & B^* & \to & P^{**} & \to & (P/A)^{**} & \end{array}$$

with exact rows. In this diagram $\sigma = \sigma_{(P/A)}$ and λ is induced by σ_P and σ. We use σ_P to identify P with P^{**}. Then the image of B^* in $P^{**} = P$ is the annihilator A'' of A'. It consists of all x in P such that $h(x) = 0$ whenever h is in P^* and $h(A) = 0$.

Lemma 1. *If $A \subset P$, then $P = A'' \oplus C$. Both A'' and C are direct products of at most countably many copies of Z and $(A''/A)^* = 0$.*

Proof. We refer to diagrams (1)–(3) preceding the lemma. Since A^* can be embedded in a product (of copies of Z) and $B \subset A^*$, B can also be embedded in a product. As an image of P^*, B is countable. These two properties together imply that B is free. Hence the sequence (2) splits. It follows that the bottom row of (3) also splits so that $P = A'' \oplus C$ with A'' isomorphic to B^* and C isomorphic to $(P/A)^{**}$. Since B and $(P/A)^*$ are free of at most countable rank, their duals are products of at most countably many copies of Z.

To show that $(A''/A)^* = 0$ suppose $h: A'' \to Z$ is such that $h(A) = 0$. Then h can be extended to P by annihilating C. From the definition of A'', $h(A'') = 0$ so that $h = 0$.

Let Z be given the discrete topology and P the associated cartesian product topology. The statement that P^* is free on the coordinate projections is equivalent to the statement that each homomorphism of P into Z is continuous. Hence every endomorphism of P is continuous and the product topology on P is independent of the way P is represented as a product of Z's. From lemma 1 we see that if $P = A \oplus C$, then both A and C are products. Moreover this splitting is topological and the induced topologies on A and C are the product topologies.

Lemma 2. *Let $A \subset P$ and let S be the subgroup of finite sequences in P.*

(a) *If A has finite rank, then A is closed.*

(b) *If A has infinite rank, there is an isomorphism of $\bar{A}$ with P which carries A onto a subgroup of P containing S.*

(c) *If A is dense in P, there is an automorphism of P which carries A onto a subgroup of P containing S.*

Proof. Let $P_n = \{x \in P \mid x_i = 0 \text{ for } i < n\}$. Then $P = P_1, P_2, \ldots$ is a base at 0 for the topology on P. There are elements a^n $(n = 1, 2, \ldots)$ in A such that

i) $a_i^n = 0$ for $i < n$;
ii) $a_n^n = 0$ if and only if $a^n = 0$;
iii) a_n^n divides x_n for all x in $A \cap P_n$.

In view of (i), (ii) the $a^n \neq 0$ are independent. If A has finite rank the set of $a^n \neq 0$ is finite and generates $\bar{A}$ (cf. [6]). Thus $A = \bar{A}$ in this case proving (a).

If A has infinite rank, the set of all n such that $a^n \neq 0$ is infinite. Let k be a one-to-one correspondence between the natural numbers and this set. There is an endomorphism h of P such that $h(x)_i = \Sigma_n \; x_n a_i^{k(n)}$. The properties (i)–(iii) show that h is an isomorphism between P and $\bar{A}$. If δ^n is the element of P whose n-th coordinate is 1 and whose other coordinates are 0, then $h(\delta^n) = a^{k(n)}$ and is in A. Thus h^{-1} is the isomorphism required to prove (b).

To prove (c) we observe that if A is dense in P it has infinite rank and then apply (b).

Suppose $A \subset P$. If h is in P^* and $h(A)=0$, then $h(\bar{A})=0$ because h is continuous. We therefore have $A \subset \bar{A} \subset A'' \subset P$. Moreover, in view of lemmas 1 and 2, $\bar{A}$ and A'' are products and A'' is a direct summand of P.

Lemma 3. *Let $A \subset P$ with A a product and $(P/A)^* = 0$. Then the map $P^* \to A^*$ induced by duality is a monomorphism. If $U = A^*/P^*$, then $U^* = 0$ and $P/A \approx$* Ext (U, Z).

Proof. The first statement is obvious. Since A is a product, A^* is free and Ext $(A^*, Z) = 0$. Dualizing the exact sequence

$$0 \to P^* \to A^* \to U \to 0$$

gives the commutative diagram

$$\begin{array}{ccccccccccc} & & 0 & \to & A & \to & P & \to & P/A & \to & 0 \\ & & & & \downarrow \sigma_A & & \downarrow \sigma_P & & \downarrow \tau & & \\ 0 & \to & U^* & \to & A^{**} & \to & P^{**} & \to & \mathrm{Ext}(U, Z) & \to & \mathrm{Ext}(A^*, Z) \to 0 \end{array}$$

with exact rows. The map τ is induced by σ_A and σ_P. Since A and P are products both σ_A and σ_P are isomorphisms. Thus τ is an isomorphism and $U^* = 0$.

As stated in the introduction the next lemma is a consequence of theorem 1 of [2]. We give here a direct proof using the theory of abelian group extensions. Moreover the method of proof used here, together with the representation theorem for Boolean σ-algebras, can be used to prove BALCERZYK's theorem.

Lemma 4. *If Q is the additive group of rational numbers, then* Ext $(Q, P/S) = 0$.

Proof. Let π: $P \to P/S$ be the natural projection. Since the sequence

$$\mathrm{Ext}(Q, P) \xrightarrow{\pi^*} \mathrm{Ext}(Q, P/S) \to 0$$

is exact, it is enough to show that the image of π^* is 0. In terms of extensions this means that, for each extension

(4) $$0 \to P \to E \to Q \to 0,$$

there is a homomorphism f: $E \to P/S$ which extends π.

For $n = 1, 2, \ldots$ let e^n be an element in E mapping onto $1/n!$ modulo P. Then E is generated by P and $e^1, e^2, \ldots$ with relations

(5) $$e^n = (n+1)e^{n+1} + a^n$$

where the a^n are in P. If f: $\{e^1, e^2, \ldots\} \to P/S$ satisfies the relations

(6) $$f(e^n) = (n+1) f(e^{n+1}) + \pi(a^n)$$

for all n, then f can be extended to a homomorphism of E into P/S with the desired properties. We therefore want to define f on the e's so as to satisfy the relations (6).

Each a^n is in P. There are elements b^n in S such that $(a^n+b^n)_i=0$ for $i<n$. We set

$$y^n = \sum_{k\geqq n} k!(a^k+b^k);$$

the sum has meaning because it is finite on each coordinate. We also define

$$x^n = y^n/n! = \sum_{k\geqq n} (k!/n!)(a^k+b^k).$$

Then

$$y^n = y^{n+1}+n!(a^n+b^n)$$

so that

$$x^n = (n+1)x^{n+1}+a^n+b^n. \tag{7}$$

If we now define $f(e^n)=\pi(x^n)$ we see that (7) implies (6) because the b's are in S. Thus the required homomorphism exists.

A group C is a *cotorsion* group if it is reduced and if $C\subset E$ and E/C torsion-free imply that C is a direct summand of E. The group C is the direct sum of a divisible group and a cotorsion group if and only if $\mathrm{Ext}\,(Q, C)=0$ (see HARRISON [4] § 2 or NUNKE [5] § 7). Since a pure subgroup of bounded order of a group is always a direct summand, every group of bounded order is cotorsion.

Theorem 5. *Each homomorphic image of P is the direct sum of a divisible group, a cotorsion group, and the direct product of at most countably many copies of Z.*

Proof. Let $A\subset P$. If A has finite rank, let B be the pure subgroup of P generated by A. The $P=B\oplus C$ where C is a product by lemma 1 and B is a finitely generated free group. Then $P/A=B/A\oplus C$ and B/A is finite, hence cotorsion.

Suppose that A has infinite rank. By lemma 1 we have $P=A''\oplus C$ where A'' and C are products and $(A''/A)^*=0$. Then $P/A=A''/A\oplus C$ so that the proof will be complete once we show that $\mathrm{Ext}\,(Q, A''/A)=0$.

We have inclusions $A\subset\bar{A}\subset A''$, hence an exact sequence

$$0\to\bar{A}/A\to A''/A\to A''/\bar{A}\to 0. \tag{8}$$

Since A has infinite rank both $\bar{A}$ and A'' are isomorphic to P. From (8) we get an exact sequence

$$\mathrm{Ext}(Q, \bar{A}/A)\to\mathrm{Ext}(Q, A''/A)\to\mathrm{Ext}(Q, A''/\bar{A}). \tag{9}$$

Since A'' and $\bar{A}$ are both isomorphic to P, lemma 3 gives $A''/\bar{A}\approx\mathrm{Ext}\,(U, Z)$ for some group U. Applying the associative law for Ext and Tor ([5] p. 225) we get

$$\mathrm{Ext}(Q, \mathrm{Ext}(U, Z))\approx\mathrm{Ext}(\mathrm{Tor}(Q, U), Z)=0.$$

The equality holds because Q is torsion-free which implies $\mathrm{Tor}\,(Q, U)=0$. Thus $\mathrm{Ext}\,(Q, A''/\bar{A})=0$.

Since A has infinite rank, there is by lemma 2 an isomorphism of $\bar{A}$ with P such that the image A_0 of A contains S. We have then $A/\bar{A} \approx P/A_0$ and, since $S \subset A_0 \subset P$, an exact sequence

$$P/S \to P/A_0 \to 0.$$

The sequence

$$\operatorname{Ext}(Q, P/S) \to \operatorname{Ext}(Q, P/A_0) \to 0$$

is then exact. By lemma 4 Ext $(Q, P/S)=0$ so that Ext $(Q, P/A_0)=0=$ Ext $(Q, \bar{A}/A)$.

Since the two end groups in (8) are 0 and (8) is exact, the middle group is also 0. Thus Ext $(Q, A''/A)=0$ as desired.

Corollary 6. *A torsion-free group is slender if and only if it is reduced, contains no copy of the p-adic integers for any prime p, and contains no copy of P.*

Proof. We note first that the group of p-adic integers is not slender for the homomorphism $x \to \Sigma_i x_i p^i$ sends each δ^n into $p^n \neq 0$. Since a subgroup of a slender group is slender, the proof in the forward direction is then easy.

According to [4] p. 371 a torsion-free cotorsion group C has the form Hom$(Q/Z, B)$ where B is a divisible torsion group. If C is not 0, it contains a subgroup isomorphic to Hom $(Z(p^\infty), Z(p^\infty))$ for some prime p. This last group is isomorphic to the p-adic integers. Thus every nonzero torsion-free cotorsion group contains a copy of the p-adic integers for some p.

Suppose G is a torsion-free group satisfying the second clause of the corollary. A group is slender if and only if every homomorphic image of P in it is slender. In view of theorem 5, the preceding paragraph, and the hypothesis on G, a homomorphic image of P in G is the product of a finite number of copies of Z and is therefore slender by [8].

A group is called $\aleph_1$*-free* if every at most countable subgroup is free.

Corollary 7. *An* $\aleph_1$*-free group is slender if and only if it contains no copy of P.*

2. In this section we apply corollary 7 to show that every B-group and every W-group is slender. Various people (see [5] or [7] for example) have shown that B-groups and W-groups are $\aleph_1$-free. If $B \subset A$, then

$$\operatorname{Ext}(A, C) \to \operatorname{Ext}(B, C) \to 0$$

is exact for every C. Hence every subgroup of a B-group (W-group) is a B-group (W-group). In view of corollary 7 slenderness will follow if we show that P is neither a B-group nor a W-group. The first of these was shown by Baer in [1]. The second is Ext $(P, Z) \neq 0$. The group structure of Ext (P, Z) is easily described.

Theorem 8. *Let* $c=2^{\aleph_0}$. *Then* Ext (P, Z) *is the direct sum of* 2^c *copies of* Q *and* 2^c *copies of* Q/Z.

Proof. Let p be a prime. If $pA=0$, then A is a vector space over Z/pZ whose dimension we shall call the p-rank of A. It is the number of summands in any representation of A as the direct sum of copies of Z/pZ. If A is the direct product of b copies of Z/pZ and b is infinite, then the p-rank of A is 2^b.

Since P/pZ is the direct product of $\aleph_0$ copies of Z/pZ, its p-rank is $c=2^{\aleph_0}$. Since Ext (, Z) carries direct sums into direct products, Ext $(P/pP, Z)$ is the direct product of c copies of Z/pZ and therefore has p-rank 2^c. The exact sequence

$$0 \to P \xrightarrow{p} P \xrightarrow{\alpha} P/pP \to 0$$

gives an exact sequence

$$\mathrm{Hom}\,(P, Z) \to \mathrm{Ext}\,(P/pP, Z) \xrightarrow{\alpha^*} \mathrm{Ext}\,(P, Z) \xrightarrow{p} \mathrm{Ext}\,(P, Z) \to 0.$$

The image of α^* is Ext (P, Z) $[p]$. Moreover Hom (P, Z) has cardinal $\aleph_0$ while Ext $(P/pP, Z)$ has cardinal 2^c. Hence Ext (P, Z) $[p]$ has p-rank 2^c.

Now Ext (P, Z) is divisible because P is torsion-free. Hence the p-primary component of Ext (P, Z) is the direct sum of copies of $Z(p^\infty)$, the number of copies being the p-rank of Ext (P, Z) $[p]$, i. e., 2^c. Since this is true for all primes, the torsion subgroup of Ext (P, Z) is the direct sum of 2^c copies of Q/Z.

Since Ext (P, Z) is divisible, it is the direct sum of its torsion subgroup and the direct sum of copies of Q equal in number to its torsion-free rank. We shall therefore be finished when we show that the rank of Ext (P, Z) is 2^c. We have $0 \to P \to P \otimes Q$ exact and $rk\,(P \otimes Q) = rk\,(P) = c$. Thus $P \otimes Q$ is the direct sum of c copies of Q. Then

$$\mathrm{Ext}(P \otimes Q, Z) \to \mathrm{Ext}(P, Z) \to 0$$

is exact and Ext $(P \otimes Q, Z)$ is the direct product of c copies of Ext (Q, Z). Moreover Ext (Q, Z) is torsion-free with rank c. Hence Ext $(P \otimes Q, Z)$ has rank 2^c. Thus Ext (P, Z) has rank $\leqq 2^c$.

The group P has a sequence of subgroups $T_1, T_2, \ldots$ each isomorphic to P such that the sum $\Sigma_n T_n$ is direct. We thus get an exact sequence

$$\mathrm{Ext}(P, Z) \to \Pi_n \,\mathrm{Ext}(T_n, Z) \to 0.$$

Let p_n be the n-th prime and let I be a set of cardinal 2^c. There exists, for each n, a family $(y_{ni})_{i \in I}$ of elements in Ext (T_n, Z) $[p_n]$ independent modulo p_n. Let y_i be the element in Π_n Ext (T_n, X) whose n-th coordinate is y_{ni}. The family $(y_i)_{i \in I}$ is independent so that Π_n Ext (T_n, Z) has rank$\geqq 2^c$. Thus Ext (P, Z) has rank $\geqq 2^c$. Its rank is therefore exactly 2^c as required to prove the theorem.

The discussion at the beginning of this section now gives

Theorem 9. *Every B-group and every W-group is slender.*

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