On unitary equivalence of unitary dilations of contractions in Hilbert space

By N. G. DE BRUIJN in Eindhoven (Holland)

1. Introduction

Let $\mathfrak{H}$ be a Hilbert space, and let T be a contraction of $\mathfrak{H}$ (i. e. a linear operator with norm $\leqq 1$). It was proved by B. SZ.-NAGY [2] that there exists a Hilbert space $\mathfrak{K}$ with $\mathfrak{K} \supset \mathfrak{H}$, and a unitary operator U in $\mathfrak{K}$ such that (P denoting the projection onto $\mathfrak{H}$)

$$T^n P = P U^n P \qquad (n = 1, 2, 3, \ldots), \tag{1.1}$$

$$T^{*n} P = P U^{-n} P \qquad (n = 1, 2, 3, \ldots). \tag{1.2}$$

((1. 2) is, of course, a consequence of (1. 1)). If we require that $\mathfrak{K}$ is *minimal*, i. e. that $\mathfrak{K}$ is the closed linear hull of the set of all $U^n h$ ($n = 0, \pm 1, \pm 2, \ldots; h \in \mathfrak{H}$), then $\mathfrak{K}$ and U are uniquely determined (if we neglect isometries which leave $\mathfrak{H}$ and T invariant). This U is called the *unitary dilation* of T.

It was shown by M. SCHREIBER [6] (see also B. SZ.-NAGY [3]) that the unitary dilation of a proper contraction (i. e. an operator with norm $\|T\| < 1$) is always unitarily equivalent to a fixed operator U_0, depending on $\mathfrak{H}$ only. U_0 can be described as the orthogonal sum of η copies of the bilateral shift operator, where $\eta = \dim \mathfrak{H}$. This means that $\mathfrak{K}$ has a complete orthogonal system $\{\varphi_{ij}\}$ (where $i = 0, \pm 1, \pm 2, \ldots$, and j runs through an index set of cardinality η) such that $U_0 \varphi_{ij} = \varphi_{i+1,j}$ for all i and j.

In this note we shall establish the unitary equivalence explicitly in matrix form, thus giving an answer to a question proposed by B. SZ.-NAGY [4], a question directly connected with J. J. SCHÄFFER's matrix representation of the unitary dilation (see [5] and [4]). We shall moreover generalize SCHREIBER's result: Instead of $\|T\| < 1$, our assumption will be only $T^n \to 0$. Under that condition we shall prove that the minimal dilation is still unitarily equivalent to the orthogonal sum of a number of copies of the bilateral shift operator, but the number of copies can be less than $\dim \mathfrak{H}$. In fact it equals $\dim \mathfrak{M}_Z$ (to be defined below).

A further discussion will be postponed to section 6. Here we only remark that the results of this paper have been generalized, and proved in a more geometric way, by I. HALPERIN.

2. Preliminaries

Throughout the paper we assume that T is a contraction, and we put

$$Z=(I-T^*T)^{\frac{1}{2}}, \quad S=(I-TT^*)^{\frac{1}{2}}; \tag{2.1}$$

Z and S are non-negative definite hermitean operators. We have (cf. [1])

$$TZ=ST, \quad T^*S=ZT^*. \tag{2.2}$$

The spaces $\mathfrak{M}_Z$ and $\mathfrak{M}_S$ are closed subspaces of $\mathfrak{H}$, defined by

$$\mathfrak{M}_Z=\overline{Z\mathfrak{H}}, \quad \mathfrak{M}_S=\overline{S\mathfrak{H}}. \tag{2.3}$$

It follows from (2.2) that

$$T\mathfrak{M}_Z=T\overline{Z\mathfrak{H}}\subset\overline{TZ\mathfrak{H}}\subset\overline{ST\mathfrak{H}}\subset\overline{S\mathfrak{H}}=\mathfrak{M}_S,$$

and a similar result for $T^*\mathfrak{M}_S$, whence

$$T\mathfrak{M}_Z\subset\mathfrak{M}_S, \quad T^*\mathfrak{M}_S\subset\mathfrak{M}_Z. \tag{2.4}$$

3. Operators and matrices

Let $\mathfrak{R}$ denote the orthogonal sum of countably many copies of $\mathfrak{H}$. Elements of $\mathfrak{R}$ are sequences $\{h_i\}$ $(-\infty<i<\infty)$ with $h_i\in\mathfrak{H}$, $\sum_{-\infty}^{\infty}\|h_i\|^2<\infty$.

Let P_i denote the projection of $\mathfrak{R}$ onto the i-th coordinate space, and let Q_i denote the natural isometric mapping of this coordinate space onto $\mathfrak{H}$ itself (the sequence $\{h_i\}$ is mapped by P_j onto $\{\dots, 0, 0, h_j, 0, 0, \dots\}$, and this one is mapped by Q_j onto h_j).

If A is an operator in $\mathfrak{R}$, then we can define a matrix of operators in $\mathfrak{H}$ by

$$A_{ij}=Q_iP_iAP_j^{-1}Q_j^{-1} \qquad (i,j=0, \pm 1, \pm 2, \dots). \tag{3.1}$$

If A, B, C are bounded operators in $\mathfrak{R}$, and if $AB=C$, then it is not difficult to establish a matrix product relation

$$\sum_{j=-\infty}^{\infty} A_{ij}B_{jk}=C_{ik}. \tag{3.2}$$

It has to be noticed that this means that the partial sums of the series on the left converge to the operator on the right in the sense of operator convergence, i. e. that for every $h\in\mathfrak{H}$

$$\lim_{N\to\infty}\lim_{M\to\infty}\sum_{-M}^{N} A_{ij}B_{jk}h=C_{ik}h.$$

Moreover we notice that, if A is bounded, we have $(A^*)_{ij}=(A_{ji})^*$ for the adjoints. For convenience, we shall occasionally use the same symbol A both for this operator and for the matrix (A_{ij}).

Conversely, if a matrix (A_{ij}) of operators of $\mathfrak{H}$ is given, it is sometimes (but not always) possible to find an operator A of $\mathfrak{K}$ such that A_{ij} is related to A by (3. 1). For example, if the A_{ij}'s satisfy, for all i, k,

$$\sum_{j=-\infty}^{\infty} (A_{ji})^* A_{jk} = \delta_{ik} I, \tag{3.2}$$

and

$$\sum_{j=-\infty}^{\infty} A_{ij} (A_{kj})^* = \delta_{ik} I, \tag{3.3}$$

then there is exactly one unitary operator A of $\mathfrak{K}$ satisfying (3. 1).

4. Reduction of the Schäffer matrix

The Schäffer matrix is (see [5])

$$U_T = \begin{pmatrix} \ddots &&&&&&& \\ & I &&&&&& \\ && I &&&&& \\ &&& Z & -T^* &&& \\ &&& \boxed{T} & S &&& \\ &&&&& I && \\ &&&&&& I & \\ &&&&&&& \ddots \end{pmatrix}$$

(in order to indicate the indices of rows and columns, we have drawn a square around the central element, i. e. the element at the intersection of 0-th row and 0-th column). It is easily seen from what was said at the end of section 3, using (2. 1) and (2. 2), that U_T defines a unitary operator in $\mathfrak{R}$. And as the matrix U_T^n has as its central element T^n (if $n \geqq 0$) or $(T^*)^{-n}$ (if $n < 0$), we find that (1. 1) and (1. 2) are satisfied with $P = P_0$ (see section 3), $U = U_T$.

If T is replaced by 0, we get $Z = S = I$, $T = T^* = 0$ whence U_0 is the operator described in section 1.

We want to establish unitary equivalence of U_T and U_0. That is, we want to find a unitary operator W satisfying

$$U_T W = W U_0. \tag{4.1}$$

We shall show that $T^n \to 0$ is a sufficient condition for the existence of such a W.

First we determine a matrix (W_{ij}) satisfying (4. 1) as a product relation. As U_T and U_0 have only a finite number of non-zero elements in each row and in each column, this is only a matter of finite sums. It turns out that we are still able to require

$$W_{ij} = \delta_{ij} \text{ if } i \text{ and } j \text{ are not both } \geqq 0. \tag{4.2}$$

The matrix (W_{ij}) is uniquely determined by (4. 1) and (4. 2); it turns out that

$$W = \begin{pmatrix} \ddots & & & & & & & & \\ & I & & & & & & & \\ & & I & & & & & & \\ & & & \boxed{Z} & T^*Z & T^{*2}Z & T^{*3}Z & \cdots & \\ & & & -T & SZ & ST^*Z & ST^{*2}Z & \cdots & \\ & & & & -T & SZ & ST^*Z & \cdots & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & \ddots & & \\ & & & & & & & \ddots & \end{pmatrix}$$

i. e.

$$\begin{aligned} &W_{0j}=T^{*j}Z && \text{if } j\geqq 0, \\ &W_{ij}=0 && \text{if } i-j\geqq 2, \\ &W_{ij}=-T && \text{if } i=j+1\geqq 1, \\ &W_{ij}=S(T^*)^{j-i}Z && \text{if } j\geqq i\geqq 1. \end{aligned}$$

These values can be obtained e. g. by introducing the formal power series $w_i=\sum_{j=0}^{\infty} W_{ij}\zeta^j$ $(i=0, 1, 2, \dots)$. Relation (4. 1) implies

$$Zw_0 - T^*w_1 = I$$

$$Tw_0 + Sw_1 = \zeta w_0$$

and $w_2=\zeta w_1$, $w_3=\zeta w_2, \dots$. Using (2. 1) and (2. 2) we easily solve these equations for w_0, w_1, and find

$$w_0 = (I-\zeta T^*)^{-1}Z, \quad w_1 = -T+\zeta S(I-\zeta T^*)^{-1}Z.$$

We have to verify that the matrix W defines a unitary operator, i. e. we have to check (3. 2) and (3. 3), with A replaced by W. For $\Sigma(W_{ji})^* W_{jk}$ this does not involve infinite series, and the result follows by some elementary computation. It can also be remarked that $W^*W=I$ directly follows from (4. 1) and (4. 2): If we put $W^*W=J$, we have, by (4. 1), $U_0J=JU_0$, or $J_{ij}=J_{i+1,\,j+1}$; now (4. 2) shows that $J=I$.

In order to prove (3. 3) (with A replaced by W), we shall assume that $T^n\to 0$. This implies that $T^{*n}T^n\to 0$ (actually, if T is a contraction, then $T^n\to 0$, $T^{*n}T^n\to 0$ and $T^{*n}T^n\rightharpoonup 0$ are mutually equivalent), whence

$$\begin{aligned} ZZ^*+T^*Z(T^*Z)^*+T^{*2}Z(T^{*2}Z)^*+\dots &= I-T^*T+T^*(I-T^*T)T+\dots = \\ &= I-T^*T+(T^*T-T^{*2}T^2)+\dots = I. \end{aligned}$$

The other relations we have to establish for proving (3. 3) easily follow from this one.

5. The unitary dilation

If T is a proper contraction then the Schäffer matrix U_T represents the unitary dilation, but if T is improper, U_T does not always satisfy the minimality condition. It was remarked by B. Sz.-Nagy [4] that the unitary dilation is still described by U_T if we only restrict ourselves to a suitable subspace of $\mathfrak{R}$, viz.

$$\mathfrak{K} = \ldots \oplus \mathfrak{M}_Z \oplus \mathfrak{M}_Z \oplus \boxed{\mathfrak{H}} \oplus \mathfrak{M}_S \oplus \mathfrak{M}_S \oplus \ldots.$$

This notation means that $\mathfrak{K}$ consists of the sequences $(\ldots, h_{-1}, h_0, h_1, \ldots)$ of $\mathfrak{R}$ for which $h_j \in \mathfrak{M}_Z$ if $j<0$, $h_j \in \mathfrak{M}_S$ if $j>0$, $h_0 \in \mathfrak{H}$. By (2. 3) and (2. 4) we have

$$U_T \mathfrak{K} \subset \mathfrak{K}, \quad U_T^* \mathfrak{K} \subset \mathfrak{K}, \tag{5. 1}$$

whence U_T provides a unitary operator of $\mathfrak{K}$, still satisfying (1. 1) and (1. 2), and moreover $\mathfrak{K}$ is minimal. So, when restricted to $\mathfrak{K}$, the operator U_T provides the unitary dilation of T.

Next we introduce a second subspace of $\mathfrak{R}$, viz.

$$\mathfrak{L} = \ldots \oplus \mathfrak{M}_Z \oplus \boxed{\mathfrak{M}_Z} \oplus \mathfrak{M}_Z \oplus \ldots.$$

We infer from the matrix representation of W, using (2. 3) and (2. 4), that

$$W\mathfrak{L} \subset \mathfrak{K}, \quad W^*\mathfrak{K} \subset \mathfrak{L}.$$

As $WW^* = W^*W = I$, we infer that W provides an isometric mapping of $\mathfrak{L}$ onto $\mathfrak{K}$. The transformed operator $W^* U_T W = U_0$ maps $\mathfrak{L}$ into itself, and we obtain

Theorem. *If the contraction T satisfies $T^n \to 0$ $(n \to \infty)$, then the unitary dilation of T is unitarily equivalent to the orthogonal sum of ω copies of the bilateral shift operator, where $\omega = \dim \mathfrak{M}_Z$.*

6. Remarks

We shall say, for a moment, that T has property (A) if the unitary dilation of T is unitarily equivalent to the orthogonal sum of a number of copies of the bilateral shift operator.

If T has property (A), then T^* has propety (A), for if U is the unitary dilation of T, then U^{-1} is the unitary dilation of T^*. With this in mind we see that our theorem has a lack of symmetry, for the conditions $T^n \to 0$ and $T^{*n} \to 0$ are not equivalent, whereas both are sufficient for T and T^* to have property (A). Meanwhile we learn that if both $T^n \to 0$ and $T^{*n} \to 0$, then $\mathfrak{M}_Z$ and $\mathfrak{M}_S$ have the same dimension.

Neither $T^n \to 0$ nor $T^{*n} \to 0$ are necessary for (A). For example, if T is itself a bilateral shift operator, then T is its own unitary dilation.

An instructive example is provided by the unilateral shift operator, defined by $T\varphi_1 = 0$, $T\varphi_2 = \varphi_1$, $T\varphi_3 = \varphi_2, \ldots$, if $\varphi_1, \varphi_2, \varphi_3, \ldots$ is some complete orthogonal system. With this operator T^n tends to zero, but T^{*n} does not. $\mathfrak{M}_Z$ consists of all multiples of φ_1, but $\mathfrak{M}_S = 0$.

A simple necessary condition for (A) is that T^n tends weakly to zero, or, what is the same thing, that T^{*n} tends weakly to zero. In order to show this, we only need to remark that the powers of the bilateral shift operator tend weakly to zero.

On the other hand, this weak convergence of T^n is by no means sufficient for (A). For it is not difficult to find unitary operators whose powers tend weakly to zero but whose spectra show gaps. Because of these gaps they cannot be of the bilateral shift type.

It is easy to see that $\mathfrak{M}_Z = \mathfrak{H}$ and $\mathfrak{M}_S = \mathfrak{H}$ are equivalent; in that case the spaces $\mathfrak{R}$ and $\mathfrak{K}$ are identical. The author thinks it possible that this condition $\mathfrak{M}_Z = \mathfrak{H}$ implies (A).

References

[1] P. R. Halmos, Normal dilations and extensions of operators, *Summa Brasil. Math.*, **2** (1950), 125–134.

[2] B. Sz.-Nagy, Sur les contractions de l'espace de Hilbert, *Acta Sci. Math.*, **15** (1953), 87–92.

[3] ——— Sur les contractions de l'espace de Hilbert. II, *Acta Sci. Math.*, **18** (1957), 1–14.

[4] ——— On Schäffer's construction of unitary dilations, *Annales Univ. Budapest*, **3–4** (1960/61), 343–346.

[5] J. J. Schäffer, On unitary dilations of contractions, *Proc. Amer. Math. Soc.*, **6** (1955), 322

[6] M. Schreiber, Unitary dilations of operators, *Duke Math. J.*, **23** (1956), 579–594.

(Received July 23, 1961)