Non-summable partial sums of orthogonal series

By DAN J. EUSTICE in Urbana (Illinois, USA)

Introduction

Let $\{\Phi_n(x)\}$ be an orthonormal system on [0, 1] and let $\{S_n(x)\}$ denote the sequence of partial sums of $\sum a_n\Phi_n(x)$, where $\sum a_n^2<\infty$. It will be shown that the Riesz summability, $R(\lambda, 1)$, almost everywhere of $\{S_n(x)\}$ does not necessarily imply the $R(\lambda, 1)$-summability of every subsequence of $\{S_n(x)\}$. A condition on the index set $\{n_k\}$ implying the $R(\lambda, 1)$-summability a. e. of $\{S_{n_k}(x)\}$ if $\{S_n(x)\}$ is $R(\lambda, 1)$-summable a. e. will also be demonstrated.

Let λ_n be a positive, strictly increasing function with $\lambda_0=0$ and $\lambda_n\to+\infty$ as $n\to+\infty$. Then $\{S_n\}$ is $R(\lambda, 1)$-summable if σ_n converges, where

$$\sigma_n=\frac{1}{\lambda_{n+1}}\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)S_k.$$

ZYGMUND [8] has shown that if the coefficients $\{a_n\}$ are such that

$$\sum a_n^2(\log\log\lambda_n)^2<\infty, \tag{1}$$

then $\{S_n(x)\}$ is $R(\lambda, 1)$-summable a. e. It was recently shown ([1], [4]) that condition (1) implies every subsequence of $\{S_n(x)\}$ is $R(\lambda, 1)$-summable a. e.

The following result is a generalization of a theorem of K. TANDORI [6] for $(C, 1)$-summability.

Theorem 1. *Let $R(\lambda, 1)$-summability be a Riesz method stronger than convergence. There exists an orthonormal system $\{\Phi_n(x)\}$, a sequence $\{a_n\}$ with $\sum a_n^2<\infty$, and an index set $\nu=\{\nu_k\}$ such that the sequence of partial sums $\{S_n(x)\}$ is $R(\lambda, 1)$-summable a. e. but $\{S_{\nu_n}(x)\}$ is nowhere $R(\lambda, 1)$-summable.*

The existence of such an example was recently conjectured by J. MEDER [4].

Theorem 2. *Let the sequence of partial sums $\{S_n(x)\}$ of an orthogonal expansion with $\sum a_n^2<\infty$ be $R(\lambda, 1)$-summable a. e. Then, if $\nu=\{\nu_k\}$ is an index set such that*

$$\lambda_{\nu_j}^2\sum_{k=j}^{\infty}\frac{\Delta\lambda_k}{\lambda_{k+1}\lambda_{\nu_k}^2}=O(1) \tag{2}$$

as $j\to\infty$, where $\Delta\lambda_k=\lambda_{k+1}-\lambda_k$, the sequence of partial sums $\{S_{\nu_k}(x)\}$ is $R(\lambda, 1)$-summable a. e.

This is a generalization of a result of ZALCWASSER [7] for $(C, 1)$-summability.

1. Proof of Theorem 1

The proof of Theorem 1 depends on two lemmas. For any index set $\nu=\{\nu_k\}$, let

$$\sigma_{\nu,n}(x) = \frac{1}{\lambda_{n+1}} \sum_{k=1}^{n} \Delta\lambda_k S_{\nu_k}(x). \tag{3}$$

Also, let $\Lambda(x)$ denote the function inverse to a continuous, monotone extension of λ_n.

Lemma 1. *For any index set* $\nu=\{\nu_k\}$,

$$\lim_{n\to\infty} \sigma_{\nu,n}(x)=f(x) \qquad \text{a. e.} \tag{4}$$

if and only if

$$\lim_{k\to\infty} S_{\nu_{p_k}}(x)=f(x) \qquad \text{a. e.,} \tag{5}$$

where $p_k = [\Lambda(2^k)]$. ($[x]$ denotes greatest integer function.)

Proof. ([4], [1, Lemma 4. 2]).

Lemma 2. *Let $R(\lambda, 1)$-summability be a Riesz method stronger than convergence. Then there exists a sequence c_ν of positive, non-increasing real numbers such that*

$$\sum c_\nu^2 (\log\log \lambda_\nu)^2 < \infty \tag{6}$$

but $\sum c_\nu^2 (\log \nu)^2$ *diverges.*

Proof. It is clear that

$$\frac{\log n}{\log\log \lambda_n} \neq O(1) \tag{7}$$

as $n\to\infty$, since if it were $O(1)$, any sequence of partial sums that is $R(\lambda, 1)$-summable would be convergent, for (6) would then imply

$$\sum c_\nu^2 \log^2 \nu < \infty \tag{8}$$

which, in turn, would imply convergence of $\sum c_\nu \Phi_\nu(x)$ by the theorem of Menchoff and Rademacher [3, p. 164].

A particular sequence $\{n_k\}$ of positive integers will now be defined. Let $n_0=1$ and let n_1 be the smallest integer such that

$$\frac{\log n_1}{\log\log \lambda_{n_1}} \geqq 2.$$

Such an integer exists by (7). In general, if $n_1, \dots, n_k$ have been defined, let n_{k+1} be the smallest integer greater than $(n_k+1)^2-1$ and such that

$$\frac{\log n_{k+1}}{\log\log \lambda_{n_{k+1}}} \geqq 2^{k+1}.$$

Now, let $\{c_\nu\}$ be defined such that

$$c_\nu = \nu^{-1/2} (\log n_k)^{-3/2}$$

when $n_{k-1} < \nu < n_k$. Then $\{c_\nu\}$ is a positive, non-increasing sequence and

$$\sum_{\nu=2}^{\infty} c_\nu^2 \log^2 \nu = \sum_{k=0}^{\infty} \sum_{\nu=n_k+1}^{n_{k+1}} c_\nu^2 \log^2 \nu = \sum_{k=0}^{\infty} (\log n_{k+1})^{-3} \sum_{\nu=n_k+1}^{n_{k+1}} \frac{\log^2 \nu}{\nu} \geqq$$

$$\geqq \sum_{k=0}^{\infty} \frac{1}{\log^3 n_{k+1}} \left[\frac{\log^3 (n_{k+1}+1)}{3} - \frac{\log^3 (n_k+1)}{3} \right] \geqq$$

$$\geqq \sum_{k=0}^{\infty} \left[\frac{1}{3} - \frac{1}{3 \cdot 2^3} \right] = \infty.$$

But

$$\sum_{\nu: \lambda_\nu \geqq 2} c_\nu^2 (\log\log \lambda_\nu)^2 = \sum_{k=1}^{\infty} \sum_{\nu=n_k+1}^{n_{k+1}} c_\nu^2 (\log\log \lambda_\nu)^2 \leqq$$

$$\leqq \sum_{k=1}^{\infty} \frac{1}{\log n_{k+1}} \sum_{\nu=n_k+1}^{n_{k+1}} \frac{1}{\nu \log^2 n_{k+1}} (\log\log \lambda_{n_{k+1}})^2 \leqq$$

$$\leqq \sum_{k=1}^{\infty} \frac{1}{2^{k+1} \log n_{k+1}} \sum_{\nu=2}^{n_{k+1}} \frac{1}{\nu} \leqq \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} < \infty.$$

P r o o f o f T h e o r e m 1. TANDORI [6] has shown that if $\{c_\nu\}$ is a sequence of positive, non-increasing real numbers then condition (8) is necessary and sufficient for the convergence a. e. of $\sum c_\nu \psi_\nu(x)$ for every orthonormal system $\{\psi_\nu(x)\}$. Moreover, if condition (8) is not satisfied, $\{\psi_\nu(x)\}$ can be determined such that for a suitably chosen index set $\{N_k\}$, the series $\sum c_\nu \psi_\nu(x)$ with the N_k-th terms deleted, diverges everywhere. This example of TANDORI is a modification of an example of MENCHOFF [5]. By Lemma 2, the sequence $\{c_\nu\}$ can be chosen such that (8) does not hold but (6) does. Let $\sum c_\nu^* \psi_\nu^*(x)$ denote the orthogonal series formed by deleting the N_k-th terms and re-indexing. Denote its partial sums by $\{S_n^*(x)\}$. Condition (6) then implies that $\{S_n^*(x)\}$ is $R(\lambda, 1)$-summable a. e. Also, if $p_\nu = [\Lambda(2^\nu)]$, then $\{S_{p_\nu}^*(x)\}$ converges a. e. The theorem will be proved by constructing an orthonormal system $\{\Phi_n(x)\}$, a sequence $\{a_\nu\}$, and an index set $\{\nu_k\}$ such that

$$S_{p_\nu}(x) = S_{p_w}^*(x) \tag{9}$$

where $w = w_\nu$ tends to ∞ as ν tends to ∞, and such also that

$$S_{\nu_{p_k}}(x) = S_k^*(x). \tag{10}$$

Property (9) will show that $\{S_n(x)\}$ is $R(\lambda, 1)$-summable a. e. and (10) will show the divergence of $\sigma_{\nu,n}(x)$ by Lemma 1.

The index set $\{\nu_k\}$ will now be constructed to have the property that for every $m \geqq M$,

$$\max\{k: \nu_{p_k} \leqq p_m\} = p_w \tag{11}$$

for some w. Let $\nu_i = i$ for $i=1, 2, \ldots, p_{p_1}$. Let μ be the smallest integer such that

$$p_{p_\mu+R+1} - p_{p_\mu+R} > p_{p_2} - p_{p_1}$$

for some R, $0 \leqq R \leqq p_{\mu+1} - p_\mu$. Such a value of μ exists since the sequence $\{p_{\nu+1} - p_\nu\}$, $\nu = 0, 1, \ldots$ cannot be bounded if $R(\lambda, 1)$-summability is stronger than convergence [2]. Now, let

$$\nu_{p_{p_1}+i} = p_{p_\mu+R} + i, \qquad i = 1, 2, \ldots, p_{p_2} - p_{p_1}.$$

In general, suppose ν_i has been defined for $i = 1, \ldots p_{p_k}$; let μ' be the smallest integer greater than p_k such that

$$p_{p_{\mu'}+R+1} - p_{p_{\mu'}+R} > p_{p_{k+1}} - p_{p_k}$$

for some R, $0 \leqq R \leqq p_{\mu'+1} - p_{\mu'}$. Now, define

$$\nu_{p_{p_k}+i} = p_{p_{\mu'}+R} + i, \qquad i = 1, 2, \ldots, p_{p_{k+1}} - p_{p_k}.$$

This defines $\{\nu_i\}$ such that for every $m \geqq p_{p_1}$, property (11) holds.

Now, define

$$a_\nu = \begin{cases} c_n, & \nu = \nu_{p_n}, \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

and let $\Phi_{p_n}(x) = \psi_n^*(x)$. Complete the definition of the orthonormal system $\{\Phi_n(x)\}$ with sucessive members of $\psi_{N_k}(x)$. It will be shown that properties (9) and (10) are satisfied for this system.

$$S_{\nu_{p_n}}(x) = \sum_{k=1}^{\nu_{p_n}} a_k \Phi_k(x) = \sum_{k=1}^{n} a_{\nu_{p_k}} \Phi_{\nu_{p_k}}(x) = \sum_{k=1}^{n} c_k^* \psi_k^*(x) = S_n^*(x).$$

Thus, (10) is proved.

$$S_{p_m}(x) = \sum_{k=1}^{p_m} a_k \Phi_k(x) = \sum_{k:\, \nu_{p_k} \leqq p_m} a_{\nu_{p_k}} \Phi_{\nu_{p_k}}(x) = \sum_{k=1}^{p_w} c_k^* \psi_k^*(x) = S_{p_w}^*(x).$$

Thus, (9) holds and the proof the theorem is completed.

2. Proof of Theorem 2

Let $f(x)$ be a square-integrable function to which $\{S_n(x)\}$ is $R(\lambda, 1)$-summable a. e.

$$\begin{gathered} \frac{1}{\lambda_{n+1}} \sum_{k=j}^{n} \Delta\lambda_k [S_{\nu_k}(x) - f(x)]^2 \leqq \\ \leqq \frac{2}{\lambda_{n+1}} \sum_{k=1}^{n} \Delta\lambda_k [S_{\nu_k}(x) - \sigma_{\nu_k}(x)]^2 + \frac{2}{\lambda_{n+1}} \sum_{k=1}^{n} \Delta\lambda_k [\sigma_{\nu_k}(x) - f(x)]^2. \end{gathered} \tag{13}$$

Since $\sigma_{\nu_k}(x)$ converges to $f(x)$ a. e., the last term on the right side of (13) converges to 0 a. e. The first term on the right side of (13) will converge to 0 a. e. if

$$I_n(x) = \sum_{k=1}^{n} \frac{\Delta\lambda_k}{\lambda_{k+1}} [S_{\nu_k}(x) - \sigma_{\nu_k}(x)]^2$$

is finite a. e. as $n \to \infty$. To show this it is seen that

$$\int_0^1 I_n(x)\,dx = \sum_{k=1}^{n} \frac{\Delta\lambda_k}{\lambda_{k+1}} \int_0^1 \left(\frac{1}{\lambda_{\nu_k+1}} \sum_{j=1}^{\nu_k} \lambda_j a_j \Phi_j(x) \right)^2 dx =$$

$$= \sum_{k=1}^{n} \frac{\Delta\lambda_k}{\lambda_{k+1}} \cdot \frac{1}{\lambda_{\nu_k+1}^2} \sum_{i=1}^{k} \sum_{j=\nu_{i-1}+1}^{\nu_i} \lambda_j^2 a_j^2 \leqq \sum_{k=1}^{n} \frac{\Delta\lambda_k}{\lambda_{k+1}} \cdot \frac{1}{\lambda_{\nu_k+1}^2} \sum_{i=0}^{k} \lambda_{\nu_i}^2 \sum_{j=\nu_{i-1}+1}^{\nu_i} a_j^2.$$

Therefore,

$$\lim_{n\to\infty} \int_0^1 I_n(x)\,dx \leqq \sum_{j=1}^{\infty} \lambda_{\nu_j}^2 \sum_{i=\nu_{j-1}+1}^{\nu_j} a_i^2 \sum_{k=j}^{\infty} \frac{\Delta\lambda_k}{\lambda_{k+1}\lambda_{\nu_k}^2} =$$

$$= O(1) \sum_{j=1}^{\infty} \sum_{i=\nu_{j-1}+1}^{\nu_j} a_i^2 = O(1) \sum_{j=1}^{\infty} a_j^2 < \infty.$$

This completes the proof of the theorem.

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UNIVERSITY OF ILLINOIS

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