Remark to the preceding paper of J. Feldman*

By B. SZ.-NAGY in Szeged and C. FOIAŞ in Bucharest

At the end of his paper, J. FELDMAN raises a question. We show that the answer to this is in the negative, i. e. we exhibit *a strong operator measure* $F(\sigma)$ *in Hilbert space* H, *such that the spectrum of* $A=\int_0^{2\pi} e^{it}F(dt)$ *covers the unit circle* C *in the complex plane, nevertheless the map*

$$f \to \int_0^{2\pi} f(t)\,F(dt) \tag{1}$$

from $L^\infty(F)$ *into* $B(H)$ *is not isometric.*

Let M be an open set in $(0,2\pi)$ such that (i) $m(M)=\pi$, (ii) $m(M\cap\Delta)>0$ for any interval $\Delta\subset(0,2\pi)$, m denoting Lebesgue measure. Such an M may be constructed e. g. by taking first the open interval of length $2\pi/3$ from the middle of $(0,2\pi)$, the two open intervals each of length $2\pi/(2\cdot 3^2)$ from the middle of the two remaining parts of $(0,2\pi)$, then the four open intervals each of length $2\pi/(2^2\cdot 3^3)$ from the middle of the four remaining parts of $(0,2\pi)$, and so on.

Now consider the Hilbert space $H=L^2(M)\oplus X$ where X is one-dimensional. For any Borel subset σ of $[0,2\pi)$ define

$$F(\sigma)\big(u(\theta)\oplus\xi\big) = \chi(\sigma;\,\theta)u(\theta)\oplus\frac{m(\sigma)}{2\pi}\,\xi,$$

where $\chi(\sigma;\theta)$ denotes the characteristic function of σ. F is evidently an operator measure on H; it is a strong one since

$$\int_0^{2\pi} e^{int}F(dt)\big(u(\theta)\oplus\xi\big) = \int_0^{2\pi} e^{int}\chi(dt;\,\theta)u(\theta)\oplus\int_0^{2\pi} e^{int}\frac{dt}{2\pi}\,\xi = e^{in\theta}u(\theta)\oplus 0$$

$(n=1,2,\ldots)$, thus if

$$A=\int_0^{2\pi} e^{it}F(dt) \text{ then } A^n=\int_0^{2\pi} e^{int}F(dt) \qquad (n=1,2,\ldots).$$

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* J. FELDMAN, On the functional calculus of an operator measure, *Acta Sci. Math.*, **23** (1962), 268–271.

The spectrum of A contains the spectrum of the part of A in $L^2(M)$, i. e. the spectrum of the unitary operator $Uu(\theta)=e^{i\theta}u(\theta)$ on $L^2(M)$. The spectral measure E of U is given by

$$E(\sigma)u(\theta) = \chi(\sigma;\theta)u(\theta),$$

thus, for any interval Δ in $[0,2\pi)$,

$$\|E(\Delta)1\|^2 = \int_{M\cap\Delta} d\theta = m(M\cap\Delta) > 0$$

by (ii). It follows that no interval Δ is of E-measure 0, thus the spectrum of A covers C.

Nevertheless the map (1) is no $L^\infty(F)\to B(H)$ isometry. For if $M' = [0,2\pi)-M$ then

$$F(M')\big(u(\theta)\oplus\xi\big) = \chi(M';\theta)u(\theta)\oplus\frac{1}{2}\xi = 0\oplus\frac{1}{2}\xi,$$

thus $\|F(M')\| = \frac{1}{2}$. Taking $f(\cdot) = \chi(M';\cdot)$ we get

$$\left\|\int_0^{2\pi} f(t)F(dt)\right\| = \|F(M')\| = \frac{1}{2}$$

whereas

$$\|f\|_{L^\infty(F)} = 1$$

since the set on which f assumes the value 1, i. e. the set M', has not F-measure 0 (indeed, $\|F(M')\| = \frac{1}{2}$).

(Received October 24, 1961)