The generation of affine hulls[1])

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0. Introduction. Throughout this paper, E denotes a vector space over a field Φ of characteristic zero, the case of special interest being that in which Φ is the real number field R. A subset A of E is called *affine* iff $\alpha u+\beta v\in A$ whenever $u, v\in A$, $\alpha, \beta\in\Phi$, and $\alpha+\beta=1$. (When $\Phi=R$, this requires that A contain each line determined by any two of its points.) Each intersection of affine sets is affine, and the *affine hull* ($\operatorname{aff} X$) of a set X is defined as the intersection of all affine sets containing X. Equivalently, $\operatorname{aff} X$ is the set of all *affine combinations* of X, these being points of the form $\sum_1^n \alpha_i x_i$ with $n\in N$ (natural numbers), $x_i\in X$, $\alpha_i\in\Phi$, and $\sum_1^n \alpha_i=1$. This relationship between *blank hulls* and *blank combinations* is valid not only when *blank* means *affine*, but also when it means *linear*, *positive*, or *convex* (where, for the last two, Φ should be an *ordered* field). If bla denotes the operation of forming the blank hull, then $\operatorname{bla} X=\bigcup_{n\in N}\operatorname{bla}_n X$, where $\operatorname{bla}_n X$, denotes the set of all blank combinations of n (or fewer) points of X.

The individual operations bla_n are also of interest. When $\Phi=R$, $\operatorname{aff}_2 X$ is (X together with) the union of all lines determined by two points of X, $\operatorname{aff}_3 X$ is ($\operatorname{aff}_2 X$ together with) the union of all planes determined by three points of X, etc. It is easily verifield that

$$\operatorname{bla}_m(\operatorname{bla}_n X)=\operatorname{bla}_{mn} X \quad \text{for} \quad \operatorname{bla}\neq\operatorname{aff}. \quad \text{(See 1. 2.)}$$

The present paper is motivated by the fact that while

$$\operatorname{aff}_m(\operatorname{aff}_n X)\subset\operatorname{aff}_{mn} X,$$

the two sets need not be equal. For example, if the affinely independent set $Z\subset E$ consists of four points $z_1, \ldots, z_4$, then

$$\operatorname{aff}_4 Z\sim\operatorname{aff}_2(\operatorname{aff}_2 Z)=\left\{\frac{1}{2}\left(\sum_1^4 z_i\right)-z_j:\ 1\leqq j\leqq 4\right\}.$$

To describe the same example more geometrically, let a, b, c, and d be the vertices of a tetrahedron in R^3. For each permutation u, v, w, x of these four vertices, let $\Pi(u, v; w, x)$ denote the plane which contains the line $\overline{wx}$ and is parallel to the

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line $\overline{uv}$. Then

$$\Pi(a, b; c, d) \cap \Pi(a, c; b, d) \cap \Pi(a, d; b, c) = \left\{\frac{1}{2} b + \frac{1}{2} c + \frac{1}{2} d - \frac{1}{2} a\right\},$$

and the set $\mathrm{aff}_4\{a, b, c, d\} \sim \mathrm{aff}_2(\mathrm{aff}_2\{a, b, c, d\})$ consists of this point together with three others which are similarly situated.

We study here sets of the form $\mathrm{aff}_{n_1}(\mathrm{aff}_{n_2}(\ldots(\mathrm{aff}_{n_k}X)\ldots))$ and others which are formed in a similar way. Since many of the results are rather technical in nature, the reader is referred to the text for full statements. However, the general spirit of our results is indicated by the following corollaries (3. 3 and 3. 4):

For all X, $\mathrm{aff}_{mn-1}X \subset \mathrm{aff}_m(\mathrm{aff}_n X)$; *and if* $m \neq n$, $\mathrm{aff}_{mn}X = \mathrm{aff}_m(\mathrm{aff}_n X) \cup \mathrm{aff}_n(\mathrm{aff}_m X)$.

1. Results on $\mathrm{bla}_m(\mathrm{bla}_n X)$. Let us begin by extending the definition of $\mathrm{bla}_n X$. For $n_1, \ldots, n_m \in N$, $\mathrm{bla}_{(n_1,\ldots,n_m)} X$ will denote the set of all points of the form $\sum_1^m \alpha_i y_i$ for $y_i \in \mathrm{bla}_{n_i} X$ and $(\alpha_1, \ldots, \alpha_m) \in B_m$, where for bla = lin the last condition imposes no restriction, for bla = aff it means that $\sum_1^m \alpha_i = 1$, for bla = pos it means that $\alpha_i \geqq 0$, and for bla = con it means that $\alpha_i \geqq 0$ and $\sum_1^m \alpha_i = 1$. Thus, in particular, $\mathrm{bla}_m X = \mathrm{bla}_{(1,\ldots,1)} X$ with m 1's and $\mathrm{bla}_m(\mathrm{bla}_n X) = \mathrm{bla}_{(n,\ldots,n)} X$ with m n's.

1.1. Proposition. *For all four types of operation (and for all X),* $\mathrm{bla}_{(n_1,\ldots,n_m)} X \subset \mathrm{bla}_{\sum_1^m n_i} X$; *in particular,* $\mathrm{bla}_m(\mathrm{bla}_n X) \subset \mathrm{bla}_{mn} X$.

Proof. For $p \in \mathrm{bla}_{(n_1,\ldots,n_m)} X$, let $p = \sum_1^m \alpha_i y_i$ with $y_i \in \mathrm{bla}_{n_i} X$ and $(\alpha_1, \ldots, \alpha_m) \in B_m$. For each i, y_i can be expressed in the form $\sum_{j=1}^{n_i} \beta_{ij} x_{ij}$ with $x_{ij} \in X$ and $(\beta_{i1}, \ldots, \beta_{in_i}) \in B_{n_i}$. But then of course

$$p = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_i \beta_{ij} x_{ij},$$

where it is easily verified that

$$(\alpha_1 \beta_{11}, \ldots, \alpha_1 \beta_{1n_1}, \ldots, \alpha_m \beta_{m1}, \ldots, \alpha_m \beta_{mn_m}) \in B_{\sum_1^m n_i}. \blacksquare$$

The next observation is due jointly to W. E. Bonnice and the author.[1])

1.2. Proposition. *For bla* $\neq$ aff, $\mathrm{bla}_{(n_1,\ldots,n_m)} X = \mathrm{bla}_{\sum_1^m n_i} X$; *in particular,* $\mathrm{bla}_m(\mathrm{bla}_n X) = \mathrm{bla}_{mn} X$.

Proof. Since this is obvious for linear or positive combinations, we discuss only the case of convex combinations. Let $k_0 = 0$ and $k_i = \sum_{r=1}^{i} n_i$ for $1 \leqq i \leqq m$.

[1]) See also William Bonnice and Victor Klee, The generalisation of convex hulls, *Math. Annalen,* **149** (1963), to appear.

Consider a point $p \in \operatorname{con}_{k_m} X$ — say $p = \sum_1^{k_m} \alpha_j x_j$ with $x_j \in X$, $a_j \geqq 0$, and $\sum_1^{k_m} \alpha_j = 1$. For $1 \leqq i \leqq m$, let $\sigma_i = \sum_{k_{i-1}+1}^{k_i} \alpha_j$. For $k_{i-1}+1 \leqq j \leqq k_i$, define

$$\beta_j = \alpha_j / \sigma_i \text{ when } \sigma_i \neq 0, \quad \beta_j = \alpha_j \text{ when } \sigma_i = 0.$$

Then $p = \sum_{i=1}^{m} \left(\sigma_i \sum_{j=k_{i-1}+1}^{k_i} \beta_j x_j \right)$, where all coefficients are $\geqq 0$, $\sum_1^m \sigma_i = 1$, and

$$\sum_{j=k_{i-1}+1}^{k_i} \beta_i = 1 \text{ when } \sigma_i \neq 0,$$

$$\sum_{j=k_{i-1}+1}^{k_i} \beta_j x_j = 0 \text{ when } \sigma_i = 0.$$

Consequently $p \in \operatorname{con}_{(n_1, \ldots, n_m)} X$. ∎

The proof of 1. 2 depends on partitioning the α_j's into m groups such that there are n_i of them in the i^{th} group and such that those in each group are all zero or have nonzero sum. The same problem arises in connection with affine combinations, but there the desired partition may not exist. In order to discuss the situation efficiently, we shall introduce the notion of a weighted set and shall study partitions of such sets.

2. Partitions of weighted sets. Here and subsequently, Γ denotes a fixed (but arbitrary) ordered abelian group, while $<$, $+$, and $-$ are used for the ordering, addition, and subtraction in both Γ and N. A *weighted point* is an ordered pair $w = (\bar{w}, w')$ for which $w' \in \Gamma$ ($\bar{w}$ arbitrary); w' is called the *weight* of w. A *weighted set* is a finite set W of weighted points such that for $u, v \in W$, $\bar{u} = \bar{v} \Rightarrow u' = v'$. The *weight* $\mu(W)$ of a weighted set W is the sum of the weights of its points $\left(\mu(W) = \sum_{w \in W} w'\right)$; $(W, \mu(W))$ is a weighted point. A weighted set will be called *good* unless its weight is zero while at least one of its points has nonzero weight, where zero is the neutral element of Γ.

A *partition* of a set S is a finite family of pairwise disjoint subsets of S whose union is S. For $n \in N$, an *n-partition* is one in which each member consists of n points. An $(n_1, \ldots, n_m)$*-partition* is one consisting of m sets which can be ordered in such a way that (for $1 \leqq i \leqq m$) the i^{th} set is of cardinality n_i. A partition $\mathcal{P}$ of a weighted set will be called *nice* iff each of its members is good; thus $\mathcal{P}$ is nice unless there exist $P \in \mathcal{P}$ and $w \in P$ such that $w' \neq 0 = \mu(P)$.

2. 1. Theorem. *Suppose W is a weighted set and $n_1, \ldots, n_m \in N$ with $\sum_1^m n_i = \operatorname{card} W$. Then W admits a nice $(n_1, \ldots, n_m)$-partition if and only if the following three statement are all false:*

(S_1) *$m = 1$ and W is not good;*

(S_2) *$n_i = 2$ for all i; W is the union of two sets of odd cardinality such that all points of one set have the same nonzero weight α and all points of the other set have weight $-\alpha$;*

(S_3) *there exists $n \geqq 3$ such that $n_i = n$ for all i, all but one point of W have the same nonzero weight α, and the exceptional point has weight $(1-n)\alpha$.*

Proof. It is easily verified that if S_1, S_2, or S_3 is true, W does not admit a nice $(n_1, \ldots, n_m)$-partition. We now assume that W does not admit such a partition, and wish to show that S_1, S_2, or S_3 is true. If $m=1$, it is evident that W is not good and S_1 holds, so we assume $m>1$. Let $k=\operatorname{card} W$ and consider an enumeration of the points of W in order of increasing weight:

$$w_1' \leqq w_2' \leqq \ldots \leqq w_k'.$$

Let P_1 be (the set consisting of) the first n_1 of the w_j's, P_2 the next n_2 of them, ..., P_m the last n_m of them. Then some set P_r fails to be good, and from the method of construction it is clear that $w'<0$ for all $w \in \bigcup_1^{r-1} P_i$, while $w'>0$ for all $w \in \bigcup_{r+1}^{m} P_i$. Since $m>1$, there are three cases to be considered:

$$1 = r<m; \; 1<r<m; \; 1<r = m.$$

However, the first case is treated like the third, so it suffices to consider the second and third cases. We assume, then, that $1<r$.

Note that if $u \in P_r$ with $u' \leqq 0$ and $w \in P_i$ for $i<r$, then $w'=u'$, for otherwise a nice $(n_1, \ldots, n_m)$-partition of W results from the partition $\{P_1, \ldots, P_m\}$ upon interchanging u and w. Since $1<r$, this implies the existence of $\alpha>0$ such that $w' = -\alpha$ whenever $w \in W$ with $w' \leqq 0$. Further, if $i<r$ and $v \in P_r$ with $v'>0$, then $\mu(P_i) = -n_i\alpha$ and hence $v' = (n_i-1)\alpha$, for otherwise a nice $(n_1, \ldots, n_m)$-partition of W results from interchanging v with a point of P_i. Thus there exists $n \geqq 2$ such that $n_i=n$ for all $i<r$, and $v' = (n-1)\alpha$ whenever $v \in P_r$ with $v'>0$.

We wish next to show that $n_r=n$, and for this purpose will consider another $(n_1, \ldots, n_m)$-partition of W. Let Q_r be the first n_r of the w_j's (in terms of the given ordering), Q_1 the next n_1 of them, Q_2 the next n_2 of them, ..., Q_{r-1} the next n_{r-1} of them, Q_{r+1} the next n_{r+1} of them, ..., Q_m the last n_m of them. Then $Q_i=P_i$ for $i>r$ and (since $1<r$) there exists $s<r$ such that the weighted set Q_s is not good. If $s>1$ it follows from reasoning in the preceding paragraph that $\operatorname{card} Q_r = \operatorname{card} Q_1$, whence $n_r=n_1=n$. If $s=1$, let R_1 be the first n_1 of the w_j's, R_r the next n_r of them, and $R_i=Q_i$ for $i \notin \{1, r\}$. By hypothesis, some R_i fails to be good, and clearly it is R_r. But then from the method of construction (using the ordering of the w_j's) we conclude that $R_r=Q_1$, whence $n_r=n_1=n$.

We have now established that $n_i = n \geqq 2$ for $1 \leqq i \leqq r$, that $w' = -\alpha$ for all $w \in W$ with $w' \leqq 0$, and that $v' = (n-1)\alpha$ for all $v \in P_r$ with $v'>0$. Let l denote the number of points of P_r which are of positive weight. Then

$$0 = \mu(P_r) = l(n-1)\alpha = (n-l)\alpha = (l-1)n\alpha,$$

whence $l=1$. Thus S_3 holds (with its α the negative of our present α) if $r=m$ and $n \geqq 3$, while S_2 holds if $r=m$ and $n=2$. If $r<m$, the reasoning of the above paragraphs shows that $n_i=n$ for *all* i and that $\alpha=(n-1)\beta$ whenever $\beta=w'>0$ for some $w \in W$. But then $\alpha = (n-1)^2\alpha$, whence $n=2$ and S_2 holds. ∎

3. The basic theorem on $\mathrm{aff}_{(n_\alpha, \ldots, n_m)}X$.

3. 1. Theorem. *Let E be a vector space over a field Φ of characteristic zero, $n_1, \ldots, n_m \in N$, and $p \in \mathrm{aff}_{\Sigma_1^m n_i} X$. Then $p \in \mathrm{aff}_{(n_1, \ldots, n_m)} X$ unless one of the following statements is true:*

(i) $m>1$; $n_i=2$ *for all* i; *for each expression of* p *in the form* $\sum_1^{2m}\alpha_i x_i$ *with* $x_i\in X$, $\alpha_i\in\Phi$, *and* $\sum_1^{2m}\alpha_i=1$, *there is an odd number* $l\in N$ *such that* $m<l<2m$, l *of the* α_i's *are equal to* $\frac{1}{2(l-m)}\in\Phi$, *and the remaining* $2m-l$ α_i's *are equal to* $-\frac{1}{2(l-m)}\in\Phi$;

(ii) $m>1$; *all* n_i's *have the same value* $n\geqq 3$; *for each expression of* p *in the form* $\sum_1^{mn}\alpha_i x_i$ *with* $x_i\in X$, $\alpha_i\in\Phi$, *and* $\sum_1^{mn}\alpha_i=1$, *one* α_i *is equal to* $\frac{1-n}{(m-1)n}\in\Phi$ *and the others are all equal to* $\frac{1}{(m-1)n}\in\Phi$.

(If 1 is the unit element of Φ and $a,b\in N$, the point $\underbrace{(1+1+\ldots+1)}_{a\text{ terms}}\div\underbrace{(1+1+\ldots+1)}_{b\text{ terms}}\in\Phi$ is denoted simply by $\frac{a}{b}$.

Proof. Let us suppose first that $p\in\operatorname{aff}_{(n_1,\ldots,n_m)}X$, whence there exist $y_i\in\operatorname{aff}_{n_i}X$ and $\beta_i\in\Phi$ such that $\sum_1^m\beta_i=1$ and $\sum_1^m\beta_i y_i=p$. For each i, there exist points $z_1^i,\ldots,z_{n_i}^i$ (not necessarily distinct) of X and numbers $\gamma_1^i,\ldots,\gamma_{n_i}^i\in\Phi$ such that $\sum_{j=1}^{n_i}\gamma_j^i=1$ and $\sum_{j=1}^{n_i}\gamma_j^i z_j^i=y_i$. Now with $s=\sum_1^m n_i$,

$$(x_1,\ldots,x_s)=(z_1^1,\ldots,z_{n_1}^1,\ldots,z_1^m,\ldots,z_{n_m}^m),$$

and
$$(\alpha_1,\ldots,\alpha_s)=(\beta_1\gamma_1^1,\ldots,\beta_1\gamma_{n_1}^1,\ldots,\beta_m\gamma_1^m,\ldots,\beta_m\gamma_{n_m}^m),$$

we have
$$p=\sum_1^s\alpha_r x_r,\ x_r\in X,\ \text{and}\ \sum_1^s\alpha_r=1.$$

Further, in their natural ordering the indices $1,\ldots,s$ are partitioned into m sets such that there are n_i indices in the i^{th} set, and the sum $\beta_i\left(=\sum_{r=1}^{n_i}\beta_i\gamma_r^i\right)$ of the α_r's corresponding to the i^{th} set of incides is different from zero unless all of these α_r's are zero. From this it is easily verified that (i) and (ii) are both false.

Conversely, we assume (i) and (ii) to be false and want to prove that $p\in\operatorname{aff}_{(n_1,\ldots,n_m)}X$. Since the field Φ is of characteristic zero, its additive group Γ is isomorphic with a subgroup of a direct sum of a (possibly infinite) number of copies of the additive group of rational numbers. Since this direct sum is an ordered group under the lexicographic ordering based on the natural ordering of rational numbers, we may assume without loss of generality that Γ is an ordered group (not implying, of course, that Φ is an ordered field).

Now taking Γ as an ordered group, we see from 2. 1 (and the assumption that (i) and (ii) are false) that p admits an expression in the form $p=\sum_1^s\alpha_r x_r$ with $x_r\in X$, $\alpha_r\in\Phi$, $\sum_1^s\alpha_r=1$, and such that the weighted set $\{(r,\alpha_r):1\leqq r\leqq s\}$ admits a nice $(n_1, \dots, n_m)$-partition. We may assume without loss of generality that the members of the partition are the sets $\{(r, \alpha_r): s_{i-1} < r \leqq s_i\}$ for $1 \leqq i \leqq m$, where $s_0 = 0$ and $s_i = \sum_1^i n_r$. Defining $\sigma_i = \sum_{s_i - 1}^{s_i} \alpha_r$, we see that either $\sigma_i \neq 0$ or $\alpha_r = 0$ for all r with $s_{i-1} < r \leqq s_i$. It then follows as in the proof of 1. 2 that $p \in \operatorname{aff}_{(n_1, \dots, n_m)} X$. ∎

3. 2. Corollary. *If the numbers $n_1, \dots, n_m (\in N)$ are not all the same,* $\operatorname{aff}_{(n_1, \dots, n_m)} X = \operatorname{aff}_{\sum_1^m n_i} X$.

3. 3. Corollary. *For all X and all $n_1, \dots, n_m \in N$,* $\operatorname{aff}_{\left(\sum_1^m n_i\right)-1} X \subset \operatorname{aff}_{(n_1, \dots, n_m)} X \subset \subset \operatorname{aff}_{\sum_1^m n_i} X$; *in particular,* $\operatorname{aff}_{mn-1} X \subset \operatorname{aff}_m (\operatorname{aff}_n X) \subset \operatorname{aff}_{mn} X$.

3. 4. Corollary. *For $m \neq n$,* $\operatorname{aff}_{mn} X = \operatorname{aff}_m (\operatorname{aff}_n X) \cup \operatorname{aff}_n (\operatorname{aff}_m X)$.

3. 5. Corollary. *If X is affinely independent and consists of k points, and $m \geqq 2$, the cardinality of the set*

$$X' = \operatorname{aff}_{mn} X \sim \operatorname{aff}_m (\operatorname{aff}_n X)$$

is equal to
$$\begin{cases} \binom{k}{mn} mn & \text{when } n \geqq 3, \\ \binom{k}{2m} 2^{2m-2} & \text{when } n = 2 \text{ and } m \text{ is even}, \\ \binom{k}{2m} \left(2^{2m-2} - \frac{1}{2}\binom{2m}{m}\right) & \text{when } n = 2 \text{ and } m \text{ is odd}. \end{cases}$$

3. 6. Corollary. *If X is finite, so is X'. If* $\dim (\operatorname{aff} X) < mn - 1$, *$X'$ is empty. If* $\dim (\operatorname{aff} X) = mn - 1$, $\operatorname{card} X' \leqq c(m, n)$, *where*

$$c(m, n) = \begin{cases} mn & \text{when } n \geqq 3, \\ 2^{2m-2} & \text{when } n = 2 \text{ and } m \text{ is even}, \\ 2^{2m-2} - \frac{1}{2}\binom{2m}{m} & \text{when } n = 2 \text{ and } m \text{ is odd}. \end{cases}$$

If $\dim E \geqq mn$ *and $m \geqq 2 \leqq n$, then E containts a set X for which X' consists of $c(m, n)$ distinct parallel „lines" (genuine lines when $\Phi = R$).*

Proofs. The Corollaries 3. 2, 3. 3, and 3. 4 follow immediately from 3. 1. For the first part of 3. 5, apply 3. 1 (ii) to show that card $X' = \binom{k}{mn}\binom{mn}{1}$. For the second and third parts of 3. 5, apply 3. 1 (i) to show that card X' is equal to $\binom{k}{mn}$ times the number of sets $Y \subset \{1, \dots, mn\}$ for which card Y is odd and card $Y < mn -$ $-$ card Y. The first three assertions of 3. 6 follow from 3. 1, 3. 3, and 3. 5 respectively. For the fourth, let F be an $(mn-1)$-dimensional linear subspace of E, Y an affinely independent set in F with card $Y=mn$, $z\in E\sim F$, and $X=Y+\Phi z$. It is easily verified that

$$\mathrm{aff}_{mn}X\sim \mathrm{aff}_m(\mathrm{aff}_n X) = (\mathrm{aff}_{mn}Y\sim \mathrm{aff}_m(\mathrm{aff}_n Y))+\Phi z,$$

whence the desired conclusion follows from 3. 5. ▮

4. A qualitative approach. For $Y\subset E$ and $n_i^j\in N$, consider the set

$$\mathrm{bla}_{(n_1^1,\dots,n_{m(1)}^1)}(\mathrm{bla}_{(n_1^2,\dots,n_{m(2)}^2)}(\dots(\mathrm{bla}_{(n_1^k,\dots,n_{m(k)}^k)}Y)\dots)).$$

With $n=\prod_{j=1}^{k}\left(\sum_{i=1}^{m(j)} n_i^j\right)$, it follows from 1. 2 that this set is equal to $\mathrm{bla}_n Y$ when bla $\neq$ $\neq$ aff. For bla $=$ aff, the situation is much more complex and a full analysis would probably cost more than it is worth. In any case, the problem of describing the above set reduces to one concerning the interaction of operations aff_n for various values of n, since (by 3. 2) $\mathrm{aff}_{(n_1,\dots,n_m)}X=\mathrm{aff}_{\sum_1^m n_i}X$ for all X if the n_i's assume at least two different values, while of course $\mathrm{aff}_{(n_1,\dots,n_m)}X=\mathrm{aff}_m(\mathrm{aff}_n X)$ if all the n_i's have the same value n.

From 3. 3 it follows that always

$$\mathrm{aff}_l(\mathrm{aff}_m(\mathrm{aff}_n X))\subset \mathrm{aff}_{lmn-l-1}X\cup \mathrm{aff}_{lmn-n-1}X.$$

However, this is a crude approach and becomes cruder as the number of operations increases. The present section shows by means of a qualitative approach that always

(1) $$\mathrm{aff}_{n_1}(\mathrm{aff}_{n_2}(\dots(\mathrm{aff}_{n_k}X)\dots))\subset \mathrm{aff}_{(n_1 n_2\dots n_k)-1}X,$$

and that if X is finite, so is the set

(2) $$\mathrm{aff}_{n_1 n_2\dots n_k}X\sim \mathrm{aff}_{n_1}(\mathrm{aff}_{n_2}(\dots(\mathrm{aff}_{n_k}X)\dots)).$$

Section 5 contains a more quantitative analysis, leading to a description of sets of the form (2) for $k=3$ which is similar in completeness to that of Section 3 for the case $k=2$ (cf. 6. 6).

A basic tool is the notion of a weighted partition. When $\mathscr{P}$ is a partition of a weighted set, the corresponding *weighted partition* is the weighted set $\mathscr{P}^* = \{(P,\mu(P)):P\in\mathscr{P}\}$. To illustrate the combinatorial problem which is involved in the study of sets of the form (2), let us consider a weighted set W consisting of twelve points, ten of weight $1/6$ and two of weight $-2/6$. Though W admits a nice 3-partition $\mathscr{P}$, the weighted partition $\mathscr{P}^*$ must consist of four „points" (the sets $P\in\mathscr{P}$), two of weight $1/2$ and one of weight $-1/2$, whence $\mathscr{P}^*$ does not admit a nice 2-partition. This corresponds to the fact that if an affinely independent set $X\subset E$ consists of twelve distinct points $x_1,\dots,x_{12}$, and if $p=\left(\sum_1^{10}\frac{1}{6}x_i\right)-\frac{1}{3}x_{11}-\frac{1}{3}x_{12}$, then $p\notin \mathrm{aff}_2(\mathrm{aff}_2(\mathrm{aff}_3 X))$, even though $p\in \mathrm{aff}_m(\mathrm{aff}_n X)$ whenever $mn=12$. Thus in studying sets of the form (2), trouble is caused (speaking roughly) not only by weighted sets which admit no nice partitions but also by those whose nice partitions admit no nice partitions, and so on down the line. To establish (1) we must show that if W is a weighted set of cardinality $n_1 n_2 \ldots n_k$ with at least one point of zero weight, then W admits a nice n_k-partition $\mathscr{P}_{(k)}$ such that $\mathscr{P}^*_{(k)}$ admits a nice n_{k-1}-partition $\mathscr{P}_{(k-1)}$ such that $\mathscr{P}^*_{(k-1)} \ldots$ such that $\mathscr{P}^*_{(3)}$ admits a nice n_2-partition $\mathscr{P}_{(2)}$.

The basic lemma is easy to prove, but its statement requires some additional notation. Let T be a finite set, $\mathfrak{S}$ the class of all nonempty subsets of T, and Ξ the class of all functions on T to Γ. For $S \in \mathfrak{S}$ and $\xi \in \Xi$, let S_ξ denote the weighted set $\{(s, \xi s): s \in S\}$ and let $\mu_\xi S$ denote its weight $((\mu_\xi S = \mu(S_\xi) = \sum_{s \in S} \xi s))$. Let $n_1, \ldots, n_m \in N$ with $m \geqq 2$ and $\sum_1^m n_i = \operatorname{card} T$, and let $\mathfrak{P}$ denote the class of all $(n_1, \ldots, n_m)$-partitions of T. For each $\mathscr{P} \in \mathfrak{P}$ and $\xi \in \Xi$, let $\mathscr{P}_{(\xi)}$ denote the corresponding partition of the weighted set T_ξ; that is, $P_{(\xi)} = \{S_\xi : S \in \mathscr{P}\}$.

4. 1. **Lemma.** *Suppose Δ is a finite subset of Γ and H is the set of all $\xi \in \Xi$ such that T_ξ admits at least one nice $(n_1, \ldots, n_m)$-partition $\mathscr{P}_{(\xi)}$, with $\mathscr{P}^{*\prime}_{(\xi)} \subset \Delta$ for all such $\mathscr{P}_{(\xi)}$. Then the set H is finite.*

Proof. Let I be the class of all ordered triples $(\mathfrak{Q}, f, g)$ for which $\mathfrak{Q}$ is a nonempty subset of $\mathfrak{P}$, f is a function whose domain is $\mathfrak{P} \sim \mathfrak{Q}$, and the following conditions are satisfied:

for each $\mathscr{Q} \in \mathfrak{Q}$, $f_{\mathscr{Q}}$ is a function on $\mathscr{Q}$ to Δ;

for each $\mathscr{P} \in \mathfrak{P} \sim \mathfrak{Q}$, $g\mathscr{P}$ is a nonempty subset of $\mathscr{P}$.

For each $\iota = (\mathfrak{Q}, f, g) \in I$, let H_ι denote the set of all $\eta \in H$ which have the following two properties:

$\mathfrak{Q} = \{\mathscr{Q} \in \mathfrak{P} : \mathscr{Q}_{(\eta)}$ is nice$\}$; whenever $S \in \mathscr{Q} \in \mathfrak{Q}$, then $\mu_\eta S = f_{\mathscr{Q}} S$;

for each $\mathscr{P} \in \mathfrak{P} \sim \mathfrak{Q}$, $g\mathscr{P} = \{S \in \mathscr{P} : \mu_\eta S = 0\}$.

It is evident that $H = \bigcup_{\iota \in I} H_\iota$ and that I is finite. To complete the proof it suffices to show (for $\iota \in I$) that the difference of any two functions in H is constant on X, for then it is apparent that each set H_ι has at most one member.

Let $\iota = (\mathfrak{Q}, f, g) \in I$ and consider two functions $\xi, \eta \in H_\iota$. Choose $\mathscr{Q} \in \mathfrak{Q}$. To show that $\xi - \eta$ is constant it suffices to show that whenever u_1 and u_2 are points of T which lie in different members U_1 and U_2 of $\mathscr{Q}$, then $\xi u_1 - \eta u_1 = \xi u_2 - \eta u_2$. For such U_i it follows from the definition of H_ι that $\mu_\xi U_i = \mu_\eta U_i$ $(i = 1, 2)$. Let $\mathscr{P}$ denote the partition of T which is obtained from $\mathscr{Q}$ by interchanging u_1 and u_2. Then

$$\mathscr{P} = (\mathscr{Q} \sim \{U_1, U_2\}) \cup \{V_1, V_2\},$$

where

$$V_i = (U_i \sim \{u_i\}) \cup \{u_j\} \qquad (i \neq j).$$

Clearly

$$\mu_\xi V_i = \mu_\xi U_i - \xi u_i + \xi u_j$$

and

$$\mu_\eta V_i = \mu_\eta U_i - \eta u_i + \eta u_j.$$

If $\mathscr{P} \in \mathfrak{Q}$, then (for $i = 1$ *and* $i = 2$) $\mu_\xi V_i = f_P V_i = \mu_\eta V_i$, and (recalling that $\mu_\xi U_i = \mu_\eta U_i$) we conclude that $\xi u_j - \eta u_j = \xi u_i - \eta u_i$. Suppose, on the other hand, that $\mathscr{P} \in \mathfrak{P} \sim \mathfrak{Q}$. Then by the definition of H_ι, neither $\mathscr{P}_{(\xi)}$ nor $\mathscr{P}_{(\eta)}$ is nice. Since $\mathscr{Q}$ was nice it follows that $\mu_\xi V_i = 0$ for $i = 1$ *or* $i = 2$ (but not necessarily both), whence $V_i \in g\mathscr{P}$ and $\mu_\eta V_i = 0$. Then, as before, $\xi u_j - \eta u_j = \xi u_i - \eta u_i$. ∎

For a finite set T and for $\gamma\in\Gamma$, let $A_\gamma(T)$ denote the set of all functions ξ on T to Γ such that $\sum_{t\in T}\xi t=\gamma$. For $n_1,\dots,n_k\in N$ with $\prod_{i=1}^{k} n_i=\operatorname{card} T$, let $A_\gamma(T;n_1,\dots,n_k)$ denote the set of all $\xi\in A_\gamma(T)$ for which there exist weighted sets $T_\xi = W_{k+1}, W_k, \dots, W_2$ with $W_i=\mathscr{P}_i^*$ for some nice n_i-partition $\mathscr{P}_i$ of W_{i+1} $(2\leqq i\leqq k)$.

4. 2. Theorem. *Suppose T is a finite set, $\gamma\in\Gamma$, and $n_1,\dots,n_k\in N$ with $\prod_{i=1}^{k} n_i=$ card T. Then the set $A_\gamma(T)\sim A_\gamma(T;n_1,\dots,n_k)$ is finite.*

Proof. When $k=2$, the assertion follows from 2. 1. Suppose it is known for $k=j-1\geqq 2$ and consider the case $k=j$. Let S be a set of cardinality $\prod_{i=1}^{j-1} n_i$ and let $B=A_\gamma(S)\sim A_\gamma(S;n_1,\dots,n_{j-1})$. Then B is finite by the inductive hypothesis, so the set $\Delta=\bigcup_{\eta\in B}\eta S$ is also finite. Now with card $T=\prod_{i=1}^{j} n_i$, let G denote the set of all $\xi\in A_\gamma(T)$ such that T_ξ admits no nice n_j-partition. The set G is finite by 2. 1. Let H denote the set of all $\xi\in A_\gamma(T)$ such that T_ξ admits at least one nice n_j-partition $\mathscr{P}_{(\xi)}$, but $\mathscr{P}_\xi^{*\prime}\subset\Delta$ for all such $\mathscr{P}_{(\xi)}$. Then H is finite by 4. 1, and it is easily verified that

$$A_\gamma(T)\sim A_\gamma(T;n_1,\dots,n_k)\subset G\cup H. \blacksquare$$

4. 3. Theorem. *For each set $X\subset E$,*

$$\operatorname{aff}_{(n_1n_2\dots n_k)-1}X\subset \operatorname{aff}_{n_1}(\operatorname{aff}_{n_2}(\dots(\operatorname{aff}_{n_k}X)\dots)).$$

If X is finite, so is the set

$$\operatorname{aff}_{n_1n_2\dots n_k}X\sim \operatorname{aff}_{n_1}(\operatorname{aff}_{n_2}(\dots(\operatorname{aff}_{n_k}X)\dots)).$$

Proof. Let $r=\prod_{i=1}^{k} n_i$ and let $T=\{1,\dots,r\}$. As in the proof of 3. 1, we see that if $\xi\in A_1$ $(T;n_1,\dots,n_k)$ and if $x_1,\dots,x_r$ are (not necessarily distinct) points of X, then

$$\sum_1^r \xi(i)x_i\in \operatorname{aff}_{n_1}(\operatorname{aff}_{n_2}(\dots(\operatorname{aff}_{n_k}X)\dots)).$$

The second statement of 4. 3 follows at once from this fact in conjunction with 4. 2. For the first part of 4. 3, consider an arbitrary point $p=\sum_1^{r-1}\alpha_i x_i$ with $x_i\in X$, $\alpha_i\in\Gamma$, and $\sum_1^{r-1}\alpha_i=1$. For each $\beta\in\Gamma$, let the function $\xi\beta\in A_1(T)$ be defined as follows:

$$\xi_\beta(i)=\alpha_i \quad\text{for}\quad 1\leqq i\leqq r-2;\quad \xi_\beta(r-1)=\alpha_{r-1}-\beta;\ \xi_\beta(r)=\beta.$$

Since Γ is infinite, 4. 2 implies the existence of $\beta\in\Gamma$ for which $\xi_\beta\in A_1$ $(T;n_1\dots,n_k)$. With $x_r=x_{r-1}$, we have

$$p=\sum_{i=1}^{r}\xi_\beta(i)x_i\in \operatorname{aff}_{n_1}(\operatorname{aff}_{n_2}(\dots(\operatorname{aff}_{n_k}X)\dots)). \blacksquare$$

5. Troublesome sets: Lemmas. For a weighted set W, W' will denote the set $\{w' : w \in W\} \subset \Gamma$. W will be said to *have the form* $(\gamma_1)^{a_1} \dots (\gamma_k)^{a_k}$ iff $a_i \in \Gamma$, $a_i \in N$, $\sum_1^k a_i = \text{card } W$, and $a_i = \text{card } \{w \in W : w' = \gamma_i\}$ for $1 \leqq i \leqq k$; and W *has the crude form* $(\gamma_1)^{a_1} \dots (\gamma_k)^{a_k}$ iff $\gamma_i \in \Gamma$, $a_i \in N \cup \{0\}$, $\sum_1^k a_i = \text{card } W$, and W admits a partition into pairwise disjoint sets $P_1, \dots, P_k$ such that card $P_i = a_i$ and $\subset \{\gamma_i\}$ for $1 \leqq i \leqq k$. With $a_i > 0$, the first condition requires that $W' = \{\gamma_i : 1 \leqq i \leqq k\}$ and the γ_i's are distinct; the second condition requires that $W' \subset \{\gamma_i : 1 \leqq i \leqq k\}$ but permits $a_i = 0$ (with of course $P_i = \emptyset$) and $\gamma_i = \gamma_j$, for $i \neq j$.

A weighted set W will be called *troublesome* iff W has the form

$$\text{(T)} \qquad (\alpha)^r (\beta_1)^{r_1} \dots (\beta_s)^{r_s} \textit{ with } r \geqq 3, s \geqq 1,$$

and

$$0 < \alpha \leqq \min \{-\beta_i : 1 \leqq i \leqq s\} \quad \textit{or} \quad 0 > \alpha \geqq \max \{-\beta_i : 1 \leqq i \leqq s\}.$$

We shall often refer to the expression (T), using its notation without further explanation.

A weighted set W will be called *positively* ⟨resp. *negatively*⟩ *troublesome* iff W has the form (T) with $\alpha > 0$ ⟨resp. $\alpha < 0$⟩, *doubly troublesome* iff it has the form (T) with $s = 1$ and $\beta_1 = -\alpha$, *singly troublesome* iff it has the form (T) with $s = 1 = r_1$, and *t-singly troublesome* (for $t \in N$) iff it has the form (T) with $s = 1 = r_1$ and $\beta_1 = -t\alpha$. In connection with 2. 1 and with the principal goal of this section, the doubly and t-singly troublesome sets are of special interest; unification in the treatment of these two special types is achieved through the more general notion. Note that a set which is both positively and negatively troublesome must be doubly troublesome, but not conversely, and that a troublesome set may be both doubly and singly troublesome but need not be either.

A partition $\mathscr{P}$ of a weighted set will be called *troublesome* ⟨resp. *doubly troublesome* etc.⟩ iff the weighetd set $(\mathscr{P}^*)'$ is troublesome ⟨resp. doubly troublesome, etc.⟩. When $\mathscr{P}$ is a partition of W and $\gamma \in \Gamma$, we define $\mathscr{P}_\gamma = \{P \in \mathscr{P} : \mu(P) = \gamma\}$, $\mathscr{P}_- = \{P \in \mathscr{P} : \mu(P) < 0\}$, and $\mathscr{P}_+ = \{P \in \mathscr{P} : \mu(P) > 0\}$. For any family $\mathscr{F}$ of sets, $\amalg \mathscr{F}$ will denote the union of all members of $\mathscr{F}$. Thus (for example) $\amalg(\mathscr{P}_-)$ is the union of all members of $\mathscr{P}$ which have negative weight, while $((\mathscr{P}_-)^*)'$ is the set of all negative weights attained by members of $\mathscr{P}$. Since the danger of confusion is slight, we shall usually omit the parentheses in expressions such as these.

When $\mathscr{P}$ is a partition of W and x and y are points of W, $\mathscr{P}(x, y)$ will denote the partition which results from $\mathscr{P}$ upon interchanging x and y. Thus for $x \in X \in \mathscr{P}$ and $y \in Y \in \mathscr{P}$,

$$\mathscr{P}(x, y) = (\mathscr{P} \sim \{X, Y\}) \cup \{(X \sim \{x\}) \cup \{y\}, (Y \sim \{y\}) \cup \{x\}\}.$$

When more complicated interchanges are required, they will be described explicitly.

For the remainder of this section, we make the

STANDING HYPOTHESES: *W is a weighted set and $n_1, \dots, n_m \in N$, with $m \geqq 4$ and $\sum_1^m n_i = \text{card } W$. W admits a nice $(n_1, \dots, n_m)$-partition, but all such partitions are troublesome.*

Partition *will mean an* $(n_1, \ldots, n_m)$*-partition of* W. *A partition* $\mathscr{P}$ *will be called an* α-partition *iff* $\mathscr{P}^*$ *has the form* (T) *and in addition*

(†) $x'-y' \in \{-\alpha, 0, \alpha\}$ *whenever* x *and* y *are points of distinct members of* $\mathscr{P}_\alpha$. *An* α*-partition* $\mathscr{P}$ *will be called a* minimal α-partition *iff there is no* α*-partition* $\mathscr{Q}$ *for which* $\mathscr{Q}_\alpha$ *is a proper subset of* $\mathscr{P}_\alpha$.

* * *

The first lemma is

5. 1. *If* $\mathscr{P}$ *is a partition and* $\mathscr{P}^*$ *has the form* (T) *(but requiring only* $r \geqq 2$*), then* $\max \cup\mathscr{P}' \leqq \min \cup\mathscr{P}'_+$.

Proof. It suffices to consider the case $\alpha > 0$. If $u \in \cup\mathscr{P}_{\beta_j} \subset \cup\mathscr{P}_-$ and $x \in \cup\mathscr{P}_\alpha = \cup\mathscr{P}_+$, then $\mathscr{P}(u, x)$ is a partition for which

$$\mathscr{P}(u, x)^{*\prime} = B \cup \{\beta_i : i \neq j\} \cup \{\beta_j - (u'-x'), \alpha + (u'-x'), \alpha\},$$

where $B \subset \{\beta_j\}$. If $u'-x' > 0$, the partition $\mathscr{P}(u, x)$ is nice but cannot be troublesome, for $\beta_j - (u'-x') < \beta_j \leqq -\alpha < 0 < \alpha < \alpha + (u'-x')$. The contradiction shows that $u'-x' \leqq 0$ and yields the desired conclusion. ∎

5. 2. *If* $\mathscr{P}$ *is a partition and* $\mathscr{P}^*$ *has the form* (T) *with* $|\alpha| < \max\{|\beta_i| : 1 \leqq i \leqq s\}$, *then* $\mathscr{P}$ *is an* α*-partition.*

Proof. We assume without loss of generality that $\alpha > 0$. If x and y lie in different members of $\mathscr{P}_\alpha$, and $x' > y'$, then

$$\mathscr{P}(x, y)^{*\prime} = \{\beta_i : 1 \leqq i \leqq s\} \cup \{\alpha - (x'-y'), \alpha + (x'-y'), \alpha\},$$

and $\mathscr{P}(x, y)$ is not positively troublesome since $0 < \alpha < \alpha + (x'-y')$. If $\mathscr{P}(x, y)$ is negatively troublesome, then $s = 1$ and $0 > \beta_1 \geqq -\alpha$. Since we knew already that $\alpha \leqq -\beta_1$, it follows that

$$|\alpha| = |\beta_1| = \max\{|\beta_i| : 1 \leqq i \leqq s\},$$

contradicting the hypothesis of 5. 2. Thus $\mathscr{P}(x, y)$ must fail to be nice, whence $\alpha - (x'-y') = 0$. ∎

5. 3. *For some* α, W *admits an* α*-partition.*

Proof. Let $\mathscr{P}$ be a nice partition, whence $\mathscr{P}^*$ has the form (T). Suppose $\mathscr{P}$ is not an α-partition, whence there exists points x and y in different members of $\mathscr{P}_\alpha$ such that $x'-y' \notin \{-\alpha, 0, \alpha\}$. The partition $\mathscr{P}(x, y)$ is nice and hence troublesome. We assume without loss of generality that $\alpha > 0$ and $x'-y' > 0$, whence $\mathscr{P}(x, y)^{*\prime}$ contains at least two positive weights and $\mathscr{P}(x, y)$ must be negatively troublesome; this implies $s = 1$ and $\beta_1 \geqq -\alpha$, whence $\beta_1 = -\alpha$. With $0 \neq \alpha - (x'-y') < \alpha$, it follows that $\alpha - (x'-y') = -\alpha$, whence $x'-y' = 2\alpha$ and $\mathscr{P}(x, y)^*$ has the form $(-\alpha)^{r_1+1}(\alpha)^{r-2}(3\alpha)^1$. But then $r_1 \geqq 2$ and $\mathscr{P}(x, y)$ is a $(-\alpha)$-partition by 5. 2. ∎

5. 4. *For each* α*-partition* $\mathscr{P}$ *there is a minimal* α*-partition* $\mathscr{Q}$ *with* $\mathscr{Q}_\alpha \subset \mathscr{P}_\alpha$.

Now we add to the

STANDING HYPOTHESES: $\mathscr{Q}$ *is a minimal* α*-partition of* W, *with* $\alpha > 0$; $z \in Z \in \mathscr{Q}_\alpha$ *with* $z' = \min \cup\mathscr{Q}'_\alpha$; $\gamma = z'$. (The assumption $\alpha > 0$ is only for convenience, since the case $\alpha < 0$ can be treated in the same way.)

From (†) there follows

5. 5. *Either* (i) ${}_{\text{II}}\mathcal{Q}'_\alpha = \{\gamma, \gamma+\alpha\}$

or (ii) $\{\gamma, \gamma+2\alpha\} \subset Z' \subset \{\gamma, \gamma+\alpha, \gamma+2\alpha\}$ *and* ${}_{\text{II}}(\mathcal{Q}_\alpha \sim \{Z\})' = \{\gamma+\alpha\}$.

The discussion is now divided into three cases, as follows:

(A) $\gamma \geqq 0$; (B) $\gamma < 0$; $\gamma+2\alpha \in {}_{\text{II}}\mathcal{Q}'_\alpha$; (C) $\gamma < 0$; $\gamma+2\alpha \notin {}_{\text{II}}\mathcal{Q}'_\alpha$.

By adding the appropriate letter to the number of a lemma, we indicate the addition of one of these three conditions to the standing hypotheses.

* * *

5. 6_A. *There exists* $n \in N$ *such that* $\alpha = n\gamma$ *and each member of* $\mathcal{Q}_\alpha$ *has the form* $(\gamma)^n$. *In particular,* $\gamma > 0$.

Proof. Since $\gamma \geqq 0$ by condition (A), it follows from the definition of γ that all points of ${}_{\text{II}}\mathcal{Q}_\alpha$ have non-negative weight. Consider two points x and y lying in different members of $\mathcal{Q}_\alpha$. The partition $\mathcal{Q}(x, y)$ is nice but is not troublesome if $x' \neq y'$, for then $\mathcal{Q}(x, y)^{*\prime}$ contains β_1 as well as three different non-negative weights, and one of the latter is $< \alpha \leqq -\beta_1$. This shows that $x' = y'$ and consequently ${}_{\text{II}}\mathcal{Q}'_\alpha = \{\gamma\}$. The desired conclusions follow. ▮

5. 7_A. *If* $Q \in \mathcal{Q}_{\beta_j}$, *then* $Q' \subset \{\gamma, \gamma-\alpha, \gamma-2\alpha, \gamma+\beta_j, \gamma+\beta_j-\alpha\}$.

Proof. By 5. 1, $\max {}_{\text{II}}\mathcal{Q}'_- \leqq \gamma$. Let $U_0 = \{u \in Q : u' < \gamma\}$, and define the subsets U_i of U_0 by saying that if $u \in U_0$, then

$u \in U_1$ iff $\mathcal{Q}(u, z)$ is not nice;

$u \in U_2$ iff $\mathcal{Q}(u, z)$ is positively troublesome;

$u \in U_3$ iff $\mathcal{Q}(u, z)$ is negatively troublesome.

Obviously $U_0 = U_1 \cup U_2 \cup U_3$. For $u \in U_0$, we have

$$\mathcal{Q}(u, z)^{*\prime} = B \cup \{\beta_i : i \neq j\} \cup \{\beta_j - u' + \gamma, \alpha - \gamma + u', \alpha\},$$

with $B \subset \{\beta_j\}$. Clearly $u \in U_1$ implies $u' = \gamma+\beta_j$ or $u' = \gamma-\alpha$. If $u \in U_2 \cup U_3$, then $\alpha-\gamma+u' < 0$, for otherwise $\mathcal{Q}(u, z)^{*\prime}$ contains the positive weights α and $\alpha-\gamma+u'$ with

$$\alpha - \gamma + u' < \alpha \leqq \min\{-\beta_i : 1 \leqq i \leqq s\},$$

and $\mathcal{Q}(u, z)$ is not troublesome. If $u \in U_2$, then (since $\mathcal{Q}$ is a *minimal* α-partition) $\beta_j - u' + \gamma > 0$, whence $\beta_j - u' + \gamma = \alpha$ and $u' = \gamma + \beta_j - \alpha$. If $u \in U_3$, then $\alpha - \gamma + u' = \beta_i = -\alpha$ (for $i \neq j$, where in fact this situation entails $s = 2$ and $r_j = 1$). We have now proved that $U'_0 \subset \{\gamma-\alpha, \gamma-2\alpha, \gamma+\beta_j, \gamma+\beta_j-\alpha\}$. ▮

5. 8_A. *W is troublesome when* $n \geqq 2$.

Proof. Use 5. 6, 5. 7, and the fact that

$$\max\{\gamma-\alpha, \gamma-2\alpha, \gamma+\beta \quad \gamma+\beta_j-\alpha\} = \gamma-\alpha = (1-n)\gamma. ▮$$

5. 9_A. *With* $Q \in \mathcal{Q}_{\beta_j}$, *let a, b, and c denote the number of points of Q which are of weight* γ, $\gamma-\alpha$, *and* $\gamma-2\alpha$ *respectively. Let* $d = 0$ *if* $\beta_j \in \{-\alpha, -2\alpha\}$ *and otherwise* $d = \text{card}\{u \in Q : u' = \gamma+\beta_j\}$. *Let* $e = 0$ *if* $\beta_j = -\alpha$ *and otherwise* $e = \text{card}\{u \in Q : u' = \gamma+\beta_j-\alpha\}$. *Then one of the following statements is true:*

(i) $d=1, c=e=0;\ a+1 = (n-1)b;$
(ii) $e=1, c=d=0;\ a+1 = (n-1)b+n;$
(iii) $d=e=0;\ (a-(n-1)b-(2n-1)c) = \beta_j.$

Proof. Clearly

$$\gamma(a+b+c+d+e)-\alpha(b+2c+e)+\beta_j(d+e) = \mu(Q) = \beta_j,$$

and since $\alpha=n\gamma$ it follows that

$$(a+(1-n)b+(1-2n)c+d+(1-n)e)\gamma = (1-d-e)\beta_j.$$

To gain more information about the numbers $a, \ldots, e$, we consider the partition $\mathcal{Q}_{(u,v)}$ which is obtained from $\mathcal{Q}$ by interchanging two points u and v of Q with two points which lie in different members of $\mathcal{Q}_\alpha$. Then

$$\mathcal{Q}^{*\prime}_{(u,v)} = \beta_j \cup A(u,v) \cup \{\alpha\},$$

where

$$\{\beta_i : i \neq j\} \subset B_j \subset \{\beta_i : 1 \leqq i \leqq s\}$$

and

$$A(u,v) = \{\beta_j - u' - v' + 2\gamma, \alpha-\gamma+u', \alpha-\gamma+v'\}.$$

The possibilities of special interest are described in the following table:

	u'	v'	$A(u,v)$
$(d \geqq 2)$	$\gamma+\beta_j$	$\gamma+\beta_j$	$\{-\beta_j, \alpha+\beta_j\}$
$(e \geqq 2)$	$\gamma+\beta_j-\alpha$	$\gamma+\beta_j-\alpha$	$\{2\alpha-\beta_j, \beta_j\}$
$(d \geqq 1 \leqq e)$	$\gamma+\beta_j$	$\gamma+\beta_j-\alpha$	$\{\alpha-\beta_j, \alpha+\beta_j, \beta_j\}$.
$(c \geqq 1 \leqq d)$	$\gamma-2\alpha$	$\gamma+\beta_j$	$\{2\alpha, -\alpha, \alpha+\beta_j\}$
$(c \geqq 1 \leqq e)$	$\gamma-2\alpha$	$\gamma+\beta_j-\alpha$	$\{3\alpha, -\alpha, \beta_j\}$.

Recalling that $d\neq 0$ implies $\beta_j \notin \{-\alpha, -2\alpha\}$, we see that $\mathcal{Q}_{(u,v)}$ is nice in each case and hence must be troublesome. In the first case, $\mathcal{Q}_{(u,v)}$ cannot be positively troublesome since $-\beta_j \neq \alpha$ and cannot be negatively troublesome since $0 > \alpha+\beta_j \neq -\alpha$. In the second case, $Q_{(u,v)}$ cannor be positively troublesome since $0<\alpha<2\alpha-\beta_j$ and cannot be negatively troublesome since (with $e \neq 0$) $\beta_j < -\alpha$. Similar contradictions ensue in the other three cases. It follows that $d+e \leqq 1$, and $d+e=1$ implies $c=0$, whence the remaining possibilities for $a, \ldots, e$ are exactly as described in 5. 9. ∎

5. 10_A. *If $n=1$, each member of $\mathcal{Q}_{\beta_j}$ has the form $(\beta_j)^1$ or the crude form $(\gamma)^a(-\gamma)^c$. Thus W is troublesome.*

Proof. With $n=1$, 5. 9 (i) is impossible, 5. 9 (ii) implies $a=0$, and 5. 9 (iii) becomes $(a-c)\gamma=\beta_j$. The corresponding possibilities for the crude form of $Q\in\mathcal{Q}_{\beta_j}$ are $(0)^b\ (\beta_j)^1$ and $(\gamma)^a(0)^b(-\gamma)^c$; to establish 5. 10 we must prove $b=0$. Suppose $b>0$ and let $u\in Q$ with $u'=0$. Then $\mathcal{Q}(u,z)^{*\prime}=B_j\cup\{\beta_j+\gamma, 0, \gamma\}$, so $\mathcal{Q}(u,z)$ is not troublesome and hence not nice. Since $u\in\{u\}\in\mathcal{Q}(u,z)$, the fact that $u'=0$ does not account for $\mathcal{Q}(u,z)$'s lack of niceness, and it follows that $\beta_j = -\gamma$. Thus $\mu(Q) = -\gamma$, Q contains a point v with $v' = -\gamma$, and $\mathcal{Q}^{*\prime}_{(u,v)}=B_j\cup\{2\gamma, 0, -\gamma, \gamma\}$. Since 0 appears only as the weight of a onepointed member of $\mathcal{Q}_{(u,v)}$, $\mathcal{Q}_{(u,v)}$ is nice but not troublesome. The contradiction implies $b=0$. ∎

5.11$_A$. *Suppose $n=1$ and there exists $Q\in\mathcal{Q}_-$ with* card $Q>1$. *Then Q has the crude form $(\gamma)^a(-\gamma)^c$ for $c\in\{a+1, a+2, a+3\}$ and each member of $\mathcal{Q}_-\sim\{Q\}$ has the form $(-\gamma)^1$.*

Proof. Clearly $c\geqq a+1$, for $(c-a)\gamma=\mu(Q)<0$. Suppose $\mu(Q)=\beta_j$ and let $v_1\in Q$ with $v_1'=-\gamma$. Then $\mathcal{Q}(v_1, z)^{*\prime}=B_j\cup\{(a-c+2)\gamma, -\gamma, \gamma\}$. If $\mathcal{Q}(v_1, z)$ is not nice, $a-c+2=0$. If $\mathcal{Q}(v_1, z)$ is positively troublesome, then (since $\mathcal{Q}$ is a minimal α-partition) $a-c+2=1$. If $\mathcal{Q}(v_1, z)$ is negatively troublesome, then $a-c+2=-1$ when $a-c+2<0$ and $-1\geqq -a+c-2$ when $a-c+2>0$. It follows that $c-a\in\in\{1, 2, 3\}$, with $c=a+3$ only when $\mathcal{Q}(v_1, z)$ is negatively troublesome.

Now suppose $c=a+3$ and let v_2 and v_3 be distinct points of $Q\sim\{v_1\}$ such that $v_2'=v_3'=-\gamma$. Suppose some member P of $\mathcal{Q}_-\sim\{Q\}$ has form other than $(-\gamma)^1$. Since $\mathcal{Q}(v_1, z)$ is negatively troublesome, it is evident that $\mu(P)=-\gamma$ and hence (using 5.10) if P does not have the form $(-\gamma)^1$ there exists $w\in P$ with $w'=\gamma$. But then

$$\mathcal{Q}_{(v_1, v_2)}(v_3; w)^{*\prime}=B\cup\{-3\gamma, 3\gamma, -\gamma, \gamma\}$$

with $B\subset\{\beta_i: 1\leqq i\leqq 1\}$, whence the partition $\mathcal{Q}_{(v_1, v_2)}(v_3, w)$ is nice but not troublesome and the contradiction shows that P has the form $(-\gamma)^1$.

Alternatively, suppose $c-a\in\{1, 2\}$ and note that $c\geqq 2$ since card $Q>1$. Suppose some member P of $\mathcal{Q}_-\sim\{Q\}$ has form other than $(-\gamma)^1$. With $v_1, v_2\in Q$ and $v_1'=v_2'=-\gamma$, we have

$$\mathcal{Q}^{*\prime}_{(v_1, v_2)}=B\cup\{\mu(P), (a-c+4)\gamma, -\gamma, \gamma\},$$

so $\mathcal{Q}_{(v_1, v_2)}$ is nice and the fact that it is troublesome implies $\mu(P)=-\gamma$. Thus there exists $w\in P$ with $w'=\gamma$ and we have

$$\mathcal{Q}(v_1, z)(v_2, w)^{*\prime}=B\cup\{-3\gamma, (a-c+4)\gamma, -\gamma, \gamma\},$$

a contradiction which yields the desired conclusion. ∎

5.12$_A$. *If all the n_i's have the same value $n\geqq 2$, each member of $\mathcal{Q}_{\beta_j}$ has one of the following crude forms:*

$(\beta_j\notin\{-\alpha, -2\alpha\})$	(i) $(\gamma)^{n-2}(\gamma-\alpha)^1(\gamma+\beta_j)^1$;	
$(\beta_j\neq-\alpha)$	(ii) $(\gamma)^{n-1}(\gamma+\beta_j-\alpha)^1$;	
$(\beta_j=-\alpha)$	(iii) $(\gamma)^{n-2}(\gamma-\alpha)^2$;	(iv) $(\gamma)^{n-1}(\gamma-2\alpha)^1$;
$(\beta_j=-2\alpha)$	(v) $(\gamma)^{n-3}(\gamma-\alpha)^3$;	(vi) $(\gamma)^{n-2}(\gamma-\alpha)^1(\gamma-2\alpha)^1$;
$(\beta_j=-3\alpha)$	(vii) $(\gamma)^{n-4}(\gamma-\alpha)^4$;	(viii) $(\gamma)^{n-3}(\gamma-\alpha)^2(\gamma-2\alpha)^1$;
	(ix) $(\gamma)^{n-2}(\gamma-2\alpha)^2$.	

If some member Q of $\mathcal{Q}_-$ has the crude form (vii), (viii), *or* (ix), *then $\mathcal{Q}_-\sim\{Q\}$ is nonempty and all its members have the crude form* (iii) *or* (iv).

Proof. Here 5.9 (i) becomes

$$n=a+b+d=(n-1)b-1+b+1=nb,$$

whence $b=1$, $a=n-2$, and Q has the crude form (i) above. And 5. 9 (ii) becomes

$$n = a+b+e = (n-1)b+n-1+b+1 = nb+n,$$

whence $b=0$, $a=n-1$, and Q has the form (ii) above.

For 5. 9 (iii) we have $n = a+b+c$ (and of course $\alpha=n\gamma$), so $\beta_j = (1-b-2c)\alpha$. Now with $g\leqq b$, $h\leqq c$, and $g+2h>1$, let $\mathscr{R}_{(g,h)}$ denote a partition which is obtained from $\mathscr{Q}$ by interchanging g points of weight $\gamma-\alpha$ in Q and h points of weight $\gamma-2\alpha$ in Q with $g+h$ points of weight γ in a single member of $\mathscr{Q}_\alpha$. (When all n_i's have the same value, such an interchange is possible.) Then

$$\mathscr{R}^{*\prime}_{(g,h)} = B_j\cup\{\beta_j+(g+2h)\alpha, (1-g-2h)\alpha, \alpha\}.$$

Note that $1-g-2h<0$. Thus if $\mathscr{R}_{(g,h)}$ is not nice, $\beta_j = -(g+2h)\alpha$ and $g+2h= =b+2c-1$. If $\beta_j+(g+2h)\alpha<0$, then (since $\mathscr{Q}$ is a minimal α-partition) $\mathscr{R}_{(g,h)}$ is negatively troublesome and

$$\beta_j+(g+2h)\alpha = (1-g-2h)\alpha = -\alpha,$$

whence $g+2h = 2$, $\beta_j = -3\alpha$, and $b+2c = 4$. If $\beta_j+(g+2h)\alpha>0$, then $g+2h = =b+2c$ when $\mathscr{R}_{(g,h)}$ is positively troublesome, while negative troublesomeness of $\mathscr{R}_{(g,h)}$ implies

$$-\beta_j-(g+2h)\alpha\leqq(1-g-2h)\alpha = -\alpha,$$

whence $g+2h = 2$ and $-\beta_j\leqq\alpha$. But then $\beta_j = -\alpha$ and $b+2c = 2$.

The preceding paragraph shows that if $Q(\in\mathscr{Q}_-)$ has the crude form $(\gamma)^{n-b-c}$ $\cdot(\gamma-\alpha)^b(\gamma-2\alpha)^c$, the pair (b, c) must be such that $b+2c>1$, and such that whenever $g\leqq b$, $h\leqq c$, and $1<g+2h<b+2c$, then $g+2h = b+2c-1$, or $g+2h = 2$ and $b+2c = 4$. It is obvious that $b\leqq 4$ and $c\leqq 2$, and a closer examination shows that

$$(b, c)\in\{(0, 1), (0, 2), (1, 1), (2, 0), (2, 1), (3, 0), (4, 0)\},$$

whence Q has one of the crude forms (iii)—(ix). Note also that if $(b, c)\in\{(0, 2), (2, 1), (4, 0)\}$, then $\mu(Q) = -3\alpha$ and there exist g and h as described for which $\mu(Q)+(g+2h)\alpha<0$. But then $\mathscr{R}_{(g,h)}$ is negatively troublesome, whence $\mathscr{Q}_-\sim\{Q\}$ is nonempty and all members of $\mathscr{Q}_-\sim\{Q\}$ have weight $-\alpha$. ∎

* * *

We consider now the case in which condition (B) is satisfied. In this case, $\gamma<0$ and a single member Z of $\mathscr{Q}_\alpha$ contains points of weights γ and $\gamma+2\alpha$ (perhaps also $\gamma+\alpha$), while the other members of $\mathscr{Q}_\alpha$ consist exclusively of points of weight $\gamma+\alpha$.

5. 13_B. *$\alpha = -2\gamma$. Z has the form $(\gamma)^1(-3\gamma)^1$, while all other members of $\mathscr{Q}_\alpha$ have the form $(-\gamma)^2$. All members of $\mathscr{Q}_-$ have the form (γ) with at most one exception, and the exceptional member Q (if there is one) has the form $(\gamma)^1(3\gamma)^1$ or the form $(\gamma)^4$. If there is such a Q, then $\mathscr{Q}_-\sim\{Q\}$ is nonempty.*

Proof. Note the existence of $n\in N$ such that each member of $\mathscr{Q}_\alpha\sim\{Z\}$ has the form $(\gamma+\alpha)^n$; with $\alpha=n(\gamma+\alpha)$, we have $(1-n)\alpha=n\gamma$ and thus $n\geqq 2$.

Let $x, z\in Z$ and $y\in\cup\mathscr{Q}_\alpha\sim Z$ with $x' = \gamma+2\alpha$, $y'=\gamma+\alpha$, and $z'=\gamma$. For each $u\in\cup\mathscr{Q}_{\beta_j}\subset\cup\mathscr{Q}$, let $\mathscr{Q}_{(u)}$ denote the partition which results from $\mathscr{Q}$ under cyclic permutation of u, x, and y (replacing x by u, y by x, and u by y). Then

$$\mathscr{Q}^{*\prime}_{(u)} = B\cup\{\beta_i: i\neq j\}\cup\{\beta_j-u'+\gamma+\alpha, -\gamma-\alpha+u', 2\alpha, \alpha\}$$

with $B \subset \{\beta_j\}$, so $\mathcal{Q}_{(u)}$ is not positively troublesome. Thus ${}_{\mathrm{u}}\mathcal{Q}_-' = \bigcup_{1 \leqq j \leqq s} (U_j^1 \cup U_j^2)$, where U_j^1 is the set of all $u \in {}_{\mathrm{u}}\mathcal{Q}_{\beta_j}$ for which $\mathcal{Q}_{(u)}$ is negatively troublesome and U_j^2 is the set of all $u \in {}_{\mathrm{u}}\mathcal{Q}_{\beta_j}$ for which $\mathcal{Q}_{(u)}$ is not nice.

Now we claim that for $1 \leqq j \leqq s$,

(1) $$(U_j^2)' = \{\gamma + \beta_j + \alpha\}$$

and

(2) $$(U_j^1)' = \{\gamma\}.$$

The statement (1) is immediate from the definition of U_j^2 and the form of $\mathcal{Q}^*_{(u)}$, since (by 5. 1) $-\gamma - \alpha + u' \leqq -\alpha$ for all $u \in {}_{\mathrm{u}}\mathcal{Q}_-$. Now suppose $u \in U_j^1$. Since $\alpha \in \mathcal{Q}^{*\prime}_{(u)}$, it follows that $-\gamma - \alpha + u' = -\alpha$, whence $u' = \gamma$ and $\beta_j + \alpha \in \mathcal{Q}^{*\prime}_{(u)}$. This establishes (2) and (continuing with the assumption that $u \in U_j^1$) since $\beta_j + \alpha \leqq 0$ we see that $\beta_j + \alpha = -\alpha$, whence it follows that $\beta_j = -2\alpha$, $r_j = 1$, $\beta_i = -\alpha$ for $i \neq j$, and $s = 2$.

If $u \in Q \cap U_j^1$ (where $Q \in \mathcal{Q}_{\beta_j}$), the above reasoning shows that $u' = \gamma$ and there exists $k \in N$ such that each member of $\mathcal{Q}_- \sim \{Q\}$ has the form $(\gamma)^k$, with $k\gamma = -\alpha$. Recalling that $n\gamma = (1-n)\alpha$, we see that $k = n/(n-1)$, whence $n = 2 = k$ and $\alpha = -2\gamma$. Thus each member of $\mathcal{Q}_\alpha \sim \{Z\}$ has the form $(-\gamma)^2$ while each member of $\mathcal{Q}_- \sim \{Q\}$ has the form $(\gamma)^2$. We want to show that Z has the form $(\gamma)^1(-3\gamma)^2$ while Q has the form $(\gamma)^1(3\gamma)^1$ or the form $(\gamma)^4$.

From (1) and (2) we know that Q consists of a points of weight γ and b of weight 3γ, with $a\gamma + 3b\gamma = \mu(Q) = -2\alpha = 4\gamma$. Hence $a = 1$ and $b = 1$ or $a = 4$ and $b = 0$; in either case, Q has the desired form. A simple interchange shows that if some such Q occurs with $\mathcal{Q}_- = \{Q\}$, then W admits a partition $\mathcal{S}$ for which $\mathcal{S}^*$ has the form $(2\gamma)^2(-2\gamma)^c(-4\gamma)$, an impossibility since $\mathcal{S}$ is nice but not troublesome.

We know that Z consists of c points of weight γ, d of weight $-\gamma$, and e of weight -3γ, with $c \geqq 1 \leqq e$. Now suppose $e \geqq 2$ $\langle$resp. $d \geqq 1\rangle$ and let $\mathcal{R}$ denote the partition which is obtained from $\mathcal{Q}$ by interchanging two points of weight -3γ $\langle$resp. one of weight -3γ and one of weight $-\gamma\rangle$ from Z with the two points of weight γ from a single member of $\mathcal{Q}_- \sim \{Q\}$. Then $\mathcal{R}^{*\prime} = \{4\gamma, 2\gamma, -2\gamma\} \cup A$, where $A = \{-6\gamma, 6\gamma\}$ $\langle$resp. $A = \{-4\gamma, 4\gamma\}\rangle$. Thus $\mathcal{R}$ is nice but not troublesome, and the contradiction implies that $d = 0$ and $e = 1$, whence $c = 1$ and Z has the desired form.

For the proof of 5. 13, it remains to consider the case in which $\bigcup_{1 \leqq j \leqq s} U_j^2 = {}_{\mathrm{u}}\mathcal{Q}_-'$, whence (for all j) each member of $\mathcal{Q}_{\beta_j}$ has the form $(\gamma + \beta_j + \alpha)^{k_j}$ for some $k_j \in N$. With $(1-k_j)\beta_j = k_j(\gamma + \alpha)$ and $\alpha = n(\gamma + \alpha)$, we have $n(1-k_j)\beta_j = k_j\alpha$. Now for $u \in {}_{\mathrm{u}}\mathcal{Q}_{\beta_j}$, note that

$$\mathcal{Q}(u, z)^{*\prime} = B \cup \{\beta_i : i \neq j\} \cup \{-\alpha, \beta_j + 2\alpha, \alpha\}$$

with $B \subset \{\beta_j\}$. If $\mathcal{Q}(u, z)$ is not nice, then $\beta_j = -2\alpha$ and $2n = k_j/(k_j - 1)$. This implies $k_j = 2$ and $n = 1$, whence $\gamma = 0$ in contradiction of our basic assumption that $\gamma < 0$. If $\beta_j + 2\alpha < 0$, then (since $\mathcal{Q}$ is a minimal α-partition) $Q(u, z)$ must be negatively troublesome, whence $\beta_j + 2\alpha = -\alpha$ and $3n = k_j/(k_j - 1)$, an impossibility. Suppose, finally, that $\beta_j + 2\alpha > 0$. If $\mathcal{Q}(u, z)$ is positively troublesome, then $\beta_j + 2\alpha = \alpha$, while negative troublesomeness of $\mathcal{Q}(u, z)$ implies $-\alpha \geqq -\beta_j - 2\alpha$. But we know already that $\beta_j \leqq -\alpha$, so both possibilities imply $\beta_j = -\alpha$. From this it follows that $n = k_j/(k_j - 1)$, whence $k_j = 2$, $n = 2$, and ${}_{\mathrm{u}}\mathcal{Q}_-' = \{\gamma\}$. Thus each member of $\mathcal{Q}_-$ has the form $(\gamma)^2$ while each member of $\mathcal{Q}_\alpha \sim \{Z\}$ has the form

$(-\gamma)^2$. The argument of the preceding paragraph shows that Z has the form $(\gamma)^1(-3\gamma)^1$, and this completes the proof of 5. 13. ∎

* * *

We turn finally to the case in which condition C is satisfied.

5. 14_C. *Each set $X \in Q_\alpha$ has the crude form $(\gamma)^{a(X)}(\gamma+\alpha)^{b(X)}$ with $(a(X)+b(X))\gamma = (1-b(X))\alpha$ and $b(X) \geqq 2$. Of course $a(Z) \geqq 1$.*

Proof. With $a\gamma + b(\gamma+\alpha) = \mu(X) = \alpha$, the equality $(a+b)\gamma = (1-b)\gamma$ is immediate; $b \geqq 2$ because $\gamma < 0 < \alpha$. Further, $z \in Z$ with $z' = \gamma$ (as part of the standing hypotheses). ∎

5. 15_C. *Suppose $a(Z_i) \geqq 1$ for at least two different members Z_1 and Z_2 of $\mathcal{Q}_\alpha$, or $a(Z_3) \geqq 2$ for some $Z_3 \in \mathcal{Q}_\alpha$. Then $\mathcal{Q}$ is doubly troublesome,* card $\mathcal{Q}_- \geqq 2$, $\text{ш}\mathcal{Q}_- = \{\gamma\}$, *and W admits a nice $(n_1, \ldots, n_m)$-partition which is neither doubly nor singly troublesome.*

Proof. In the first instance, let $z_1 \in Z_1$ and $z_2 \in Z_2$, and in the second $z_1, z_2 \in Z_3$, with $z_1' = \gamma = z_2'$ in each case. Let Y be a member of $\mathcal{Q}_\alpha$ different from the Z_i's, and $y_1, y_2 \in Y$ with $y_1' = \gamma + \alpha \doteq y_2'$.

Let $\mathcal{R} = \mathcal{Q}(y_1, z_1)(y_2, z_2)$, whence

$$\mathcal{R}^{*\prime} = \{\beta_i : 1 \leqq i \leqq s\} \cup \{-\alpha\} \cup A$$

with $\{2\alpha\} \subset A \subset \{\alpha, 2\alpha\}$ or $A = \{\alpha, 3\alpha\}$. In the first case, 2α appears as the weight of two different members of $\mathcal{R}$, so in neither case is $\mathcal{R}$ doubly or singly troublesome. On the other hand, $\mathcal{R}$ is nice and hence troublesome, which can happen only if $\beta_i = -\alpha$ for all i (whence $\mathcal{Q}$ is doubly troublesome) and card $\mathcal{Q}_- \geqq 2$.

Now suppose $u \in \text{ш}\mathcal{Q}_-$ with $u' \neq \gamma$. Then $u' < \gamma$ by 5. 1, whence $\alpha - \gamma + u' < \alpha$. Since

$$\mathcal{Q}(u, z_1) = \{-\alpha, -\alpha - u' + \gamma, \alpha - \gamma + u', \alpha\},$$

it follows easily that $\alpha - \gamma + u' = 0$ or $\alpha - \gamma + u' = -\alpha$. Now if a member of $\mathcal{Q}_{-\alpha}$ contains a point of weight $< \gamma - \alpha$, or two points of weight $\leqq \gamma - \alpha$, then by interchanging these with points of weight $\gamma + \alpha$ in a single member of $\mathcal{Q}_\alpha$ we obtain a nice partition whose members have weights $< -\alpha$, $= -\alpha$, $= \alpha$, and $> \alpha$. Since this is impossible, we conclude that each member of $\mathcal{Q}_{-\alpha}$ has the crude form $(\gamma - \alpha)^c(\gamma)^d$ with $c \in \{0, 1\}$. But then $(c+d)\gamma - c\alpha = -\alpha$, so $c = 1$ implies $\gamma = 0$. This contradiction completes the proof. ∎

5. 16_C. *W is troublesome.*

Proof. By 5. 14, W contains at least six points of weight $\gamma + \alpha$. By 5. 1, $w' \leqq \gamma$ whenever $w \in W$ with $w' < \gamma + \alpha$. Thus W is surely troublesome if $\gamma + \alpha \leqq -\gamma$. Suppose, on the other hand, that $\alpha > 2(-\gamma)$. Then for each $X \in \mathcal{Q}_\alpha$, $(a(X) + b(X)) > 2(b(X) - 1)$, whence $a(X) \geqq 1$. It then follows from 5. 15 that W has the form $(\gamma)^e(\gamma+\alpha)^f$ with $\gamma < 0 < \gamma + \alpha$ and $e \geqq 3 \leqq f$, so of course W is troublesome. ∎

5. 17_C. *If all n_i' s have the same value n, then $n \geqq 3$; each member of $\mathcal{Q}_-$ has the form $(\gamma)^n$ while each member of $\mathcal{Q}_\alpha$* has the form $(\gamma)^{n-2}(\gamma+\alpha)^2$.

Proof. For each $X \in \mathcal{Q}_\alpha$ we have $a(X) + b(X) = n$, whence (by 5. 14) $(1 - b(X))\,\alpha = n\gamma$ and $b(X)$ has the same value for all $X \in \mathcal{Q}_\alpha$. Thus the same is also true of $a(X)$, and 5. 15 applies to show that $-\alpha = n\gamma$. But then $1-b(X) = -1$ and the desired conclusions follow. ∎

6. Troublesome sets: Theorems. The results of this section are based on the lemmas of Section 5.

6. 1. Theorem. *Suppose W is a weighted set and $n_1, \ldots, n_m \in N$ with $m \geqq 4$ and $\sum_1^m n_i = \text{card } W$. If all nice $(n_1, \ldots, n_m)$-partitions of W are troublesome, then either W itself is troublesome or all n_i's are equal to 2 and W has the form $(-3\gamma)^1(-\gamma)^{4+2a}(\gamma)^{4+2b}(3\gamma)^1$ for some $\gamma \in \Gamma \sim \{0\}$ and $a, b \in N \cup \{0\}$.*

Proof. If W admits no nice $(n_1, \ldots, n_m)$-partition, 2. 1 implies that W is troublesome. Suppose, then, that W admits a nice $(n_1, \ldots, n_m)$-partition, and let $\mathcal{Q}$ be a minimal α-partition of W as described in Section 5. Referring to 5. 8, 5. 10, 5. 13, and 5. 16, we see that W can fail to be troublesome only if $\mathcal{Q}$ satisfies the condition (B). By 5. 13, the only non-troublesome possibility for this case is that described above. ∎

It would be interesting to have an intrinsic *characterization* of those weighted sets W and m-tuples $(n_1, \ldots, n_m$ such that all nice $(n_1, \ldots, n_m)$-partitions of W are troublesome. (This is not provided by 6. 1, for a troublesome set may admit nice partitions which are not troublesome). Relevant information is supplied by 5. 6_A, 5. 7_A, 5. 9_A, 5. 11_A, 5. 12_A, 5. 13_B, 5. 15_C and 5. 17_C. The picture is complete for condition (B) and could probably be completed without difficulty for (C), but the case of (A) seems more complicated. We have a complete solution only when all the n_i's have the same value. For $m \geqq 4$ and $n \geqq 2$, let $\mathfrak{T}(m, n)$ denote the class of all weighted sets W of cardinality mn such that all nice n-partitions of W are troublesome. Let $\mathfrak{T}_N(m, n)$ denote the class of all $W \in \mathfrak{T}(m, n)$ such that W admits no nice n-partition, and for $D \in \{A, B, C\}$ let $\mathfrak{T}_D(m, n)$ denote the class of all $W \in \mathfrak{T}(m, n)$ such that for some $\alpha \in \Gamma \sim \{0\}$, W admits a minimal α-partition which satisfies condition (D). Then

$$\mathfrak{T}(m, n) = \mathfrak{T}_N(m, n) \cup \mathfrak{T}_A(m, n) \cup \mathfrak{T}_B(m, n) \cup \mathfrak{T}_C(m, n).$$

The class $\mathfrak{T}_N(m, n)$ is completely described in 2. 1, and the other classes are described in the following result.

6. 2. Theorem. *Suppose $m \geqq 4$, $n \geqq 2$, and W is a weighted set of cardinality mn. Then*

(a) $W \in \mathfrak{T}_A(m, n)$ iff W has the crude form

$$(\gamma)^{kn+a(n-2)+b(n-1)+c(n-2)+d(n-1)+e(n-3)+f(n-2)+g(n-4)+h(n-3)+i(n-2)} \ldots$$

$$\ldots(\gamma-\alpha)^{a+2c+3e+f+4g+2h}(\gamma-2\alpha)^{d+f+h+2i}(\gamma+\beta_0)^a(\gamma+\beta_1-\alpha)^1 \ldots (\gamma+\beta_b-\alpha)^1$$

for some $\gamma \in \Gamma \sim \{0\}$, $\alpha = n_\gamma$, β_j of opposite sign from α but of greater absolute value $(0 \leqq j \leqq b)$, $\beta_0 \neq -2\alpha$, $3 \leqq k < m$, and $a, b, \ldots h, i \in N \cup \{0\}$ with $e > 0 \Rightarrow n \geqq 3$, $g > 0 \Rightarrow n \geqq 4$, $h > 0 \Rightarrow n \geqq 3$, and one of the following four conditions satisfied:

(a_1) $0 = e = f = h = i$, $a = 1$;
(a_2) $0 = a = g = h = i$, $b \geqq 1$, $e + f \leqq 1$;
(a_3) $0 = a = b = g = h = i$, $c + d \geqq 1$, $e + f \leqq 2$;
(a_4) $0 = a = b = e = f = 0$, $c + d \geqq 1$, $g + h + i = 1$;

(b) $W \in \mathfrak{T}_B(m, n)$ *iff* $n=2$ *and* W *has the form* $(-3\gamma)^1(-\gamma)^{4+2a}(\gamma)^{3+2b}$ *or the form* $(-3\gamma)^1(-\gamma)^{4+2a}(\gamma)^{4+2b}(3\gamma)^1$ *for some* $\gamma \in \Gamma \sim \{0\}$ *and* $a, b \in N \cup \{0\}$.

(c) $W \in \mathfrak{T}_C(m, n)$ *iff* $n \geqq 3$ *and* W *has the form* $(\gamma-\alpha)^{6+2a}(\gamma)^{5n-6+a(n-2)+bn}$ *for some* $\gamma \in \Gamma \sim \{0\}$, $\alpha = n\gamma$, *and* $a, b \in N \cup \{0\}$.

Proof. It is tedious but not difficult to verify that if W has one of the stated forms, then W is a member of the appropriate class $\mathfrak{T}_D(m, n)$. This task is left to the reader. That the members of $\mathfrak{T}_B(m, n)$ and $\mathfrak{T}_C(m, n)$ must have the indicated forms is an almost immediate consequence of 5. 13 and 5. 17 respectively, with a slight change of notation in the latter case and use of 5. 15 to show that card $\mathcal{Q}_- \geqq 2$ when $\alpha > 0$. This takes care of (*b*) and (*c*). For (*a*) we use 5. 12, but some additional argument is necessary.

Let Q be as in 5. 12, whence $\mathcal{Q}_\alpha$ consists of k sets of the form $(\gamma)^n$, a sets of the crude form (i) (for various $\beta_j \notin \{-\alpha, -2\alpha\}$), b sets of the form (ii) (for various $\beta_j \neq -\alpha$), c sets of the crude form (iii), ..., i sets of the crude form (ix), where $3 \leqq k < m$ and the designations (i)...(ix) refer to the statement of 5. 12. From 5. 12 it follows that if $g+h+i \geqq 1$, then $g+h+i = 1$, $a=b=e=f=0$, and $c+d \geqq 1$. And $e+f \leqq 2$ in any case, for if $e+f \geqq 3$ a simple interchange leads from $\mathcal{Q}$ to another minimal α-partition of W for which $g+h+i \geqq 1$ and $e+f \geqq 1$, in contradiction of 5. 12. Note also that if $a \geqq 1$, then $a=1$ and $e=f=g=h=i=0$, for otherwise a simple interchange leads from $\mathcal{Q}$ to another minimal α-partition one of whose members has a crude form other than those indicated in 5. 12. We now see further that if $b \geqq 1$, then $e+f \leqq 1$, for otherwise an interchange leads from $\mathcal{Q}$ to another minimal α-partition for which $a \geqq 1$ and $e+f \geqq 1$. A review of the assembled facts shows that one of the four conditions $(a_1)-(a_4)$ must be satisfied. ∎

We next discuss weighted sets all of whose nice n-partitions are doubly or singly troublesome. While the discussion could be based on 6. 2, it will be simpler to apply the relevant lemmas.

6. 3. Theorem. *Suppose* $m \geqq 4$, $n \geqq 2$, *and* W *is a weighted set of cardinality* mn *which admits a nice n-partition. Then all nice n-partitions of W are doubly troublesome iff W has the crude form* $(\gamma)^{an+b(n-2)+c(n-1)}(\gamma-\alpha)^{2b}(\gamma-2\alpha)^c$ *for some* $\gamma \in \Gamma \sim \{0\}$, $\alpha = n\gamma$, *and* $a, b, c \in N \cup \{0\}$ *such that* $a+b+c = m$ *and one of the following additional restrictions is satisfied:*

$n=2$; $3 \leqq a < m$ *and* $c \leqq 1$ *or* $m \in \{4, 5\}$, $a = m-3$,; $b=2$ *and* $c=1$

$n=3$; $3 \leqq a < m$; $b=0$ *and* $c=1$ *or* $b \in \{1, 2\}$ *and* $c=0$;

$n \geqq 4$; $a = m-1$; $b=0$ *and* $c=1$ *or* $b=1$ *and* $c=0$.

Proof. The stated crude form for W is equivalent to W's being the union of a sets of the form $(\gamma)^n$, b of the crude form $(\gamma)^{n-2}(\gamma-\alpha)^2$, and c of the form $(\gamma)^{n-1}(\gamma-2\alpha)^1$. If a, b, and c are subject to the restrictions given above, it can be verified that all nice n-partitions of W are doubly troublesome.

Now suppose conversely that all nice n-partitions of W are doubly troublesome, and let $\mathcal{Q}$ be a minimal α-partition of W. From 5. 15 it follows that $\mathcal{Q}$ satisfies condition (*A*) or condition (*B*) of Section 5, whence 5.12_A and 5.13_B will apply. Since $\mathcal{Q}$ is doubly troublesome, the form $(\gamma)^1(3\gamma)^1$ (for a member of $\mathcal{Q}$) of 5. 13

is eliminated, as are all the forms mentioned in 5. 12 except for (iii) and (iv). Thus if 5. 12 holds, W clearly has the desired form with $3 \leqq a < m$ (but ignoring, for the moment, the restrictions on b and c). And with the aid of a simple substitution (the $-\gamma$ of 5. 13 being the γ of 6. 3), W as described under 5. 13 is seen to have one of the two forms listed above for $n=2$. It remains only to justify the restrictions on b and c.

If $c=2$, we may interchange two points of weight $\gamma-2\alpha$ in ${}_{\rm II}\mathcal{Q}_{-\alpha}$ with two points of weight γ in a single member of $\mathcal{Q}_\alpha$ to obtain from $\mathcal{Q}$ a nice n-partition $\mathcal{R}$ of W for which $\mathcal{R}^{*\prime} \supset \{-3\alpha, \alpha\}$, contradicting the assumption that all nice n-partitions of W are doubly troublesome. If $n \geqq 3$ and $c \geqq 1 \leqq b$, a similar contradiction arises from an interchange involving one point of weight $\gamma-2\alpha$, two of weight $\gamma-\alpha$, and three of weight γ. If $n \geqq 3$ and $b \geqq 3$, then interchanging the two points of weight $\gamma-\alpha$ in one member of $\mathcal{Q}_{-\alpha}$ with points of weight γ in two other members of $\mathcal{Q}_{-\alpha}$ leads to a nice n-partition $\mathcal{S}$ with $\mathcal{S}^{*\prime} \supset \{-2\alpha, \alpha\}$, again an impossibility. Finally, if $n \geqq 3$ and $b \geqq 2$, a contradictory partition is obtained in a similar way by choosing the two points of weight γ from a single member of $\mathcal{Q}_{-\alpha}$. The stated restrictions have now been justified. ▌

Note that if $n \geqq 4$ and all nice n-partitions of W are doubly troublesome, then all are singly troublesome.

6. 4. Theorem. *Suppose $m \geqq 4$, $n \geqq 2$, and W is a weighted set of cardinality mn which admits a nice n-partition. Then all nice n-partitions of W are singly troublesome iff W has one of the following forms for some $\gamma \in \Gamma \sim \{0\}$, $\alpha = n\gamma$, δ and ε of opposite sign from α but $|\delta| > |\alpha|$ and $|\varepsilon| \geqq 2|\alpha|$:*

$$(\gamma)^{mn-1}(\gamma-\varepsilon)^1; \; (\gamma)^{mn-2}(\gamma-\alpha)^1(\gamma-\delta)^1; \; (\gamma)^{mn-2}(\gamma-\alpha)^2;$$

$$(\gamma)^{mn-3}(\gamma-\alpha)^3 \qquad (\textit{only for } n \geqq 3);$$

$$(3\gamma)^1(\gamma)^{mn-4}(-\gamma)^3 \qquad (\textit{only for } n=2).$$

Proof. Again, case (C) is eliminated by 5. 15. Under 5. $12_{\rm A}$, the forms (vii), (viii) and (ix) are eliminated by the fact that card $\mathcal{Q}_- = 1$ (since $\mathcal{Q}$ is singly troublesome). Combining the representations of (i) and (vi) and of (ii) and (iv), we see that W has one of the first four forms listed above.

Under 5. $13_{\rm B}$, W is seen to have the last form listed. Finally, it can be verified that if W has one of the five stated forms, then all nice n-partitions of W are singly troublesome. ▌

6. 5. Corollary. *Suppose $m \geqq 4$, $n \geqq 2$, and W is a weighted set of cardinality mn which admits a nice n-partition. Then all nice n-partitions of W are t-singly troublesome (for $t \in N$) iff W has one of the following forms for some $\gamma \in \Gamma \sim \{0\}$:*

$$(\gamma)^{mn-1}(\gamma-(1+t)n\gamma)^1; \quad (\gamma)^{mn-2}(\gamma-n\gamma)^1(\gamma-tn\gamma)^1;$$

$$(\gamma)^{mn-3}(\gamma-n\gamma)^3 \qquad (\textit{only for } n \geqq 3, \; t=2);$$

$$(3\gamma)^1(\gamma)^{mn-4}(-\gamma)^3 \qquad (\textit{only for } n=2, \; t=1).$$

With the aid of 2. 1, 6. 3, and 6. 5, it is possible to give a detailed description of sets of the form $\mathrm{aff}_l(\mathrm{aff}_m(\mathrm{aff}_n X))$. By way of illustration, we prove

6. 6. Theorem. *Suppose X is an affinely independent subset of E and* card $X = lmn$, *where $l, m, n \in N \sim \{1\}$. Then the cardinality of the set* $\operatorname{aff}_{lmn} X \sim \operatorname{aff}_l(\operatorname{aff}_m(\operatorname{aff}_n X))$ *is equal to $c(l, m, n)$ as given by the following formulae:*

when $n \geqq 3$ *and* $m \geqq 4$, $c(l, m, n) = lmn(lmn+1)$;

when $n \geqq 3$ *and* $m = 3$, $c(l, 3, n) = \frac{1}{2} ln(9(ln)^2 + 9ln + 8)$;

when $n \geqq 3$ *and* $m = 2$, $c(l, 2n) = ln(2ln+3)$;

when $n = 2$ *and* $m \geqq 3$, $c(l, m, 2) = 2^{2lm-2} + (2lm)^2 - f$, *where* $f = 0$ *when* lm *is even and* $f = \frac{1}{2}\binom{2lm}{lm}$ *when* lm *is odd;*

when $n = 2$ *and* $m = 2$, $c(l, 2, 2) = 2^{4l-2} + \sum_{i=1}^{l}\left(\binom{4l}{4i-2} + 4l\binom{4l-1}{4i-1}\right) - g$, *where* $g = -280$ *when* $l = 2$, $g = 0$ *when* l *is even but* > 2, *and* $g = \binom{4l}{2l} + 4l\binom{4l-1}{2l-2}$ *when* l *is odd.*

Proof. Let A denote the set of all functions ξ on X to Φ such that $\sum_{x \in X} \xi x = 1$, and for each $\xi \in A$ let X_ξ denote the weighted set $\{(x, \xi x) : x \in X\}$. Then $c(l, m, n)$ is equal to card B + card C, where B is the set of all $\xi \in A$ such that X_ξ admits no nice n-partition and C is the set of all $\xi \in A \sim B$ such that for each nice n-partition $\mathcal{P}$ of X_ξ the weighted set $\mathcal{P}^*$ admits no nice m-partition. From 2. 1 and 3. 5 it follows that

$$\text{card } B = \begin{Bmatrix} lmn \\ 2^{2lm-2} \\ 2^{2lm-2} - \frac{1}{2}\binom{2lm}{lm} \end{Bmatrix} \quad \text{when} \quad \begin{cases} n \geqq 3 \\ n = 2 \text{ and } lm \text{ is even} \\ n = 2 \text{ and } lm \text{ is odd.} \end{cases}$$

When $m \geqq 3$, the set C is determined by 6. 5 (with $t = m-1$) in conjunction with 2. 1, whence it is seen that

$$\text{card } C = \begin{Bmatrix} lmn + lmn(lmn-1 \\ lmn + lmn(lmn-1) + \binom{lmn}{3} \end{Bmatrix} \quad \text{when} \quad \begin{cases} m \geqq 4 \text{ or } n = 2 \\ m = 3 \text{ and } n \geqq 3. \end{cases}$$

When $m = 2$, the set C is determined by 6. 3 in conjunction with 2. 1. For $n \geqq 3$, we see that $\xi \in C$ iff X_ξ has the form $(\gamma)^{2ln-2}((1-n)\gamma)^2$, or the form $(\gamma)^{2ln-1}((1-2n)\gamma)^1$ (where $\gamma = 1/(2ln-2)$), and it follows that

$$\text{card } C = \binom{2ln}{2} + \binom{2ln}{1}.$$

When $m = 2 = n$, the above considerations show that $\xi \in C$ iff X_ξ has the form $(\gamma)^{4l-2b}(-\gamma)^{2b}$ with b odd, $b \neq l$, and $1 \leqq b \leqq 2l-1$, or the crude form $(\gamma)^{4-2b-1}$

$(-\gamma)^{2b}(-3\gamma)^1$ with b even, $b \neq l-1$, and $0 \leqq b \leqq 2l-2$ or (only when also $l=2$) the form $(\gamma)^3(-\gamma)^4(-3\gamma)^1$, where in each case the value of γ is determined by the fact that $\mu(X_\xi)=1$. Thus for $l \geqq 3$,

$$\text{card } C = \Sigma' \binom{4l}{2b} + 4l\Sigma'' \binom{4l-1}{2b},$$

where $'$ and $''$ indicate the appropriate range and restrictions for b, while for $l=2$, there must be added a term equal to $8\binom{7}{3}=280$. It can be verified that

$$\text{card } C = \sum_{i=1}^{l} \left(\binom{4l}{4i-2} + 4l \binom{4l-1}{4i-4} \right) + g,$$

where g is as described in the statement of 6. 6.

A review of the assembled facts shows that the value of $c(l, m, n)$ is indeed given by the stated formulae. ∎

We conclude with the following table:

l	m	n	$c(l, m, n)$	l	m	n	$c(l, m, n)$
2	2	2	688	3	2	2	3148
2	2	3	90	3	2	3	189
2	3	2	1168	3	3	2	41550
2	3	3	1158	3	3	3	3681

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