Semi-Carleman operators*

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1. VON NEUMANN [1] showed in 1935 that a self-adjoint operator on Hilbert space may be represented on $L_2(-\infty, \infty; dx)$ as an integral operator of Carleman type if and only if 0 is a limit point of its spectrum. In this note we show that this result survives in the non-self-adjoint case. In so doing we are lead to the consideration of what we shall call *semi-Carleman* integral operators. They are operators T on $L_2(-\infty, \infty; dx)$, given by a kernel $K(x, y)$ by the relation

$$Tf(x) = \int_{-\infty}^{\infty} K(x, y) f(y)\, dy,$$

such that

(1) $$\int_{-\infty}^{\infty} |K(x, y)|^2\, dx \equiv M^2(y) < \infty, \quad \text{a. e. in } y.$$

According to standard usage (see [2, p. 397]) Carleman integral operators have symmetric kernels $(K(y, x) = \overline{K(x, y)})$. We drop the requirement of symmetry. There is a natural choice of domain for such an operator making it closed and densely defined. We shall prove that such operators always have the point 0 as a limit point of their spectra, extending (and simplifying the proof of) [1, Theorem IV], and we shall obtain also a converse to this statement.

We are indebted to Dr. L. GROSS for a number of interesting and helpful conversations on this subject.

2. We shall say that a complex number λ is a limit point of the spectrum of an operator T if there exist unit vectors x_n $(n = 1, 2, \dots)$ which converge weakly to 0 and such that

$$(T - \lambda) x_n \to 0.$$

(*Cf.* [5, n° 133].) Suppose that T is closed and densely defined, and that 0 is a limit point of its spectrum (which we are implicitly assuming is not empty). We know that we may express T in the form $T = U(T^*T)^{1/2}$ where $(T^*T)^{1/2}$ is self-adjoint and U is a partial isometry whose initial domain is the closure of the range of $(T^*T)^{1/2}$ [3, p. 53], and we claim that 0 is a limit point of $(T^*T)^{1/2}$. For if we have unit vectors x_n tending weakly to 0 such that $Tx_n \to 0$ then $\|(T^*T)^{1/2} x_n\| = \|U(T^*T)^{1/2} x_n\| = \|Tx_n\| \to 0$, so that $(T^*T)^{1/2} x_n \to 0$, as required. Hence by the

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von Neumann theorem [1, Theorem III] $(T^*T)^{1/2}$ is unitarily equivalent to an operator H of Carleman type on $L_2(-\infty, \infty; dx)$, and therefore T may be represented as a partial isometry times a Carleman operator. We can in fact say more, by following VON NEUMANN's method adapted to the present (non-self-adjoint) circumstances. By [1, Theorem I] there exists a self-adjoint operator X of arbitrarily small Hilbert—Schmidt norm such that $A = (T^*T)^{1/2}+X$ is a *pure point operator* (in the sense that it has a complete orthonormal set of eigenvectors) having 0 as a limit point of its spectrum. Thus there exists a complete orthonormal set $\{\varphi_n\}_{n=1}^{\infty}$ of vectors and real numbers $\{\lambda_n\}_{n=1}^{\infty}$ (not necessarily distinct) such that $A\varphi_n=\lambda_n\varphi_n$ $(n=1, 2, \ldots)$. We know that 0 is a limit point of $\{\lambda_n\}_{n=1}^{\infty}$.

If it happens that $\sum_{n=1}^{\infty}|\lambda_n|^2<\infty$, then, writing $\psi_n=U\varphi_n$, we may choose a basis $\{\tilde{\varphi}_n\}_{n=1}^{\infty}$ of $L_2(-\infty, \infty; dx)$ and, defining the unitary operator W by $W\varphi_n=\tilde{\varphi}_n$, we write $\tilde{\psi}_n=W\psi_n$, and finally we define $K(x, y)=\sum_n \lambda_n\tilde{\psi}_n(x)\bar{\tilde{\varphi}}_n(y)$. Since U is not unitary, but only the partial isometry from [Range $(T^*T)^{1/2}$] to [Range (T)], we cannot conclude that the family $\{\psi_n\}$ (and the same applies to $\{\tilde{\psi}_n\}$) is orthonormal. All we know is $\|\tilde{\psi}_n\|=\|\psi_n\|=\|U\varphi_n\|\leqq 1$. Nevertheless $\iint\tilde{\psi}_n(x)\bar{\tilde{\varphi}}_n(y)\bar{\tilde{\psi}}_m(x)\tilde{\varphi}_m(y)\,dxdy=$ $=\delta_{nm}\|\tilde{\psi}_n\|^2\leqq\delta_{nm}$, which is to say that the functions $F_n(x, y)=\tilde{\psi}_n(x)\bar{\tilde{\varphi}}_n(y)$ are orthogonal and of norm $\leqq 1$ on the plane. Hence the series defining K is L_2 convergent on the plane and $\iint|K(x, y)|^2dxdy=(K, K)=\left(\sum\lambda_nF_n, \sum\lambda_mF_m\right)=\sum|\lambda_n|^2\|F_n\|^2\leqq$ $\leqq\sum|\lambda_n|^2$. Hence K is a Hilbert—Schmidt kernel, and the operator B it determines has the property $B\varphi_n=\lambda_n\tilde{\psi}_n$ That is, $B=W\{UA\}W^{-1}$. (This argument, proving that UA has a representation on $L_2(-\infty, \infty)$ as a Hilbert—Schmidt integral operator, is slightly different from the usual argument (see [4, p. 35]) because of the perturbation X, so that U is not necessarily isometric on the range of $A=(T^*T)^{1/2}+X$ Note that the argument shows that *such a representation is achieved no matter what basis* $\{\tilde{\varphi}_n\}$ *is chosen in* $L_2(-\infty, \infty)$.) Hence $WTW^{-1}=B-WUXW^{-1}$. Now UX is of Hilbert—Schmidt type since X is (see [5, p. 157]), so that, as pointed out above, $WUXW^{-1}$ is an integral operator with a Hilbert—Schmidt kernel L. Hence WTW^{-1} is an integral operator of Hilbert—Schmidt type with (nonsymmetric) kernel $K-L$.

If $\sum|\lambda_n|^2=\infty$, and we know only that $\{\lambda_n\}_{n=1}^{\infty}$ has 0 as a limit point, then we employ the following rearrangement of $\{\lambda_n\}$ (in which we are following VON NEUMANN exactly). Let $|\lambda_{m_\nu}|\leqq\frac{1}{\nu}$ $(\nu=1, 2, \ldots)$, and let $\{\lambda_{n_\nu}\}$ be the remaining members of $\{\lambda_n\}_{n=1}^{\infty}$. Set $l(\nu, k)=m_{2^{k-2}(2\nu-1)}$ for $k=2, 3, \ldots,$ and $l(\nu, 1)=n_\nu$. Then $|\lambda_{l(\nu,k)}|\leqq\frac{1}{2^{k-2}(2\nu-1)}\leqq\frac{1}{2^{k-2}}$ $(k=2, 3, \ldots)$, so that $\sum_k|\lambda_{l(\nu,k)}|^2<\infty$ for all $\nu=1, 2, \ldots$. Renumber the system so that $\nu=0, \pm1, \pm2, \ldots$. Define $U\varphi_n=\psi_n$. Choose a basis of uniformly bounded functions $\{\tilde{\varphi}_n\}_{n=1}^{\infty}$ of $L_2(0, 1; dx)$ and define a unitary operator W_1 by $W_1\varphi_n=\tilde{\varphi}_n$, and write $\tilde{\psi}_n=W_1\psi_n$. Now define, for $\nu=0, \pm1,$ $\pm2, \ldots,$

$$\Phi_{\nu,n}(t)=\begin{cases}\tilde{\varphi}_n(t-\nu), & \nu\leqq t\leqq\nu+1\\ 0 & \text{, otherwise.}\end{cases}$$

Then $\{\Phi_{\nu,n}\}$ is a complete uniformly bounded orthonormal system in $L_2(-\infty, \infty; dx)$ and the map V: $L_2(0, 1; dx) \to L_2(-\infty, \infty; dx)$ defined as

$$V\tilde{\varphi}_{l(\nu,k)} = \Phi_{\nu,k}$$

is unitary. Let $\Psi_{\nu,k} = V\tilde{\psi}_{l(\nu,k)}$, and define kernels K_ν by the relation

$$K_\nu(x, y) = \sum_k \lambda_{l(\nu,k)} \Psi_{\nu,k}(x)\overline{\Phi_{\nu,k}(y)}. \tag{2}$$

Now K_ν is square integrable on the strip $(\nu \leqq y \leqq \nu+1, -\infty < x < \infty)$ because for each fixed ν, $\sum |\lambda_{l(\nu,k)}|^2 < \infty$ (see above), so K_ν defines a Hilbert—Schmidt integral operator T_ν from $L_2(\nu, \nu+1; dx)$ to $L_2(-\infty, \infty; dx)$, with the property that $T_\nu\Phi_{\nu,k} = \lambda_{l(\nu,k)}\Psi_{\nu,k}$. If we now write

$$K(x, y) = \sum_\nu K_\nu(x, y) \tag{3}$$

as we may since the summands are supported on disjoint strips, we have a kernel defined on the whole plane which defines an operator S such that $S\Phi_{\nu,k} = \lambda_{l(\nu,k)}\Psi_{\nu,k}$ for all ν, k. S is densely defined, since $\mathfrak{D}_S$ contains the linear span of the basis $\{\Phi_{\nu,k}\} = \{VW_1\varphi_{l(\nu,k)}\}$, UA is defined on the linear span of the basis $\{\varphi_{l(\nu,k)}\} = \{\varphi_n\}$, and we have clearly

$$VW_1[UA](VW_1)^{-1} = S$$

on these dense sets. Now $|K(x, y)|^2 = \sum_\nu |K_\nu(x, y)|^2$, $\int |K_\nu(x, y)|^2\, dx = \sum_k |\lambda_{l(\nu,k)}|^2 |\Phi_{\nu,k}(y)|^2 < \infty$ for a. e. y, and for each fixed y, $\sum_\nu \int |K_\nu(x, y)|^2\, dx = \int |K_{\nu_0}(x, y)|^2\, dx$ where $\nu_0 \leqq y \leqq \nu_0+1$, so that $\int |K(x, y)|^2\, dx < \infty$ for a. e. y. Thus UA has a representation on $L_2(-\infty, \infty; dx)$ as a semi-Carleman operator.

To summarize, we have $T = U[(T^*T)^{1/2} + X] - UX = UA - UX$, where X is self-adjoint of Hilbert—Schmidt type, and UA is representable on $L_2(-\infty, \infty; dx)$ as the semi-Carleman operator S above. Now, just as before, UX goes over by the same representation on $L_2(-\infty, \infty; dx)$ (i. e., VW_1) into an integral operator of Hilbert—Schmidt type. Hence, upon adding the kernels, we arrive at the following

Theorem 1. *If T is closed, $\mathfrak{D}_T$ is dense, and 0 is a limit point of the spectrum of T, then T may be represented on $L_2(-\infty, \infty; dx)$ by a semi-Carleman integral operator.*

We have noted above that operators of Hilbert—Schmidt type have kernels no matter what representation on L_2 is chosen, and this is true even if l_2 is chosen as the representation space (here the kernel is the matrix). We do not assert this invariance of representation for the more general operators considered in Theorem 1. Indeed, every bounded operator A has a representation on l_2 as a semi-Carleman operator, where the kernel is the matrix. For, denoting by $\{x_n\}_{n=-\infty}^{\infty}$ the usual basis in l_2, T has the matrix representation $((Tx_n, x_m))$, and $\sum_n |(Tx_n, x_m)|^2 = \sum_n |(x_n, T^*x_m)|^2 = \|T^*x_m\|^2 \leqq \|T\|^2$. But it is not true that every bounded operator has a semi-Carleman representation on L_2. The identity operator may be offered as a counterexample (as may be verified just as for Hilbert—Schmidt operators, but we shall not do it that way since the same conclusion will follow from our Theorem 2 below). Thus it is essential that we employ a non-atomic measure space in Theorem 1.

3. Suppose we are given a measurable function $K(x, y)$ defined on the whole plane and satisfying the semi-Carleman condition (1). Let us write (with M as defined in (1))

$$\mathfrak{D}=\left\{f\in L_2(-\infty,\infty;\,dx)\,\Big|\int M(x)\,|f(x)|\,dx<\infty\right\}.$$

Let $\sigma_n=\{x\,|\,M(x)\leqq n\}$ $(n=1, 2, \dots)$ and let $\alpha\subset\sigma_n$ be an arbitrary measurable set of finite positive measure. Then the characteristic function of α is in $\mathfrak{D}$. Suppose $\int g\bar{f}dx=0$ for all $f\in\mathfrak{D}$. Then $\int_\alpha gdx=0$, so that $g(x)=0$ for a. e. $x\in\sigma_n$. But the complement of $\bigcup_n\sigma_n$ has measure 0, whence $g(x)=0$ for a. e. x. Hence $\mathfrak{D}$ is dense in $L_2(-\infty, \infty; dx)$. (This is essentially the argument used in [2, p. 398] for Carleman operators, and we have included it for the sake of completeness.) If $f\in\mathfrak{D}$ then

$$\int\left|\int K(x,y)f(y)dy\right|^2dx\leqq\iint dydz\,|f(y)|\,|f(z)|\int dx\,|K(x,y)|\,|K(x,z)|\leqq$$
$$\leqq\iint dydz\,|f(y)|\,|f(z)|\left(\int|K(x,y)|^2\,dx\right)^{1/2}\left(\int|K(x,z)|^2\,dx\right)^{1/2}=$$
$$=\iint dydz\,|f(y)|\,|f(z)|\,M(y)M(z)=\left[\int|f(u)|\,M(u)du\right]^2<\infty.$$

Hence the operator T given by $(Tf)(x)=\int K(x,y)f(y)dy$ for $f\in\mathfrak{D}$ is a densely defined semi-Carleman operator. Let $\sigma_n=\{x\,|\,0\leqq x\leqq1,\ M(x)\leqq n\}$. By (1) the measure of $\bigcup_n\sigma_n$ is 1, so there exists n_0 such that $0<$ measure of $\sigma_{n_0}\leqq1$. Then

$$\int_{\sigma_{n_0}}\int_{-\infty}^{\infty}|K(x,y)|^2\,dxdy=\int_{\sigma_{n_0}}dy\int_{-\infty}^{\infty}|K(x,y)|^2\,dx=\int_{\sigma_{n_0}}M(y)^2\,dy\leqq n_0^2,$$

so that we may regard K as an element of $L_2((-\infty,\infty)\times\sigma_{n_0})$ and $K^*(x,y)=\overline{K(y,x)}$ as an element of $L_2(\sigma_{n_0}\times(-\infty,\infty))$. As such, K and K^* define operators $S\colon L_2(\sigma_{n_0})\to L_2(-\infty,\infty)$ and $S^*\colon L_2(-\infty,\infty)\to L_2(\sigma_{n_0})$, respectively, of Hilbert—Schmidt type, with $N(S)\leqq n_0$, $N(S^*)\leqq n_0$, where N denotes the Hilbert—Schmidt norm. Let $\{\varphi_n\}_{n=1}^\infty$ be a basis in $L_2(\sigma_{n_0})$. Then

$$\{N((S^*S)^{1/2})\}^2=\sum_n\|(S^*S)^{1/2}\varphi_n\|^2_{\sigma_{n_0}}=$$
$$=\sum_n(S^*S\varphi_n\,\varphi_n)_{\sigma_{n_0}}=\sum\|S\varphi_n\|^2_{(-\infty,\infty)}=N(S)^2\leqq n_0^2,$$

where the subscripts indicate the norm employed. Hence $(S^*S)^{1/2}\colon L_2(\sigma_{n_0})\to L_2(\sigma_{n_0})$ is a self-adjoint Hilbert—Schmidt operator. Since Hilbert—Schmidt operators are completely continuous (see [4, p. 32]) we know there exists a set $\{\psi_n\}_{n=1}^\infty$ of unit vectors in $L_2(\sigma_{n_0})$, which are orthogonal (because $(S^*S)^{1/2}$ is self-adjoint) such that $(S^*S)^{1/2}\psi_n \to 0$ as $n \to \infty$. Let U: [Range $(S^*S)^{1/2}$] $\to$ [Range S] be the partial isometry in the polar decomposition of S: $S = U(S^*S)^{1/2}$. Then $\|S\psi_n\|_{(-\infty,\infty)} = \|U(S^*S)^{1/2}\psi_n\|_{(-\infty,\infty)} = \|(S^*S)^{1/2}\psi_n\|_{\sigma_{n_0}} \to 0$, so $S\psi_n \to 0$ in $L_2(-\infty, \infty)$. Define functions θ_n by

$$\theta_n(x) = \begin{cases} \psi_n(x), & x \in \sigma_{n_0}, \\ 0 & , x \notin \sigma_{n_0}. \end{cases}$$

Then θ_n is an orthonormal system in $L_2(-\infty, \infty)$. We have $\theta_n \in \mathfrak{D}$ for all n, for

$$\int_{-\infty}^{\infty} M(x)|\theta_n(x)|\,dx = \int_{\sigma_{n_0}} M(x)|\psi_n(x)|\,dx \leqq n_0 \int_{\sigma_{n_0}} |\psi_n(x)|\,dx \leqq n_0\|\psi_n\|_{\sigma_{n_0}} = n_0 < \infty.$$

Further, we have

$$T\theta_n(x) = \int K(x, y)\theta_n(y)\,dy = \int_{\sigma_{n_0}} K(x, y)\psi_n(y)\,dy = [S\psi_n](x),$$

so $T\theta_n = S\psi_n \to 0$ in $L_2(-\infty, \infty)$. Since θ_n converges weakly to 0, so we have proved

Theorem 2. *An integral operator of semi-Carleman type has* 0 *as a limit point of its spectrum (which is thereby, in particular, non-empty).*

From this it follows, as we mentioned earlier, that the identity operator cannot be represented as a semi-Carleman operator.

4. To complete the circle and achieve a characterization of operators of this type we have to show that semi-Carleman operators are closed. Let T be a semi-Carleman operator with kernel K acting on the domain $\mathfrak{D}$ defined above, and write $\mathscr{E} = \left\{f \middle| \int K(x, y)f(y)\,dy \in L_2(-\infty, \infty)\right\}$. We have seen above that $\mathscr{E} \supset \mathfrak{D}$. One may verify that T^* is determined by the kernel $K^*(x, y) = \overline{K(y, x)}$ acting on $\mathscr{E}^* = \left\{f \middle| \int K^*(x, y)f(y)\,dy \in L_2\right\}$ and that T^{**} is determined by K acting on $\mathscr{E}$ (the steps in the verification are the same, *mutatis mutandis*, as in [2, Theorem 10. 1, p. 398] and we omit them). Hence T has the closed extension T^{**}, and if we adopt $\mathscr{E}$ for the domain of K at the outset then the semi-Carleman operator it determines is already closed. With this understanding, we have now shown that *an operator T is representable on $L_2(-\infty, \infty)$ as a semi-Carleman operator if and only if T is closed, densely defined, and has* 0 *as a limit point of its spectrum.*

(Any partial isometry or projection with infinite-dimensional null space satisfies the above criterion, and it is easy to see what the representation is for such operators. For a partial isometry U, we have formulae (2) and (3) above, where $\lambda_{l(\nu,k)} = 0$, $k = 2, 3, \ldots, \lambda_{l(\nu,1)} = 1$, and $\Phi_{\nu,1}, \Psi_{\nu,1}$ correspond to bases for the initial and final spaces of U. Thus $K(x, y) = \sum_{\nu} \Psi_{\nu,1}(y)\overline{\Phi_{\nu,1}(x)}$, with the ν^{th} summand supported (and square integrable) on $(\nu \leqq y \leqq \nu + 1, -\infty < x < \infty)$, $\nu = 0, \pm 1, \pm 2, \ldots$. For a projection P the representation is even simpler because then $\Psi_{\nu,1} = \Phi_{\nu,1}$, so P is represented as the direct sum (on $\oplus L_2(\nu \leqq x \leqq \nu + 1)$) of operators of rank 1).

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