Equivalence of a problem of set theory to a hypothesis concerning the powers of cardinal numbers

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To Professor Béla Szőkefalvi-Nagy on his 50th birthday

Let E be an arbitrary set of power $\aleph_\alpha$ and suppose that with every element x of E is associated a non empty set $f(x)$ such that for any $x \in E$ the power of the set $f(x)$ is smaller than a given cardinal number $\aleph_\beta$ which is smaller than $\aleph_\alpha$ and $f(x) \neq f(y)$ $(x \neq y)$. We say that the subset Γ of E has the property $T(\mathfrak{q}, \mathfrak{p})$, where $\mathfrak{q}$ and $\mathfrak{p}$ are two cardinal numbers such that $\mathfrak{p} \leqq \mathfrak{q} \leqq \aleph_\alpha$, if

$$\overline{\overline{\bigcup_{x \in \Gamma} f(x)}} = \mathfrak{q} \quad \text{and} \quad \overline{\overline{\bigcup_{\substack{x, y \in \Gamma \\ x \neq y}} (f(x) \cap f(y))}} < \mathfrak{p}.$$

We define the ordinal number β_0 as follows:

Let β_0 be the smallest ordinal number $\varrho < \beta$ such that the set $E^{(\varrho)}$ of the elements $x \in E$ for which $\overline{\overline{f(x)}} < \aleph_\varrho$ has the power $\aleph_\alpha$.

Consider now the following propositions.

(I) *Under the above conditions E has a subset Γ with the property $T(\aleph_\alpha, \aleph_\alpha)$.*

(II) *For every ordinal number $\gamma, \beta < \gamma < \alpha$, the inequality*

$$\left(\aleph_\gamma^{\underset{\smile}{\aleph_{\beta_0}}}\right)^{\underset{\smile}{\aleph_{\beta_0}}} < \aleph_\alpha$$

holds, where $\aleph_\gamma^{\underset{\smile}{\aleph_{\beta_0}}} = \sum_{\varrho < \beta_0} \aleph_\gamma^{\aleph_\varrho}$.

We shall prove in this paper the following

Theorem. *The propositions* (I) *and* (II) *are equivalent.*

We shall use the following notations. For any subset Γ of E let

$$\Pi_\Gamma = \bigcup_{\substack{x, y \in \Gamma \\ x \neq y}} (f(x) \cap f(y)).$$

For any cardinal number $\mathfrak{r}$ we denote by $\mathfrak{r}^+$ the cardinal number immediately following $\mathfrak{r}$. The symbols $\overline{\overline{S}}$ and $\overline{\gamma}$ denote the cardinal numbers of the set S and of the ordinal number γ, respectively. For every ordinal number τ, $\aleph_{\mathrm{cf}(\tau)}$ denotes the least cardinal number $\mathfrak{n}$ such that $\aleph_\tau$ can be expressed as the sum of $\mathfrak{n}$ cardinal numbers each $< \aleph_\tau$. If $\mathfrak{m}$ and $\mathfrak{n}$ cardinal numbers, then we define $\mathfrak{m}^{\underset{\smile}{\mathfrak{n}}} = \sum_{\mathfrak{r} < \mathfrak{n}} \mathfrak{m}^{\mathfrak{r}}$. Put, for every ordinal number γ, $W(\gamma) = \{\xi : \xi < \gamma\}$.

In the proof of the theorem we shall use the following theorems:

Theorem 1. *If $\aleph_\alpha$ is regular and $\bigcup_{x \in E} f(x)$ has the power $\aleph_\alpha$, then E has a subset with the property $T(\aleph_\alpha, \aleph_\alpha)$.* (See [1], theorem 1.)

Theorem 2. *Let $\aleph_\alpha$ be a singular cardinal number, $\mathfrak{r}_0$ a cardinal number which is smaller then $\aleph_\alpha$ and $\{\aleph_\xi\}_{\xi<\omega_{cf(\alpha)}}$ a sequence of regular cardinal numbers such that $\aleph_\sigma > \aleph_\tau$ $(\sigma > \tau)$; $\max\{\aleph_{\mathrm{cf}(\alpha)}, \aleph_\beta, \mathfrak{r}_0\} < \aleph_\xi < \aleph_\alpha$ and $\aleph_\alpha = \sum_{\xi<\omega_{cf(\alpha)}} \aleph_\xi$. If for every $\xi < \omega_{\mathrm{cf}(\alpha)}$, E_ξ is a subset of power $\geqq \aleph_\xi$ of E such that E_ξ has a subset E'_ξ with the property $T(\aleph_\xi, \mathfrak{r}_0)$, then E has a subset with the property $T(\aleph_\alpha, [\aleph_{\mathrm{cf}(\alpha)}\mathfrak{r}_0]^+)$.* (See [1], theorem 4.)

Theorem 3. *If M is an infinite set of power $\mathfrak{m}$, and if $\mathfrak{n} \leqq \mathfrak{m}$, then the set S of subsets $X \subset M$ with $\overline{\overline{X}} < \mathfrak{n}$ has the power $\overline{\overline{S}} = \sum_{\mathfrak{r}<\mathfrak{n}} \mathfrak{m}^{\mathfrak{r}}$.* (See for example the theorem 3 of § 34 in [2].)

Theorem 4.

$$(\mathfrak{m}^{\aleph_\varrho})^{\aleph_\mu} = \begin{cases} \mathfrak{m}^{\aleph_\varrho} & \text{for } \mu \leqq \mathrm{cf}(\varrho), \\ \mathfrak{m}^{\aleph_{\varrho+1}} & \text{for } \mathrm{cf}(\varrho) < \mu \leqq \varrho + 1, \\ \mathfrak{m}^{\aleph_\mu} & \text{for } \mu > \varrho. \end{cases}$$

(See theorem 7 of § 34 in [2].)

Theorem 5. *Let $\aleph_\alpha$ be a singular cardinal number and η an ordinal number smaller than ω_α. If to every element γ of $W(\omega_\alpha)$ there corresponds an ordinal number $h(\gamma) < \eta$, then there exists a subset M of power $\aleph_\alpha$ of $W(\omega_\alpha)$ such that*

$$\overline{\overline{h[M]}} \leqq \aleph_{\mathrm{cf}(\alpha)}.$$

Proof. Let $\{\alpha_\xi\}_{\xi<\omega_{cf(\alpha)}}$ be an increasing sequence of ordinal numbers such that $\lim_{\xi<\omega_{cf(\alpha)}} \alpha_\xi = \alpha$ for every $\xi < \omega_{\mathrm{cf}(\alpha)}$, $\omega_{\alpha_\xi} > \eta$ and ω_{α_ξ} is regular. It is clear that

$$W(\omega_\alpha) = \bigcup_{\xi<\omega_{cf(\alpha)}} W(\omega_{\alpha_\xi}).$$

Let us define $g_\xi(\gamma)$ on $W(\omega_{\alpha_\xi})$ as follows:

$$g_\xi(\gamma) = h(\gamma) \quad (\gamma \in W(\omega_{\alpha_\xi})).$$

Since ω_{α_ξ} is regular and $\omega_{\alpha_\xi} > \eta$, there exists an ordinal number $\pi_\xi \in W(\eta)$ and a subset M_ξ of power $\aleph_{\alpha_\xi}$ of $W(\omega_{\alpha_\xi})$ such that

$$g_\xi[M_\xi] = \{\pi_\xi\}.$$

Let

$$M = \bigcup_{\xi<\omega_{cf(\alpha)}} M_\xi.$$

Clearly the power of M is $\aleph_\alpha$. Let further N be the set of all distinct elements of the sequence $\{\pi_\xi\}_{\xi<\omega_{cf(\alpha)}}$. It is clear that

$$h[M] = N.$$

Since $\overline{\overline{N}} \leqq \aleph_{\mathrm{cf}(\alpha)}$, theorem 5 is proved.

Corollary. *If η is an ordinal number of the second kind and* $\mathrm{cf}(\eta) \neq \mathrm{cf}(\alpha)$, *then there exists a subset M' of power $\aleph_\alpha$ of M and an ordinal number $\eta' < \eta$ such that*

$$h[M'] \subseteqq W(\eta').$$

Proof. (i) If $\overline{\overline{N}} < \aleph_{\mathrm{cf}(\alpha)}$, then it follows from the regularity of $\omega_{\mathrm{cf}(\alpha)}$ that there exists an increasing sequence $\{\xi_\nu\}_{\nu < \omega_{cf(\alpha)}}$ of the type $\omega_{\mathrm{cf}(\alpha)}$ of ordinal numbers smaller than $\omega_{\mathrm{cf}(\alpha)}$ such that

$$\pi_{\xi_0} = \pi_{\xi_1} = \dots = \pi_{\xi_\nu} = \dots \quad (\nu < \omega_{\mathrm{cf}(\alpha)}).$$

But then $$\overline{\overline{\{\gamma \in M : h(\gamma) = \pi_{\xi_0}\}}} = \sum_{\xi_\nu < \omega_{cf(\alpha)}} \aleph_{\xi_\nu} = \aleph_\alpha.$$

and $$h[M'] = h[\{\gamma \in M : h(\gamma) = \pi_{\xi_0}\}] \subseteqq W(\pi_{\xi_0} + 1).$$

(j) If $\overline{\overline{N}} = \aleph_{\mathrm{cf}(\alpha)}$, then let $\{\eta_\nu\}_{\nu < \omega_{cf(\eta)}}$ be an increasing sequence of ordinal numbers for which $\lim\limits_{\nu < \omega_{cf(\eta)}} \eta_\nu = \eta$.

(j_1) If $\mathrm{cf}(\alpha) < \mathrm{cf}(\eta)$, then it follows from the inequality $N \subset W(\eta)$ that there exists an ordinal number $\nu_0 < \omega_{\mathrm{cf}(\eta)}$, for which

$$N \subseteqq W(\eta_{\nu_0}) \subset W(\eta).$$

(j_2) If $\mathrm{cf}(\alpha) > \mathrm{cf}(\eta)$, then let $N_\nu = N \cap W(\eta_\nu)$. It is clear that

$$\bigcup_{\nu < \omega_{cf(\eta)}} N_\nu = N.$$

Since $\omega_{\mathrm{cf}(\alpha)}$ is regular, there exists an ordinal number $\nu_0 < \omega_{\mathrm{cf}(\eta)}$ such that

$$\overline{\overline{N}}_{\nu_0} = \aleph_{\mathrm{cf}(\alpha)}.$$

It follows that there exists an increasing sequence $\{\xi_\varrho\}_{\varrho < \omega_{cf(\alpha)}}$ of the type $\omega_{\mathrm{cf}(\alpha)}$ such that

$$N_{\nu_0} = \{\pi_{\xi_\varrho}\}_{\varrho < \omega_{cf(\alpha)}}.$$

Thus we get from the definition of $\{\pi_\xi\}_{\xi < \omega_{cf(\alpha)}}$ that $M' = \bigcup\limits_{\nu < \omega_{cf(\alpha)}} M_{\xi_\nu}$ has the power $\aleph_\alpha = \sum \aleph_{\alpha_{\xi_\nu}}$ and

$$h[M'] \subset W(\eta_{\nu_0}).$$

Theorem 6. *Let $\aleph_\alpha$ be a singular cardinal number and η an ordinal number smaller than ω_α. If to every element γ of $W(\omega_\alpha)$ there corresponds an ordinal number $h(\gamma) < \eta$, then the smallest ordinal number η_0, for which there exists a subset M of power $\aleph_\alpha$ of $W(\omega_\alpha)$ such that*

$$h[M] \subset W(\eta_0) \subseteqq W(\eta),$$

is either of the first kind, i. e. $\eta_0 = \tau_0 + 1$ or of the second kind with $\mathrm{cf}(\eta_0) = \mathrm{cf}(\alpha)$.

Proof. (i) $W(\eta_0)$ has a greatest element. In this case the power of the set M', for which $h[M'] = \{\pi_0\}$, is $\aleph_\alpha$ and the power of the set M'', for which

$$h[M''] \subseteqq W(\tau_0),$$

is smaller than $\aleph_\alpha$. Thus $\eta_0 = \tau_0 + 1$.

(ii) $W(\eta_0)$ does not contain a greatest element. Then η_0 is of the second kind. It follows from the definition of η_0 and the corollary of theorem 5 that $\operatorname{cf}(\eta_0) = \operatorname{cf}(\alpha)$. Theorem 6 is proved. With the aid of theorem 6 we get

Theorem 7. *The ordinal number β_0 is either of the first kind or of the second kind with* $\operatorname{cf}(\beta_0) = \operatorname{cf}(\alpha)$.

Proof of the theorem. (A) First we prove that (I) follows from (II). Suppose also that the proposition (II) holds. Put

$$(\aleph_\gamma^{\underset{\smile}{\aleph_{\beta_0}}})^{\underset{\smile}{\aleph_{\beta_0}}} = \aleph_{\beta_0(\gamma)}.$$

It follows from theorem 4, that

$$\aleph_{\beta_0(\gamma)} = \begin{cases} \aleph_\gamma^{\underset{\smile}{\aleph_{\beta_0}}} & \text{for} \quad \operatorname{cf}(\beta_0) = \beta_0 . \\ \aleph_\gamma^{\aleph_{\beta_0}} & \text{for} \quad \operatorname{cf}(\beta_0) < \beta_0 . \end{cases}$$

This implies that

$$\aleph_{\beta_0(\gamma)}^{\underset{\smile}{\aleph_{\beta_0}}} = (\aleph_\gamma^{\underset{\smile}{\aleph_{\beta_0}}})^{\underset{\smile}{\aleph_{\beta_0}}} = \aleph_\gamma^{\underset{\smile}{\aleph_{\beta_0}}} = \aleph_{\beta_0(\gamma)}$$

for $\operatorname{cf}(\beta_0) = \beta_0$ and

$$\aleph_{\beta_0(\gamma)}^{\underset{\smile}{\aleph_{\beta_0}}} = \sum_{\varrho<\beta_0} \aleph_{\beta_0(\gamma)}^{\aleph_\varrho} = \sum_{\varrho<\beta_0} (\aleph_\gamma^{\aleph_{\beta_0}})^{\aleph_\varrho} = \sum_{\varrho<\beta_0} \aleph_\gamma^{\aleph_{\beta_0}\cdot\aleph_\varrho} = \sum_{\varrho<\beta_0} \aleph_\gamma^{\aleph_{\beta_0}} = \aleph_\gamma^{\aleph_{\beta_0}}\cdot\aleph_{\beta_0} = \aleph_{\beta_0(\gamma)}$$

for $\operatorname{cf}(\beta_0) < \beta_0$, i. e. in both cases $\aleph_{\beta_0(\gamma)}^{\underset{\smile}{\aleph_{\beta_0}}} = \aleph_{\beta(\gamma_0)}$ holds. As the sets $f(x)$ are distinct it follows from this that the set $\bigcup_{x\in E} f(x)$ has the power $\aleph_\alpha$. Thus, if $\aleph_\alpha$ is regular, we get by theorem 1, that E has a subset with the property $T(\aleph_\alpha, \aleph_\alpha)$. Suppose now that $\aleph_\alpha$ is singular. Then $E^{(\beta_0)}$ has for every γ, $\beta<\gamma<\alpha$, a subset E_γ with the property $T(\aleph_{\beta_0(\gamma)+1}, \aleph_{\beta_0(\gamma)+1})$, i. e.

$$\overline{\overline{\Pi}}_{E_\gamma} \leqq \aleph_{\beta_0(\gamma)} < \aleph_{\beta_0(\gamma)+1}.$$

Let $S(\gamma)$ be the set of subsets $X \subset \Pi_{E_\gamma}$ with $\overline{\overline{X}} < \aleph_{\beta_0}$. It follows from theorem 3 that $\overline{\overline{S(\gamma)}} \leqq \aleph_{\beta_0(\gamma)}^{\underset{\smile}{\aleph_{\beta_0}}} = \aleph_{\beta_0(\gamma)}$. Hence, since for given γ the sets $f^{(\gamma)}(x) = f(x) - \Pi_{E_\gamma}$ $(x \in E_\gamma)$ are mutually disjoint, we obtain that there exists an element X_0 of $S(\gamma)$ and to this a subset E'_γ of power $\aleph_{\beta_0(\gamma)+1}$ of E_γ such that $f^\gamma(x) \neq 0$ and

$$f(x) = f^{(\gamma)}(x) \cup X_0$$

for every $x \in E'_\gamma$, i. e. E'_γ has the property $T(\aleph_{\beta_0(\gamma)+1}, \aleph_{\beta_0})$. It follows from theorem 2 that E has a subset with the property $T(\aleph_\alpha, \aleph_{\beta_0}\aleph_{\operatorname{cf}(\alpha)+1})$.

(B) We prove now that from the proposition (I) follows the proposition (II). Suppose therefore that (II) does not hold. Then we prove that the proposition (I) is false.

Let β_0 is an ordinal number of the first kind, i. e. $\beta_0 = \tau_0 + 1$. If (II) does not hold, then there exists an ordinal number γ_0, $\beta < \gamma_0 < \alpha$ for which

$$\aleph_{\gamma_0}^{\aleph_{\tau_0}} \geqq \aleph_\alpha .$$

Let E_1 be a subset of power $\aleph_{\gamma_0}$ of E and T_1 a set of power $\aleph_\alpha$ of subsets of power $\aleph_{\tau_0}$ of E_1. Let further $f(x)$ be a one-to-one mapping of E into T_1. It follows that if Γ is a subset of E with the property $T(\mathfrak{q}, \mathfrak{p})$ then $\mathfrak{q} \leqq \aleph_{\gamma_0}$, because the sets

$$f'(x) = f(x) - \Pi_\Gamma \subset E_1$$

must be not empty and mutually disjoint for $\mathfrak{q}$ elements x of Γ.

Let β_0 be an ordinal number of the second kind. Then $\mathrm{cf}(\beta_0) = \mathrm{cf}(\alpha)$ by the theorem 7. Let $\{\alpha_\eta\}_{\eta < \omega_{cf(\alpha)}}$ and $\{\beta_\eta\}_{\eta < \omega_{cf(\alpha)}}$ be two increasing sequences of ordinal numbers such that $\lim\limits_{\eta < \omega_{cf(\alpha)}} \alpha_\eta = \alpha$ and $\lim\limits_{\eta < \omega_{cf(\alpha)}} \beta_\eta = \beta_0$. We have two cases:

(i) there exists a smallest ordinal number $\eta_0 < \omega_{\mathrm{cf}(\alpha)}$ and an ordinal number γ_0, $\beta < \gamma_0 < \alpha$, such that $\aleph_{\gamma_0}^{\aleph_{\beta_{\eta_0}}} \geqq \aleph_\alpha$;

(ii) for every $\varrho < \beta_0$ there exists an $\varrho' < \beta_0$ such that $\aleph_{\gamma_0}^{\aleph_{\varrho'}} > \aleph_{\gamma_0}^{\aleph_\varrho}$. In this case we assume that, for every $\eta < \omega_{\mathrm{cf}(\alpha)}$, β_η is the smallest ordinal number such that

$$\aleph_{\gamma_0}^{\aleph_{\beta_\eta}} \geqq \aleph_{\alpha_\eta}.$$

Let T_η be in both cases (but in the case (i) we assume that $\eta_0 \leqq \eta < \beta_0$ holds) a set of power $\aleph_{\alpha_\eta}$ of subsets of power $\aleph_{\beta_\eta}$ of E_1, where $\overline{\overline{E}}_1 = \aleph_{\gamma_0}$. It is clear that the set

$$T = \bigcup_{\eta < \omega_{cf(\alpha)}} T_\eta$$

has the power $\aleph_\alpha$. Let $f(x)$ be a one-to-one mapping of E into T. If Γ is a subset of E with the property $T(\mathfrak{q}, \mathfrak{p})$, then $\mathfrak{q} \leqq \aleph_{\gamma_0}$, because the sets $f'(x) = f(x) - \Pi_\Gamma \subset E_1$ must be non empty and mutually disjoint for $\mathfrak{q}$ elements x of Γ. The theorem is proved.

References

[1] G. Fodor, Some results concerning a problem in set theory, *Acta Sci. Math.*, **16** (1955), 232–240.

[2] H. Bachmann, Transfinite Zahlen, *Ergebnisse der Math. und ihrer Grenzgebiete*. Neue Folge, Heft 1 (Berlin–Heidelberg–Göttingen, 1955).

(Received July 15, 1962)