On commuting unitary operators in spaces with indefinite metric

By M. A. NAĬMARK in Moscow (USSR)

To Professor Béla Szőkefalvi-Nagy on his 50th birthday

Let H be a Hilbert space with the usual inner product $[x, y]$ and with an indefinite inner product (x, y) which, for some complete orthonormal system $\{e_\alpha\}$ in H, is defined by

(1) $$(x, y) = \sum_{\alpha=1}^{\varkappa} \xi_\alpha \bar{\eta}_\alpha - \sum_{\alpha > \varkappa} \xi_\alpha \bar{\eta}_\alpha,$$

where

(2) $$\xi_\alpha = [x, e_\alpha], \quad \eta_\alpha = [y, e_\alpha],$$

$\varkappa$ is a fixed positive integer and $\varkappa < \dim H$. Such a space H will be called a space $\Pi_\varkappa$ with indefinite metric. Another, axiomatic definition of the space $\Pi_\varkappa$ was given by I. S. IOHVIDOV and M. G. KREĬN [1]; we shall follow here the terminology and use the results of this paper.

A linear operator U in $\Pi_\varkappa$ is called unitary if it maps $\Pi_\varkappa$ onto $\Pi_\varkappa$, and preserves the scalar product (x, y), i. e.

$$(Ux, Uy) = (x, y) \text{ for all } x, y \in \Pi_\varkappa.$$

By a theorem of L. S. PONTRYAGIN [2], there exists, for every unitary operator U in $\Pi_\varkappa$, a $\varkappa$-dimensional non-negative subspace, which is invariant with respect to U.[1]) This theorem plays an important role in the study of unitary operators in $\Pi_\varkappa$.

It is therefore natural to expect that the following theorem 1 will be useful for the theory of unitary group representations in $\Pi_\varkappa$, for the theory of rings of operators in $\Pi_\varkappa$, and for other topics[2]).

[1]) L. S. PONTRYAGIN proved his theorem for self-adjoint operators (with respect to (x, y)); using the Cayley transformation one easily sees (*cf.* [1]) that the theorem of L. S. PONTRYAGIN is equivalent to the theorem for unitary operators cited above. Another, simpler proof of the theorem for unitary (and also for more general) operators was given by M. G. KREĬN [2] (see also I. S. IOHVIDOV and M. G. KREĬN [1]; for further generlizations of this theorem see M. BRODSKIĬ [4] and H. LANGER [5], [6]).

[2]) Theorem 1 has been announced by the author in the Note [8] and a proof was there given for $\varkappa = 1$; various applications of the theorem will be treated in further publications. We note that theorem 1 (see also proposition VI. and corollary 2 below) contains the solution for $\Pi_\varkappa$ of a problem posed by PHILLIPS [7].

Theorem 1. *Let $\mathfrak{U}$ be a set of commuting unitary operators in $\Pi_\varkappa$; then there exists in $\Pi_\varkappa$ a $\varkappa$-dimensional non-negative subspace which is invariant with respect to all $U \in \mathfrak{U}$.*

Proof 1. Let $\{e_\alpha\}$ be a complete orthonormal system (with respect to $[x, y]$) in $\Pi_\varkappa$, such that (1) and (2) hold; it follows from (1) and (2) that we also have

(3) $$\xi_\alpha = (x, e_\alpha) \text{ for } \alpha = 1, \dots, \varkappa,$$

(4) $$\xi_\alpha = -(x, e_\alpha) \text{ for } \alpha > \varkappa.$$

Put for any $x \in \Pi_\varkappa$

(5) $$x^+ = \sum_{\alpha=1}^{\varkappa} \xi_\alpha e_\alpha, \quad x^- = \sum_{\alpha>\varkappa} \xi_\alpha e_\alpha;$$

then we have the relation

(6) $$x = \sum_\alpha \xi_\alpha e_\alpha = x^+ + x^-$$

and using (1) and (2) we get:

(7) $$[x, y] = (x^+, x^+) - (x^-, x^-), \quad (x, y) = (x^+, x^+) + (x^-, x^-).$$

We note also that

(8) $$(x^+, x^+) \geqq 0, \quad (x^-, x^-) \leqq 0,$$

and the equality sign holds in (8) only if $x^+ = 0$, or $x^- = 0$, respectively.

Let $X = (x_1, \dots, x_\varkappa)$ be a system of $\varkappa$ vectors $x_1, \dots, x_\varkappa \in \Pi_\varkappa$ satisfying the following conditions:

α) $x_1, \dots, x_\varkappa$ are linearly independent;

β) the $\varkappa$-dimensional subspace $\mathfrak{M}_X$ generated by $x_1, \dots, x_\varkappa$ is non-negative.

I. *The vectors $x_1^+, \dots, x_\varkappa^+$ also are linearly independent.*

In fact, let $\sum_{\alpha=1}^{\varkappa} c_\alpha x_\alpha^+ = 0$ for some numbers c_α. Put $x = \sum_{\alpha=1}^{\varkappa} c_\alpha x_\alpha$; then $x^+ = \sum_{\alpha=1}^{\varkappa} c_\alpha x_\varkappa^+ = 0$. On the other hand, by β), (7), and (8),

$$0 \leqq (x, x) = (x^+, x^+) + (x^-, x^-) = (x^-, x^-) \leqq 0,$$

thus $(x^-, x^-) = 0$, implying $x^- = 0$. Therefore, $x = x^+ + x^- = 0$, i. e. $\sum_{\alpha=1}^{\varkappa} c_\alpha x_\alpha = 0$. By α) this implies $c_1 = c_2 = \dots = c_\varkappa = 0$ concluding the proof of I.

Each vector x_j can be considered as a column of its coordinates $\xi_{\alpha j} = [x_j, e_\alpha]$, thus X will be a matrix $X = \|\xi_{\alpha j}\|$ with $\varkappa$ columns; on the other hand the $x_1^+, \dots, x_\varkappa^+$ define a $\varkappa \times \varkappa$-matrix $X^+ = \|\xi_{\alpha j}\|_{\alpha, j=1, \dots, \varkappa}$. If X satisfies α) and β), then by I the inverse $(X^+)^{-1}$ exists. A system $X = (x_1, \dots, x_\varkappa)$ satisfying α) and β) will be called *normed*, if $X^+ = 1$, where 1 denotes the $\varkappa \times \varkappa$-identity matrix. If X is not normed, then the matrix $\tilde{X} = X(X^+)^{-1}$ will define a normed system. We denote by K the set of all normed systems satisfying α) and β). Two systems $X = (x_1, \dots, x_\varkappa)$, $X' = (x_1', \dots, x_\varkappa')$ define the same subspace if and only if $X' = XA$, where A is a non-singular $\varkappa \times \varkappa$-matrix.

Particularly, X and $\tilde{X}=X(X^+)^{-1}$ define the same subspace; thus every non-negative $\varkappa$-dimensional subspace is defined by a system $X=(x_1, \ldots, x_\varkappa)\in K$. If two systems $X, X'\in K$ define the same subspace, then $X'=XA$ and hence $X'^+=X^+A$. But $X'^+=X^+=1$ and therefore $A=1$.

In other words:

II. *If $\mathfrak{M}_X$ denotes the subspace defined by a system $X\in K$, then the correspondence $X\to\mathfrak{M}_X$ is a one-to-one mapping of K onto the set of all non-negative $\varkappa$-dimensional subspaces in $\Pi_\varkappa$.*

2. If $X=(x_1, \ldots, x_\varkappa)\in K$, then $(c_1x_1+\ldots+c_\varkappa x_\varkappa, c_1x_1+\ldots+c_\varkappa x_\varkappa)\geqq 0$ holds for any complex $c_1, \ldots, c_\varkappa$. In virtue of (7) this means that

(9) $[c_1x_1^-+\ldots+c_\varkappa x_\varkappa^-, c_1x_1^-+\ldots+c_\varkappa x_\varkappa^-]\leqq[c_1x_1^++\ldots+c_\varkappa x_\varkappa^+, c_1x_1^++\ldots+c_\varkappa x_\varkappa^+]$.

But condition $X^+=1$ implies that the right hand side of (9) is $\sum_{j=1}^{\varkappa}|c_j|^2$, so that (9) can be written as

(10) $$\sum_{\alpha,\beta=1}^{\varkappa}[x_\alpha^-, x_\beta^-]c_\alpha\bar{c}_\beta\leqq\sum_{j=1}^{\varkappa}|c_j|^2.$$

Conversely, if (10) is satisfied, and if we put $x_j=e_j+x_j^-$, $j=1, \ldots, \varkappa$, we get a system $X=(x_1, \ldots, x_\varkappa)\in K$. If

$$c_{\alpha'}=\begin{cases}1 & \text{for } \alpha'=\alpha\\ 0 & \text{for } \alpha'\neq\alpha\end{cases}$$

then (10) takes the form

(11) $$[x_\alpha^-, x_\alpha^-]\leqq 1$$

and hence

(12) $$\sum_{\alpha=1}^{\varkappa}[x_\alpha^-, x_\alpha^-]\leqq\varkappa.$$

By (5) each x_α^- can be represented in the form

(13) $$x_\alpha^-=\sum_{\beta>\varkappa}\xi_{\beta\alpha}e_\beta \quad \text{where} \quad \xi_{\beta\alpha}=[x_\alpha^-, e_\beta];$$

thus (12) can be written as

(14) $$\sum_{\alpha=1}^{\varkappa}\sum_{\beta>\varkappa}|\xi_{\beta\alpha}|^2\leqq\varkappa.$$

Denote by $\mathfrak{H}$ the Hilbert space of all sequences $\xi=\{\xi_{\beta\alpha}; \alpha=1, \ldots, \varkappa; \beta>\varkappa\}$ with the norm

$$\|\xi\|=\left(\sum_{\alpha=1}^{\varkappa}\sum_{\beta>\varkappa}|\xi_{\beta\alpha}|^2\right)^{\frac{1}{2}}<\infty$$

and let Q be the ball $\|\xi\|^2\leqq\varkappa$ in $\mathfrak{H}$. Then (13) and (14) mean:

III. *The correspondence $X\to\xi=\{\xi_{\beta\alpha}; \alpha=1, \ldots, \varkappa; \beta>\varkappa\}$ is a one-to-one mapping of K onto a set $Q_1\subset Q$.*

The ball Q is known to be bicompact in the weak topology of $\mathfrak{H}$. On the other hand Q_1 is closed[3]) and hence also bicompact. In fact by (10) and (13) $\xi \in Q_1$ if and only if

$$\sum_{\gamma > \varkappa} \left| \sum_{\alpha=1}^{\varkappa} \xi_{\gamma\alpha} c_\alpha \right|^2 \leqq \sum_{j=1}^{\varkappa} |c_j|^2. \tag{15}$$

Let Γ be a finite set $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ and G be the family of all finite sets Γ (with any number of elements). Denote by $Q(\Gamma, c_1, \ldots, c_\varkappa)$ the set of all $\xi \in \mathfrak{H}$ satisfying the inequality

$$\sum_{\alpha, \beta=1}^{\varkappa} \sum_{\gamma \in \Gamma} \xi_{\gamma\alpha} \bar{\xi}_{\gamma\beta} c_\alpha \bar{c}_\beta \leqq \sum_{j=1}^{\varkappa} |c_j|^2 \tag{16}$$

for fixed $c_1, \ldots, c_\varkappa$ and Γ, and let $Q(c_1, \ldots, c_\varkappa)$ denote the set of all $\xi \in \mathfrak{H}$ satisfying (15) for fixed $c_1, \ldots, c_\varkappa$. Each $\xi_{\gamma\alpha}$ is a continuous function of ξ, hence the left hand side of (16) also is a continuous function. Therefore, $Q(\Gamma, c_1, \ldots, c_\varkappa)$ is closed. But

$$Q(c_1, \ldots, c_\varkappa) = \bigcap_{\Gamma \in G} Q(\Gamma, c_1, \ldots, c_\varkappa)$$

and

$$Q_1 = \bigcap_{c_1, \ldots, c_\varkappa} Q(c_1, \ldots, c_\varkappa),$$

where the last intersection is taken over all systems $c_1, \ldots, c_\varkappa$ of complex numbers. Thus $Q(c_1, \ldots, c_\varkappa)$ and Q_1 are also closed and Q_1 is a bicompact set.

Now we show that Q_1 is a convex set. Denote by $l_\varkappa^2$ the $\varkappa$-dimensional Hilbert space of all $c = (c_1, \ldots, c_\varkappa)$ with the inner product $(c, c') = \sum_{j=1}^{\varkappa} c_j \bar{c}'_j$ and let l^2 be the Hilbert space of all sequences $\eta = \{\eta_\gamma, \gamma > \varkappa\}$ satisfying $\sum_{\gamma > \varkappa} |\eta_\gamma| < \infty$ with the inner product

$$(\eta, \eta') = \sum_{\gamma > \varkappa} \eta_\gamma \bar{\eta}'_\gamma.$$

Then in virtue of (15) Q_1 can be regarded as the set of all bounded operators

$$\eta_\gamma = \sum_{\alpha=1}^{\varkappa} \xi_{\gamma\alpha} c_\alpha$$

from $l_\varkappa^2$ to l^2 with norm $\leqq 1$. As the last set is convex, Q_1 is also convex.

3. Let $X = (x_1, \ldots, x_\varkappa)$ be a system satisfying α) and β) (p. 178), and let U be a unitary operator in $\Pi_\varkappa$. Then the system $Y = (Ux_1, \ldots, Ux_\varkappa)$ satisfies also α) and β).

In fact, since U is unitary and $x_1, \ldots, x_\varkappa$ are linearly independent, $Ux_1, \ldots, Ux_\varkappa$ are also linearly independent. Further, using β) for $x_1, \ldots, x_\varkappa$ we have

$$\left(\sum_{\alpha=1}^{\varkappa} c_\alpha U x_\alpha, \sum_{\alpha=1}^{\varkappa} c_\alpha U x_\alpha \right) = \left(U \sum_{\alpha=1}^{\varkappa} c_\alpha x_\alpha, U \sum_{\alpha=1}^{\varkappa} c_\alpha x_\alpha \right) = \left(\sum_{\alpha=1}^{\varkappa} c_\alpha x_\alpha, \sum_{\alpha=1}^{\varkappa} c_\alpha x_\alpha \right) \geqq 0.$$

[3]) In the following all topological notions in $\mathfrak{H}$ will be considered in the weak topology of $\mathfrak{H}$.

Particularly, U transforms every system $X \in K$ (*cf.* p. 178) into a system Y satisfying $\alpha)$ and $\beta)$, hence by I the matrix $(Y^+)^{-1}$ exists. We denote by V_U the (non-linear) operator defined by

$$(17) \qquad V_U X = Y(Y^+)^{-1} \quad \text{where} \quad Y = (Ux_1, \ldots, Ux_\varkappa).$$

As $V_U X \in K$, the operator V_U transforms K into itself. By virtue of III V_U can also be considered as an operator V_U transforming Q_1 into itself. This operator V_U is continuous in Q_1. In fact, let Π^+ (and Π^-) denote the set of all $x \in \Pi_\varkappa$ for which $x^- = 0$ (resp. $x^+ = 0$); then in virtue of (6) and (7)

$$(18) \qquad \Pi_\varkappa = \Pi^+ \oplus \Pi^-$$

where Π^+ and Π^- are orthogonal with respect to (x, y) and also with respect to $[x, y]$. According to the decomposition (18) U can be given by a matrix

$$(19) \qquad U \sim \begin{Vmatrix} A & B \\ C & D \end{Vmatrix}$$

where A, B, C, D are bounded operators; A is an operator in Π^+, D is an operator in Π^-, B is an operator from Π^- into Π^+ and C is an operator from Π^+ into Π^-. If we use the orthonormal system $\{e_\alpha\}$ and the decompositions (5), we see that Π^+ and Π^- coincide with the spaces $l^2_\varkappa$ and l^2, and A, B, C, D are represented by matrices. Moreover, the systems $X \in K$ are represented by matrices

$$X = \begin{Vmatrix} 1 \\ \xi \end{Vmatrix}$$

where 1 is the $\varkappa \times \varkappa$-identity matrix and in virtue of (19) $Y = (Ux_1, \ldots, Ux_\varkappa)$ means that

$$Y = \begin{Vmatrix} A + B\xi \\ C + D\xi \end{Vmatrix}.$$

Thus $Y^+ = A + B\xi$ and $V_U \xi = (C + D\xi)(A + B\xi)^{-1}$. As A, B, C, D are bounded, the functions $\xi \to C + D\xi$ and $\xi \to A + B\xi$ are continuous. Moreover $A + B\xi$ is a $\varkappa \times \varkappa$-matrix and $(A + B\xi)^{-1}$ exists; hence the function $\xi \to (A + B\xi)^{-1}$ is also continuous. Thus the function $\xi \to V_U \xi = (C + D\xi)(A + B\xi)^{-1}$ is also continuous. Therefore V_U is a continuous transformation into itself of the convex bicompact set Q_1 and hence V_U has a fixpoint in Q_1. Let ξ be such a fixpoint, i. e.

$$V_U \xi = \xi.$$

In virtue of (17) and III this means that

$$Y(Y^+)^{-1} = X \quad \text{hence} \quad Y = XY^+$$

i. e. the systems $Y = (Ux_1, \ldots, Ux_\varkappa)$ and $X = (x_1, \ldots, x_\varkappa)$ define the same subspace $\mathfrak{M}_X$; this means that $\mathfrak{M}_X$ is invariant with respect to U. So we have proved the *existence* of a non-negative $\varkappa$-dimensional subspace, which is invariant with respect to U.[4])

[4]) The argument in sections 2 and 3 is a slight modification of the proof of theorem 3. 1 in [1].

4. We have proved in section 3, that every fixpoint ξ of V_U in Q_1 defines a non-negative $\varkappa$-dimensional subspace $\mathfrak{M}_X$ in $\Pi_\varkappa$, which is invariant with respect to U.

Conversely, let $\mathfrak{M}$ be any non-negative $\varkappa$-dimensional subspace in $\Pi_\varkappa$ which is invariant with respect to U. In virtue of II $\mathfrak{M}=\mathfrak{M}_X$ for some uniquely defined $X\in K$ and the invariance of $\mathfrak{M}_X$ means that $X=(x_1, \ldots, x_\varkappa)$ and $Y=(Ux_1, \ldots, Ux_\varkappa)$ define the same subspace, i. e.

$$Y=XA \tag{20}$$

where A is a $\varkappa\times\varkappa$-matrix. As $X^+=1$ this implies $Y^+=X^+A=A$ and hence

$$Y=XY^+, \quad Y(Y^+)^- =X.$$

But this means that $V_U\xi=\xi$, where ξ is defined by

$$X=\begin{Vmatrix} 1 \\ \xi \end{Vmatrix}$$

i. e. ξ is a fixpoint of V_U. In other words:

IV. *The mappings $\mathfrak{M}_X\leftrightarrow X\leftrightarrow\xi$ in propositions* II *and* III *define a one-to-one correspondence $\mathfrak{M}_X\leftrightarrow\xi$ between all non-negative $\varkappa$-dimensional subspaces $\mathfrak{M}$, which are invariant with respect to U and all fixpoints ξ in Q_1 of V_U.*

5. Denote by Q_U the set of all fixpoints of V_U in Q. As V_U is continuous Q_U is closed. In virtue of IV our theorem will be proved if we show that the intersection of all Q_U $(U\in\mathfrak{U})$ is not void. But Q_1 being bicompact it suffices to prove that the intersection of every finite system $Q_{U_1}, \ldots, Q_{U_n}$ $(U_j\in\mathfrak{U})$ is not void. In virtue of IV this means that for every finite system $U_1, \ldots, U_n$ of commuting unitary operators there exists a non negative $\varkappa$-dimensional subspace, which is invariant with respect to every U_j $(j=1, \ldots, n)$.

We prove first the following weaker assertion:

V. *For any commuting unitary operators $U_1, \ldots, U_n$ in $\Pi_\varkappa$ a non-negative subspace $\mathfrak{N}\neq(0)$ (not necessarily $\varkappa$-dimensional) exists, which is invariant with respect to $U_1, \ldots, U_n$.*

We prove this proposition by induction with respect to n. For $n=1$ the assertion V follows from the assertion proved in section 3. We suppose that the assertion is true for some n and prove it to be also true for $n+1$.

Let $U_1, \ldots, U_n, U_{n+1}$ be commuting unitary operators in $\Pi_\varkappa$. By our assumption a non-negative subspace $\mathfrak{N}\neq(0)$ exists, which is invariant with respect to $U_1, \ldots, U_n$; by Lemma 1. 2 in [1] $\mathfrak{N}$ is finite dimensional and $\dim\mathfrak{N}\leqq\varkappa$. The restrictions of $U_1, \ldots, U_n$ to $\mathfrak{N}$ are commuting linear operators in the finite dimensional space $\mathfrak{N}$. Hence they have a common eigenvector, say $x_0\neq 0$ in $\mathfrak{N}$. Thus

$$U_jx_0=\lambda_jx_0 \quad \text{for } j=1, \ldots, n \tag{21}$$

where λ_j is the eigenvalue of U_j corresponding to x_0; as $x_0\in\mathfrak{N}$,

$$(x_0, x_0)\geqq 0. \tag{22}$$

Denote by $\mathfrak{N}'$ the set of all vectors $x \in \Pi_\varkappa$ satisfying

(23) $$U_j x = \lambda_j x \quad \text{for } j = 1, \dots, n.$$

Then by (24) we have $x_0 \in \mathfrak{N}'$.

Moreover we have

(24) $$U_{n+1}\mathfrak{N}' = \mathfrak{N}'.$$

In fact, if $x \in \mathfrak{N}'$, i. e. (23) holds, then $U_{n+1}U_j x = \lambda_j U_{n+1}x$, i. e. $U_j U_{n+1} x = \lambda_j U_{n+1} x$. This means that $U_{n+1}\mathfrak{N}' \subset \mathfrak{N}'$. Replacing in this argument U_{n+1} by U_{n+1}^{-1} we also get $U_{n+1}^{-1}\mathfrak{N}' \subset \mathfrak{N}'$, hence $\mathfrak{N}' \subset U_{n+1}\mathfrak{N}'$ concluding the proof of (24). As $\mathfrak{N}'$ contains the non-negative vector x_0', theorem 4. 4 in [1] can be applied to $\mathfrak{N}'$ and U_{n+1}. Thus $\mathfrak{N}'$ contains a non-negative subspace $\mathfrak{N}_0' \neq (0)$ which is invariant with respect to U_{n+1}. By (23) $\mathfrak{N}_0'$ is also invariant with respect to $U_1, \dots, U_n$. This concludes the proof of proposition V.

We prove now the following proposition:

VI. *Let* $\mathfrak{N} \neq (0)$ *be a non-negative subspace, which is invariant with respect to* $U_1, \dots, U_n$. *If* $\dim \mathfrak{N} < \varkappa$ *then a non-negative subspace* $\mathfrak{N}_1 \supset \mathfrak{N}$ *exists,* $\mathfrak{N}_1 \neq \mathfrak{N}$, *which is also invariant with respect to* $U_1, \dots, U_n$.

If proposition VI is proved, then applying it first to $\mathfrak{N}$, then to $\mathfrak{N}_1$, and so on, we get after a finite number of steps a $\varkappa$-dimensional non-negative subspace $\mathfrak{M}$ which is invariant with respect to $U_1, \dots, U_n$ and this concludes the proof of Theorem 1. So, we turn to the proof of proposition VI.

Let $\dim \mathfrak{N} = \varkappa_0 < \varkappa$. Only the following cases are possible:

a) $\mathfrak{N}$ is positive. Then $\mathfrak{N}^\perp$ is a space $\Pi_{\varkappa-\varkappa_0}$ and $\mathfrak{N}^\perp$ is also invariant[5]) with respect to $U_1, \dots, U_n$. Applying proposition V to the restrictions of $U_1, \dots, U_n$ to $\mathfrak{N}^\perp$ we get a non-negative subspace $\mathfrak{N}' \subset \mathfrak{N}^\perp$, $\mathfrak{N}' \neq (0)$, which is invariant with respect to $U_1, \dots, U_n$. Put $\mathfrak{N}_1 = \mathfrak{N} \oplus \mathfrak{N}'$. Then $\mathfrak{N} \subset \mathfrak{N}_1$, $\mathfrak{N} \neq \mathfrak{N}_1$, $\mathfrak{N}_1$ is non-negative and invariant with respect to $U_1, \dots, U_n$.

b) $\mathfrak{N}$ is a nullspace. Let G be a subspace in $\Pi_\varkappa$ skewly related to $\mathfrak{N}$ (*cf.* [1], definition 4. 1); put $F = \mathfrak{N} \dotplus G$ and $H = \mathfrak{N}^\perp$. Then F is a $2\varkappa_0$-dimensional space $\Pi_{\varkappa_0}$, hence $F^\perp$ is a space $\Pi_{\varkappa-\varkappa_0}$. Thus

$$\Pi_\varkappa = (\mathfrak{N} \dotplus G) \oplus \Pi_{\varkappa-\varkappa_0}.$$

Using the argument in the proof of Lemma 4. 1 in [1] we get

(25) $$H = \mathfrak{N}^\perp = \mathfrak{N} \oplus \Pi_{\varkappa-\varkappa_0}.$$

As $\mathfrak{N} \perp H$, relation (25) shows that the factor-space $\tilde{H} = H/\mathfrak{N}$ is isomorphic to $\Pi_{\varkappa-\varkappa_0}$ and hence is also a space $\Pi_{\varkappa-\varkappa_0}$. On the other hand, $\mathfrak{N}$ being invariant with respect to the unitary operators $U_1, \dots, U_n$ the subspace $H = \mathfrak{N}^\perp$ has the same property (see the footnote [5]); hence the U_j $(j = 1, \dots, n)$ induce commuting unitary operators $\tilde{U}_j$ $(j = 1, \dots, n)$ in $\tilde{H} = \Pi_{\varkappa-\varkappa_0}$. In virtue of V, there exists a non-negative

[5]) In fact as $\mathfrak{N}$ is finite dimensional, and U_j are unitary, we have $U_j\mathfrak{N} = \mathfrak{N}$ and therefore for $x \in \mathfrak{N}^\perp$, $y \in \mathfrak{N}$ we get

$$(U_j x, y) = (x, U_j^{-1} y) = 0$$

in virtue of $U_j^{-1} y \in \mathfrak{N}$. This shows, that $U_j x \in \mathfrak{N}^\perp$.

subspace $\tilde{\mathfrak{N}} \neq (0)$, $\tilde{\mathfrak{N}} \subset H$, which is invariant with respect to $\tilde{U}_1, \ldots, \tilde{U}_n$. Let f be the natural mapping of H onto $\tilde{H}$; put $\mathfrak{N}_1 = f^{-1}(\tilde{\mathfrak{N}})$. Then $\mathfrak{N} \subset \mathfrak{N}_1$, $\mathfrak{N} \neq \mathfrak{N}_1$, $\mathfrak{N}_1$ is non-negative and invariant with respect to $U_1, \ldots, U_n$.

c) $\mathfrak{N}$ is not a nullspace, but it contains nullvectors. By the Cauchy—Bunyakovsky inequality, valid in $\mathfrak{N}$, each such nullvector is isotropic for $\mathfrak{N}$, hence the set of all nullvectors in $\mathfrak{N}$ coincides with the isotropic subspace of $\mathfrak{N}$, which we denote by $\mathfrak{N}'$. By our assumption $(0) \neq \mathfrak{N}' \subset \mathfrak{N}$, $\mathfrak{N}' \neq \mathfrak{N}$, and therefore $0 < \varkappa' < \varkappa_0$, where $\varkappa' = \dim \mathfrak{N}'$. Let G be a subspace in $\Pi_\varkappa$, which is skewly related to $\mathfrak{N}'$. Put

$$\mathfrak{N}'' = \{x : x \in \mathfrak{N}, x \perp G\} = \mathfrak{N} \cap G^\perp.$$

Then

(26) $$\mathfrak{N} = \mathfrak{N}' \oplus \mathfrak{N}''.$$

In fact, $\mathfrak{N}', \mathfrak{N}'' \subset \mathfrak{N}$ and $\mathfrak{N}' \perp \mathfrak{N}$; hence $\mathfrak{N}' \oplus \mathfrak{N}'' \subset \mathfrak{N}$ and we have to prove the opposite inclusion $\mathfrak{N}' \oplus \mathfrak{N}'' \supset \mathfrak{N}$. By Lemma 4. 1 in [1] we have

$$\Pi_\varkappa = \mathfrak{N}' \dotplus G^\perp$$

so that any $x \in \Pi_\varkappa$ can be uniquely represented in the form $x = y + z$, where $y \in \mathfrak{N}'$, $z \in G^\perp$. If now $x \in \mathfrak{N}$, then $z = x - y \in \mathfrak{N}$ thus $z \in \mathfrak{N} \cap G^\perp = \mathfrak{N}''$ and $x = y + z \in \mathfrak{N}' \oplus \mathfrak{N}''$ concluding the proof of (26).

The subspace $\mathfrak{N}''$ is positive. In fact, if $x \in \mathfrak{N}''$ and $(x, x) = 0$ then x is an isotropic vector for $\mathfrak{N}$, hence $x \in \mathfrak{N}'$. Thus x is an element of $\mathfrak{N}'$, which is orthogonal to G; by the definition of G this is impossible if $x \neq 0$. The last argument show that $\mathfrak{N}' \cap \mathfrak{N}'' = (0)$, so that in virtue of (26)

(27) $$\varkappa_0 = \varkappa' + \varkappa'', \text{ where, } \varkappa'' = \dim \mathfrak{N}''.$$

Now put (*cf.* (26))

(28) $$F = \mathfrak{N} \dotplus G = (\mathfrak{N}'' \oplus \mathfrak{N}') \dotplus G = \mathfrak{N}'' \oplus (\mathfrak{N}' \dotplus G)$$

and

(29) $$H = \mathfrak{N}^\perp, \quad H' = F^\perp.$$

As $\mathfrak{N}''$ is a positive $\varkappa''$-dimensional subspace and $\mathfrak{N}' \dotplus G$ is a $2\varkappa'$-dimensional space $\Pi_{\varkappa'}$ equality (28) implies that F is a $2\varkappa' + \varkappa''$-dimensional space $\Pi_{\varkappa' + \varkappa''} = \Pi_{\varkappa_0}$. Therefore H' is a space $\Pi_{\varkappa - \varkappa_0}$. Moreover,

(30) $$H = H' \oplus \mathfrak{N}'.$$

In fact, as $F \supset \mathfrak{N}$, we have $H' = F^\perp \subset \mathfrak{N}^\perp = H$ and also $\mathfrak{N}' \subset H$, hence $H' \oplus \mathfrak{N}' \subset H$. So we have to prove the opposite relation $H \subset H' \oplus \mathfrak{N}'$, or what is the same $\mathfrak{N} = H^\perp \supset \supset (H' \oplus \mathfrak{N}')^\perp$. Let $x \in (H' \oplus \mathfrak{N}')^\perp$. Then $x \in H'^\perp = F$ and by (28) we have $x = y + z$, where $y \in \mathfrak{N}$, $z \in G$. On the other hand, we have $\mathfrak{N}' \perp \mathfrak{N}$, hence $y \perp \mathfrak{N}'$ and therefore $z = x - y \perp \mathfrak{N}'$. As G and $\mathfrak{N}'$ are skewly related, this implies $z = 0$; then $x = y \in \mathfrak{N}$ concluding the proof of (30).

The subspaces $\mathfrak{N}$ and $H = \mathfrak{N}^\perp$ are invariant with respect to $U_1, \ldots, U_n$. Hence $\mathfrak{N}' = \mathfrak{N}^\perp \cap \mathfrak{N} = H \cap \mathfrak{N}$ is also invariant with respect to $U_1, \ldots, U_n$ and therefore the $U_j (j = 1, \ldots, n)$ induce commuting unitary operators $\tilde{U}_j (j = 1, \ldots, n)$ in the factor-space

$$\tilde{H} = H/\mathfrak{N}'. \tag{31}$$

But in virtue of (30) $\tilde{H}$ is isomorphic to H' and therefore is a space $\Pi_{\varkappa-\varkappa_0}$.

By proposition V there exists a non-negative subspace $\tilde{\mathfrak{N}} \subset \tilde{H}$, $\tilde{\mathfrak{N}} \neq (0)$, which is invariant with respect to $\tilde{U}_1, \ldots, \tilde{U}_n$. Let f be the natural mapping of H onto $\tilde{H}$; put $\mathfrak{N}^* = f^{-1}(\tilde{\mathfrak{N}})$. Then $\mathfrak{N}^*$ is a non-negative subspace, which is invariant with respect to $U_1, \ldots, U_n$, $\mathfrak{N}^* \supset \mathfrak{N}'$, $\mathfrak{N}^* \neq \mathfrak{N}'$, and $\mathfrak{N}^* \subset H$; hence $\mathfrak{N}^* \perp \mathfrak{N}$. Put

$$\mathfrak{N}_1 = \mathfrak{N} \oplus \mathfrak{N}^*;$$

then $\mathfrak{N}_1$ is a non-negative subspace, which is invariant with respect to $U_1, \ldots, U_n$ and it remains to show that $\dim \mathfrak{N}_1 > \dim \mathfrak{N}$. To this end we note that

$$\mathfrak{N}' \subset \mathfrak{N} \cap \mathfrak{N}^* \subset \mathfrak{N} \cap H = \mathfrak{N}',$$

hence $\mathfrak{N} \cap \mathfrak{N}^* = \mathfrak{N}'$ and therefore

$$\dim \mathfrak{N}_1 = \dim \mathfrak{N} + \dim \mathfrak{N}^* - \dim \mathfrak{N}' > \dim \mathfrak{N},$$

concluding the proof of proposition VI and theorem 1.

Corollary 1. *For every family $\mathcal{H}$ of commuting bounded Hermitian operators in $\Pi_\varkappa$ there exists a $\varkappa$-dimensional non-negative subspace, which is invariant with respect to all operators of $\mathcal{H}$.*

Proof. Put for real t and $H \in \mathcal{H}$

$$U_t = e^{itH} = 1 + \frac{t}{1!}(iH) + \frac{t^2}{2!}(iH)^2 + \ldots.$$

Then the U_t form a commuting set of unitary operators in $\Pi_\varkappa$. By Theorem 1, there exsits a $\varkappa$-dimensional non-negative subspace $\mathfrak{M}$, which is invariant with respect to all e^{itH}, $H \in \mathcal{H}$, $t \in (-\infty, \infty)$. In virtue of the relation

$$\left\| \frac{1}{it}(e^{itH} - 1) - H \right\| \to 0 \quad \text{for} \quad t \to 0,$$

$\mathfrak{M}$ is also an invariant subspace for all $H \in \mathcal{H}$.

Corollary 2. *Let R be a commutative algebra of bounded operators in $\Pi_\varkappa$, satisfying the condition: $A \in R$ implies $A^* \in R$ where A^* is the adjoint operator with respect to (x, y) (i. e. $(Ax, y) = (x, A^*y)$ for all $x, y \in \Pi_\varkappa$). Then a non-negative $\varkappa$-dimensional subspace exists which is invariant with respect to all $A \in R$.*

Proof. Ler $\mathcal{H}$ be the set of all Hermitian operators from R. Then $\mathcal{H}$ satisfies the conditions of Corollary 1. Hence a $\varkappa$-dimensional non-negative subspace $\mathfrak{M}$ exists, which is invariant with respect to all $H \in \mathcal{H}$. If now $A \in R$, then also $A^* \in R$ and we have $A = H_1 + iH_2$, where $H_1 = \frac{1}{2}(A + A^*)$, $H_2 = \frac{1}{2i}(A - A^*)$. Thus

H_1, H_2 are Hermitian, $H_1, H_2 \in R$ and therefore $H_1, H_2 \in \mathfrak{H}$. As $\mathfrak{M}$ is invariant with respect to $H_1, H_2 \in \mathfrak{H}$ it is also invariant with respect to A.

The following Theorem 2 generalizes Corollary 2; assertion 2) of this theorem can be considered as an infinite dimensional generalization of the Lie theorem for solvable Lie algebras.

Theorem 2. *Let* $X_0, X_1, X_2, \ldots, X_m$ *be sets of linear bounded operators in* $\Pi_\varkappa$, *and* $H_0, H_1, \ldots, H_{m-1}$ *bounded Hermitian operators such that* a) $X_0 \supset X_1 \supset \ldots \supset X_m$; b) X_ν *is generated by* H_ν *and* $X_{\nu+1}$ *for* $\nu = 0, 1, \ldots, m-1$; c) $[H_\nu, A] = H_\nu A - AH_\nu \in X_{\nu+1}$ *for every* $A \in X_{\nu+1}$; d) X_m *is commutative and* $A \in X_m$ *implies* $A^* \in X_m$.

Then: 1) *there exists a non-negative* $\varkappa$*-dimensional subspace in* $\Pi_\varkappa$ *which is invariant with respect to all operators from* X_0; 2) *there exists a non-negative vector* $x_0 \in \Pi_\varkappa$, $x_0 \neq 0$ *which is a common eigenvector for all operators from* X_0.

Proof. We prove first by induction, that $A \in X_\nu$ implies $A^* \in X_\nu$ for $\nu = 0, 1, \ldots, m-1$. For $\nu = m$ this assertion follows from the condition d) of the theorem. Now we suppose the assertion is true for some $\nu+1$ and prove it to be true for ν. Let $A \in X_\nu$; then by condition b) $A = \alpha H_\nu + A_1$, where $A_1 \in X_{\nu+1}$, hence $A_1^* \in X_{\nu+1}$. But then $A^* = \bar{\alpha} H_\nu + A_1^* \in X_\nu$ and the assertion is proved for ν. Denote by $\mathfrak{H}_\nu$ the set of all Hermitian operators from X_ν. Using the assertion proved and applying the same argument as in the proof of Corollary 2 we see that every $A \in X_\nu$ has the form

$$A = H_1 + iH_2 \quad (H_1, H_2 \in \mathfrak{H}_\nu). \tag{32}$$

Now we prove assertion 2) by induction. For X_m the assertion follows from Corollary 1. In fact, by Corollary 1 a non-negative k-dimensional subspace $\mathfrak{M}$ exists, which is invariant with respect to all $H \in \mathfrak{H}_m$; in virtue of (32) $\mathfrak{M}$ is also invariant with respect to all $A \in X_m$. As $\mathfrak{M}$ is finite dimensional and invariant with respect to the commuting family X_m, there exists a vector $x_0 \in \mathfrak{M}$, $x_0 \neq 0$, which is a common eigenvector for all $A \in X_m$.

Now we suppose that assertion 2) holds for some $X_{\nu+1}$ and then prove it to hold for X_ν. By our assumption, there exists a non-negative vector $x_0 \neq 0$, $x_0 \in \Pi_\varkappa$, which is a common eigenvector for all $A \in X_{\nu+1}$, so that

$$Ax_0 = \lambda(A) x_0 \quad \text{for all} \quad A \in X_{\nu+1}, \tag{33}$$

where $\lambda(A)$ is a complex-valued linear function on $X_{\nu+1}$. Put

$$H_\nu^p x_0 = x_p \qquad (p = 0, 1, 2, 3, \ldots) \tag{34}$$

and

$$[A, H_\nu] = A^{(1)}, \qquad [A^{(p)}, H_\nu] = A^{(p+1)} \qquad (p = 0, 1, 2, \ldots) \tag{35}$$

where by definition $A^{(0)} = A$.

Then in virtue of condition c) of the theorem

$$A^{(p)} \in X_{\nu+1} \quad \text{for all} \quad A \in X_{\nu+1} \quad \text{and} \quad p=1, 2, 3, \ldots,$$

hence by (33) and (34)

$$Ax_1 = AH_\nu x_0 = [A, H_\nu]x_0 + H_\nu A x_0 = A^{(1)}x_0 + \lambda(A)H_\nu x_0 = \\ = \lambda(A^{(1)})x_0 + \lambda(A)x_1.$$

Repeating the same argument we easily obtain by induction, that

(36) $$Ax_p = \lambda(A)x_p + p\lambda(A^{(1)})x_{p-1} + C_p^2\lambda(A^{(2)})x_{p-2} + \cdots \\ \cdots + C_p^q\lambda(A^{(q)})x_{p-q} + \cdots + \lambda(A^{(p)})x_0$$

holds for all $A \in X_{\nu+1}$ and all $p=1, 2, 3, \ldots$.

We show that in fact $\lambda(A^{(1)}) \equiv 0$ and hence also $\lambda(A^{(p)}) \equiv 0$ for all $p=1, 2, 3, \ldots$ and $A \in X_{\nu+1}$. Suppose the contrary; let $\lambda(A^{(1)}) \not\equiv 0$; then in virtue of (32) also $\lambda(A^{(1)}) \not\equiv 0$ on $\mathcal{H}_{\nu+1}$. Only the following cases can occur:

Case α): $\lambda(A)$ is not real for some $A = A_0 \in \mathcal{H}_{\nu+1}$. Then $(x_0, x_0)=0$. We show by induction, that

(37) $$(x_q, x_r)=0$$

holds for all $q, r=1, 2, \ldots$. First we remark, that

(38) $$(x_q, x_r) = (H_\nu^q x_0, H_\nu^2 x_0) = (H_\nu^{q+2}x_0, x_0),$$

so that (x_q, x_r) depends only on $q+r$.

We have seen that $(x_0, x_0)=0$, hence our assertion holds for $q+r=0$. We suppose it is true for $q+r<p$ and prove it to be true for $q+r=p$. To this end we take the inner product of both sides of (36) with x_0. Then by our inductive assumption we get

$$(Ax_p, x_0) = \lambda(A)(x_p, x_0)$$

and on the other hand for $A \in \mathcal{H}_{\nu+1}$ we have

$$(Ax_p, x_0) = (x_p, Ax_0) = (x_p, \lambda(A)x_0) = \overline{\lambda(A)}(x_p, x_0);$$

thus

$$[\lambda(A) - \overline{\lambda(A)}](x_p, x_0) = 0.$$

But $\lambda(A_0) - \overline{\lambda(A_0)} \neq 0$, hence $(x_p, x_0)=0$ concluding the proof of (37). Denote by $\mathfrak{M}$ the closed subspace generated by all x_p $(p=0, 1, 2, \ldots)$. By (34) $\mathfrak{M}$ is invariant with respect to H_ν. In virtue of (37) $\mathfrak{M}$ is a nullspace in $\Pi_\varkappa$ and hence dim $\mathfrak{M} \leqq \varkappa$, $\mathfrak{M}$ is finite-dimensional. Relations (36) show, that $\mathfrak{M}$ is also invariant with respect to A. Let $\tilde{A}, \tilde{H}_\nu$ be the restrictions of A and H_ν to $\mathfrak{M}$; then (36) holds also for these $\tilde{A}$ and $\tilde{H}_\nu$. Put in (36) $\tilde{A}^{(1)}$ instead of $\tilde{A}$; then we obtain

$$\tilde{A}^{(1)}x_p = \lambda(A^{(1)})x_p + p\lambda(A^{(2)})x_{p-1} + \cdots + \lambda(A^{(p+1)})x_0 \qquad (p=0, 1, 2, \ldots).$$

These equalities show that by our assumption

$$\mathrm{Tr}(\tilde{A}^{(1)}) = \lambda(A^{(1)}) \dim \mathfrak{M} \neq 0 \quad \text{for some} \quad A \in \mathcal{H}_{\nu+1}$$

where $\mathrm{Tr}(\tilde{A})$ denotes the trace of $\tilde{A}$.

On the other hand we have

$$\mathrm{Tr}(\tilde{A}^{(1)}) = \mathrm{Tr}(\tilde{A}\tilde{H}_\nu - \tilde{H}_\nu \tilde{A}) = \mathrm{Tr}(\tilde{A}\tilde{H}_\nu) - \mathrm{Tr}(\tilde{H}_\nu \tilde{A})$$

and we get a contradiction which shows that $\lambda(A^{(1)}) \not\equiv 0$ is impossible in case α).

Case β): $\lambda(A)$ is real for all $A \in \mathcal{H}_{\nu+1}$. For $A \in \mathcal{H}_{\nu+1}$ we have

$$A^{(1)*} = (AH_\nu - H_\nu A)^* = H_\nu A - AH_\nu = -A^{(1)},$$

thus $A^{(1)}$ has the form

$$A^{(1)} = iA_1$$

where A_1 is Hermitian. Hence

$$\lambda(A^{(1)}) = i\mu(A^{(1)}), \tag{39}$$

where $\mu(A^{(1)}) = \lambda(A_1)$ is a real number (as $\lambda(A)$ is real on $\mathcal{H}_{\nu+1}$) which is $\not\equiv 0$ on $\mathcal{H}_{\nu+1}$ (by our assumption that $\lambda(A^{(1)}) \not\equiv 0$). We show that also in this case relations (37) hold; then repeating the argument used in case α) we also get a contradiction, proving that $\lambda(A^{(1)}) \not\equiv 0$ is impossible also in case β).

By (36), for $p=1$ we have

$$Ax_1 = \lambda(A)x_1 + \lambda(A^{(1)})x_0,$$

hence

$$(Ax_1, x_0) = \lambda(A)(x_1, x_0) + \lambda(A^{(1)})(x_0, x_0).$$

On the other hand, if $A \in \mathcal{H}_{\nu+1}$, we have

$$(Ax_1, x_0) = (x_1, Ax_0) = (x_1, \lambda(A)x_0) = \lambda(A)(x_1, x_0),$$

hence

$$\lambda(A^{(1)})(x_0, x_0) = 0.$$

As $\lambda(A^{(1)}) \not\equiv 0$ we have $(x_0, x_0) = 0$ and so (37) holds for $q+r = 0$. Now we suppose that (37) holds for $q+r<p$ and prove it to be true for $q+r = p$. To this end we take the inner product of both sides of (36) with x_1. In virtue of our inductive assumption we get

$$(Ax_p, x_1) = \lambda(A)(x_p, x_1) + p\lambda(A^{(1)})(x_{p-1}, x_1). \tag{40}$$

On the other hand if $A \in \mathcal{H}_{\nu+1}$ we have in virtue of (38) and (39)

$$\begin{aligned}(Ax_p, x_1) &= (x_p, Ax_1) = (x_p, \lambda(A)x_1 + \lambda(A^{(1)})x_0) = \\ &= \lambda(A)(x_p, x_1) + \overline{\lambda(A^{(1)})}(x_p, x_0) = \lambda(A)(x_p, x_1) - \lambda(A^{(1)})(x_{p-1}, x_1)\end{aligned} \tag{41}$$

and comparing (40) and (41) we see that

$$(p+1)\lambda(A^{(1)})(x_{p-1}, x_1) = 0.$$

As $p+1>0$, $\lambda(A^{(1)}) \not\equiv 0$ we must have $(x_{p-1}, x_2) = 0$ concluding the proof of (37).

So we have proved that in every case $\lambda(A^{(1)}) \equiv 0$ and relations (36) take the form

$$Ax_p = \lambda(A)x_p \quad \text{for} \quad p = 0, 1, 2, \dots \quad \text{and} \quad A \in X_{p+1}.$$

Hence we have also on the closed subspace $\mathfrak{M}$ generated by x_p $(p=0, 1, 2, \ldots)$

(42) $$Ax=\lambda(A)x \text{ for all } x\in\mathfrak{M} \text{ and } A\in X_{\nu+1};$$

on the other hand $\mathfrak{M}$ is invariant with respect to the Hermitian operator H_ν. The subspace $\mathfrak{M}$ contains the non-negative vector $x_0\neq 0$; hence only the following three cases $\alpha')-\gamma')$ are possible:

α') $\mathfrak{M}$ is non-negative. Then $\dim\mathfrak{M}\leqq k$ and H_ν has an eigenvector $y\neq 0$ in $\mathfrak{M}$, which in virtue of (42) is also an eigenvector of all $A\in X_{\nu+1}$; by condition b) of the theorem, y is a common eigenvector for all $A\in X_\nu$ and y is non-negative as $\mathfrak{M}$ is non-negative.

β') (x, x) changes its sign on $\mathfrak{M}$ and the inner product (x, y) is non-degenerate on $\mathfrak{M}$. Then $\mathfrak{M}$ is a space $\Pi_{\varkappa'}$ and by PONTRYAGIN's theorem (see also Corollary 1) $\mathfrak{M}$ has a $\varkappa'$-dimensional non-negative subspace $\mathfrak{N}$ which is invariant with respect to H_ν. Let $y\neq 0$ be an eigenvector of H_ν in $\mathfrak{N}$; then y is non-negative and by (42) it is also an eigenvector for all $A\in X_{\nu+1}$. Hence by condition b) it is also a common eigenvector for all $A\in X_\nu$.

γ') (x, x) changes its sign on $\mathfrak{M}$ and the scalar product (x, y) degenerates on $\mathfrak{M}$. Let $\mathfrak{N}$ be the isotropic subspace of $\mathfrak{M}$, i. e. $\mathfrak{N}=\mathfrak{M}\cap\mathfrak{M}^\perp$. As $\mathfrak{M}$ is invariant with respect to H_ν, the subspaces $\mathfrak{M}^\perp$ and $\mathfrak{N}$ have the same property.

But $\mathfrak{N}$ is a nullspace, hence $\dim\mathfrak{N}\leqq\varkappa$ and therefore H_ν has an eigenvector $y\neq 0$ in $\mathfrak{N}$. Repeating the argument at the end of β') we see, that y is a common non-negative eigenvector of all $A\in X_{\nu+1}$. This concludes the proof of assertion 2).

Assertion 2) means that a non-negative subspace (of dimension $\geqq 1$ and $\leqq\varkappa$) exists which is invariant with respect to all $A\in X$. Using this fact and repeating the argument in the proof of proposition VI we see that if $\dim\mathfrak{N}<\varkappa$, then $\mathfrak{N}\subset\mathfrak{N}_1$, $\mathfrak{N}\neq\mathfrak{N}_1$, where $\mathfrak{N}_1$ is also non-negative and invariant with respect to all $A\in X$. As in the proof of theorem 1, this proves assertion 1) of theorem 2.

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