Compactification, Baire functions, and Daniell integration

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Introduction

The purpose of this paper is to expose precise relations within and between the domains of compactification, Baire functions, and Daniell integration. The principal method exploited to this end is a new topology, called the ι-topology, which we have developed on an earlier occasion [5], which when introduced into any compact space is finer than the compact or β-topology. Our interest in compactification centers upon the more delicate notions of pseudo-, countable, and real-compactifications. It seems fair to say that up to the present time these have not always been defined and approached in such a way as to make their theories completely transparent. In terms of the ι-topology, all the essential results are quite elementary.

In order to carry through this study, it is necessary to construct the standard compactification. We abandon the traditional notion of compactifying a completely regular space with respect to the set of *all* bounded continous functions in favor of the much more precise setting made available by contemporary topological algebra of compactifying an arbitrary set of points with respect to a Banach algebra of functions defined over it. The compactification is of course the structure space. This procedure gives *all* compactifications, not merely the Stone—Čech compactification, a very real advantage related to the fact that the notion of an algebra is finer than that of a topology (distinct algebras correspond to the same topology). The standard compactification is carried through using the notion of maximal positive cones of functions. This method is essentially equivalent to that of maximal convex ideals. It may be pointed out in passing that the method of filters of zero-sets has the very real disadvantage that a most interesting class of singular functions, precisely those defining the pseudocompactification have no zeros and therefore are automatically excluded from any filter procedure.

It should be mentioned in this connection that the question of obtaining all compactifications of a given completely regular space has been studied in detail by YU. SMIRNOV who based his results on the theory of proximities. We shall not elaborate on the rapport between these two formulations of the problem.

The compactification of a set $\mathscr{E}$ with respect to a Banach algebra of functions $\mathbf{B}$ is obtained by adjoining to $\mathscr{E}$ all maximal positive cones. These are of two types,

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the strong which contain functions (singular!) with empty zero-sets and the residual or weak cones. The realcompactification is obtained by adjoining to the base space $\mathscr{E}$ the set of points consisting of the weak cones. If one adjoins the strong cones, one obtains a pseudocompactification (equivalent to countable compactification for zero-sets). Adjoing to $\mathscr{E}$ both types, one obtains the compact space $\mathscr{E}^\frown$ and the algebra $\mathbf{B}^\frown$ of all continuous functions over $\mathscr{E}^\frown$. The β-topology over $\mathscr{E}^\frown$ is the weak topology induced by the functions of $\mathbf{B}^\frown$.

Starting with the functions in $\mathbf{B}^\frown$, one constructs the algebra $\mathbf{I}$ of all bounded Baire functions over $\mathscr{E}^\frown$. The ι-topology is the weak topology induced over $\mathscr{E}$ by the functions of $\mathbf{I}$. It is central to all the problems we consider. For example, the realcompactification of $\mathscr{E}$ introduced by E. HEWITT is the ι-closure in $\mathscr{E}^\frown$ of $\mathscr{E}$. Or again, $\mathscr{E}$ is pseudocompact if and only if it is ι-dense in $\mathscr{E}^\frown$. These topological facts explain why there is a unique realcompactification but no unique pseudocompactification. The latter is characterized in nine distinct ways.

While on this subject one or two remarks may be made. The points corresponding to weak maximal cones are topologically very complicated to describe and for all purposes of denumerable analysis seem superfluous. Thus it may be that for a large class of problems pseudocompactness rather than compactness is desirable. For another thing, it would seem that the common distinction in the literature of two algebras over $\mathscr{E}$, that of the bounded and that of the unbounded functions, seems largely unnecessary. A space admits an unbounded continuous function if and only if it admits a bounded singular function without zeros.

Turning to integration, the nexus of questions concerning Daniell integrals can be elucidated in the frame of this investigation. Necessary and sufficient conditions are given on the measure distribution over $\mathscr{E}^\frown$ of a linear functional F in order that F be a Daniell integral or an anti-integral over $\mathscr{E}$. Finally, it is shown that the space $\mathscr{E}^\frown$ is realcompact with respect to the algebra $\mathbf{I}$.

For general information concerning rings of continuous functions, the reader is referred to the comprehensive work of GILLMAN and JERISON [1]. The author wishes to express his thanks to LEONARD GILLMAN and to HING TONG for reading the manuscript carefully and making many suggestions. In particular, TONG has pointed out that the principal purpose of the algebraic device of a cone $\mathfrak{C}$ is to introduce the class of $\mathscr{E}$-vicinities associated with it. This class contains more information than the standard filters. Once the family of these classes is obtained the proofs proceed essentially along topological lines.

1. Notations

We introduce the fundamental structures to be considered throughout this paper. We consider an arbitrary set $\mathscr{E}$ of points $x, y, z, \ldots$. Real valued functions on $\mathscr{E}$ will be represented by $f, g, h, \ldots$; thus $f(x)$ is real for each $x \in \mathscr{E}$. We consider, in particular, a Banach algebra $\mathbf{B}$ of bounded functions of this type. Note that $\mathbf{B}$ is commutative. The norm in $\mathbf{B}$ is the uniform one: $\|f\| = \sup_{x \in \mathscr{E}} |f(x)|$. The set $\mathbf{B}$ is assumed to contain the constants; the constant function "one" is indicated by $e: e(x) = 1$, $x \in \mathscr{E}$. Real numbers will frequently be represented by $\alpha, \beta, \lambda, \mu, \ldots$.

We shall suppose that **B** distinguishes points of $\mathscr{E}$: given $x, y \in \mathscr{E}$, $x \neq y$, there exists $f \in \mathbf{B}$ such that $f(x) \neq f(y)$.

The algebra **B** is, as is well known, a lattice. The lattice operations are indicated by $f \vee g$ and $f \wedge g$. Also, $f^+ = f \vee 0$; $f^- = (-f) \vee 0$; and $|f| = f^+ + f^-$. If $f \in \mathbf{B}$, we say f is *singular* in case there exists no g in **B** such that $f \cdot g = e$. As usual, $f \geqq 0$ if and only if $f = |f|$. If f is not singular, it is called *regular*.

We shall use frequently the following fact: *Let $f \in \mathbf{B}$ and suppose $f \geqq 0$. Then f is singular if and only if for each $\varepsilon > 0$ there exists an $x \in \mathscr{E}$ such that $f(x) < \varepsilon$.* We prove the non-trivial portion of this statement. Suppose f is bounded away from zero. To simplify the computation, assume also that $\|f\| = 1$. Thus for $x \in \mathscr{E}$, $0 \leqq 1 - f(x) \leqq 1 - \varepsilon$ for some ε, $0 < \varepsilon < 1$, and $\|e - f\| < 1$. Thus $f^{-1} = \sum_{n=0}^{\infty} (e-f)^n$.

We shall write $f \rhd 0$ to indicate that $f \geqq 0$, that f is singular, and that for each $x \in \mathscr{E}, f(x) > 0$. For any $f \in \mathbf{B}$, the set of zeros of f, $\{x : x \in \mathscr{E}, f(x) = 0\}$, will be denoted by $\mathfrak{Z}(f)$. Thus $f \rhd 0$ implies $\mathfrak{Z}(f) = \varnothing$. We shall refer to $\mathfrak{Z}(f)$ as the zero set of f.

An elementary but critically important fact about zero-sets is: *The intersection of denumerably many zero-sets is a zero-set.* Proof: Let $\{f_n\}$ be a sequence in **B**. Replacing f_n by $g_n = |f_n| \wedge e$, we see that $\mathfrak{Z}(f_n) = \mathfrak{Z}(g_n)$ and that $\|g_n\| \leqq 1$.

The function $f = \sum_{n=1}^{\infty} 2^{-n} g_n$ is clearly in **B** and $\mathfrak{Z}(f) = \bigcap_{n=1}^{\infty} \mathfrak{Z}(f_n)$.

2. Cones of functions

The fundamental idea basic to our further discussion is that of cone of positive functions to which, for short, we shall refer as a positive cone. *A non-empty set $\mathfrak{C}$ of functions f in **B** is said to form a positive cone* (p. c) if: (1) $f \in \mathfrak{C}$ *implies* $f \geqq 0$; (2) $f \in \mathfrak{C}$ *implies f is singular*; (3) $f, g \in \mathfrak{C}$ imply $\alpha f + \beta g \in \mathfrak{C}$ for $\alpha > 0$, $\beta > 0$. (Actually, positive homogeneity is not necessary in the definition; it is automatic for maximal positive cones.)

The totality of positive cones may be partially ordered by set inclusion. By ZORN's lemma, there exists a maximal positive cone (m. p. c) containing any given positive cone. By a *weak maximal positive cone* $\mathfrak{C}$ (w. m. p. c) is meant a m. p. c. such that $f \in \mathfrak{C}$ implies that there exists $x \in \mathscr{E}$ such that $f(x) = 0$. In other words $\mathfrak{C}$ is a w. m. p. c. if and only if $\mathfrak{C}$ is a m. p. c. and $\mathfrak{C}$ contains no $f \rhd 0$. By a strong maximal positive cone $\mathfrak{C}$ (s. m. p. c.) is meant a m. p. c. which is not weak, hence one which contains at least one f for which $f \rhd 0$.

A p. c. $\mathfrak{C}$ is called a very strong positive cone (v. s. p. c) providing $f \in \mathfrak{C}$ implies $f \rhd 0$. By ZORN's lemma, there exists a maximal very strong positive cone (m. v. s. p. c.) containing any given v. s. p. c. Note that if $\mathfrak{C}$ is a m. v. s. p. c. then $\mathfrak{C}$ is not a m. p. c. since $0 \notin \mathfrak{C}$ whereas 0 is in every m. p. c.

Let $\mathfrak{C}$ be a m. p. c. Suppose $f \in \mathfrak{C}$ and that g in **B** is such that $0 \leqq g \leqq f$. Then $g \in \mathfrak{C}$. This may be seen from fact that $\{\alpha g + h : \alpha \geqq 0, h \in \mathfrak{C}\}$ is a p. c. including $\mathfrak{C}$. Since $\mathfrak{C}$ is maximal, $g \in \mathfrak{C}$. In particular, if $\mathfrak{C}$ is a m. p. c., and if we have $f \in \mathfrak{C}$, $g \in \mathbf{B}$, $g \geqq 0$, then $f + g \in \mathfrak{C}$ implies $g \in \mathfrak{C}$.

Suppose $\mathfrak{C}$ *is a* m. p. c. *Let* $\{f_n\}$ *be a sequence such that* $f_n \in \mathfrak{C}$ *and* $f = \sum_{n=1}^{\infty} f_n$ *converges uniformly. Then* $f \in \mathfrak{C}$. The proof follows. Note that $f \geqq 0$; also, since $f_1 + \ldots + f_n$ (which is in $\mathfrak{C}$) is singular and by the uniform convergence of the above series, f is singular. The set $\{h: h = \alpha f + g, \alpha \geqq 0, g \in \mathfrak{C}\}$ is clearly a p. c. containing $\mathfrak{C}$. Since $\mathfrak{C}$ is maximal, $f \in \mathfrak{C}$.

Let x_0 be a fixed point in $\mathscr{E}$ and let $\mathfrak{C} = \{f: f \geqq 0, f(x_0) = 0\}$. Then $\mathfrak{C}$ is evidently a p. c. It is also a m. p. c. (and hence a w. m. p. c) as we shall show. For if $\mathfrak{C}$ were not maximal, it would be contained properly in a m. p. c. $\mathfrak{C}'$ and hence there would exist in $\mathfrak{C}'$ a function g such that $g(x_0) = 1$. Since $\mathfrak{C}'$ is maximal $e \wedge g \in \mathfrak{C}'$. Writing $h = e - e \wedge g$ we see that $h \geqq 0$ and that $h(x_0) = 0$ and hence $h \in \mathfrak{C}$. Thus $h + e \wedge g = e$ is in $\mathfrak{C}'$. This is impossible since e is regular. Thus $\mathfrak{C}$ is maximal.

A m. p. c. such as $\mathfrak{C}$ in the preceding paragraph is called a *fixed* m. p. c. All other m. p. c. are *free*. From now on, unless the contrary is expressly indicated, the phrase m. p. c. will always refer to a free m. p. c.

Let $\mathfrak{W}$ denote the class of w. m. p. c.; let $\mathfrak{S}$ denote the class of s. m. p. c.; let $\mathfrak{VS}$ denote the class of m. v. s. p. c. Note that by definition, $\mathfrak{W} \cap \mathfrak{S} = \emptyset$. We shall write also, abusing somewhat notational niceties, $\mathfrak{S} \cap \mathscr{E} = \emptyset$ and $\mathfrak{W} \cap \mathscr{E} = \emptyset$. We shall establish a $1-1$ correspondence between $\mathfrak{S}$ and $\mathfrak{VS}$.

Theorem 1. *Let* $\mathfrak{C} \in \mathfrak{S}$ *and let* $\mathfrak{C}' = \{f: f \in \mathfrak{C}, f \rhd 0\}$. *Then* $\mathfrak{C}'$ *is a* m. v. s. p. c. *The mapping* $\Phi: \mathfrak{S} \to \mathfrak{VS}$ *defined by* $\Phi(\mathfrak{C}) = \mathfrak{C}'$ *is a bijection.*

Proof. We show first that $\mathfrak{C}'$ is a m. v. s. p. c. If $f, g \in \mathfrak{C}, f \rhd 0$ and $g \rhd 0$ then $f + g \rhd 0$ (note that $f + g \in \mathfrak{C}$ and hence is singular) and $\alpha f \rhd 0$ for $\alpha > 0$. Thus $\mathfrak{C}'$ is a v. s. p. c. It is easy to see that $\mathfrak{C}$ may be reconstructed from $\mathfrak{C}'$ as follows: For $f \in \mathfrak{C}'$ consider the set $\mathfrak{M}_f$ of all $g \geqq 0$ such that $g \leqq f$. Then $\mathfrak{C} = \bigcup_{f \in \mathfrak{C}'} \mathfrak{M}_f$. (Note that if $g \in \mathfrak{C}, f \in \mathfrak{C}'$ then $g + f \in \mathfrak{C}'$ and $g \leqq g + f$.) This argument shows that the mapping Φ is an injection (one-to-one) into the set of v. s. p. c.

Let $\mathfrak{D}$ be any m. v. s. p. c. which contains $\mathfrak{C}'$. Let $\mathfrak{E}$ be any m. p. c. which contains $\mathfrak{D}$. Then $\Phi(\mathfrak{E}) \supset \mathfrak{D}$ and since $\mathfrak{D}$ is a m. v. s. p. c., $\Phi(\mathfrak{E}) = \mathfrak{D}$. By the preceding paragraph, since $\mathfrak{D} \supset \mathfrak{C}'$, $\mathfrak{E} \supset \mathfrak{C}$. Since $\mathfrak{C}$ is maximal $\mathfrak{E} = \mathfrak{C}$ and $\mathfrak{D} = \mathfrak{C}'$. This shows that the range of Φ lies in $\mathfrak{VS}$; a slight variation to the above argument shows that Φ is in fact a bijection on $\mathfrak{VS}$.

By virtue of this theorem we may identify the sets $\mathfrak{S}$ and $\mathfrak{VS}$.

3. Adjunction of ideal points

In this section we adjoin "ideal" points to $\mathscr{E}$ produce various compactifications. Also we show how to extend any function f in **B** to these new points. The adjunction of points is a simple matter. Each m. p. c. $\mathfrak{C}$ will be called a point and will be denoted, according to need, by $\mathfrak{C}$, $x_{\mathfrak{C}}$ or x. The extension of the functions to these new points requires some investigation. In the meanwhile we may introduce the following

Definition 1. *The set* $\mathscr{E} \cup \mathfrak{W}$ *will be denoted by* $\mathscr{E}'$. *The set* $\mathscr{E} \cup \mathfrak{S}$ *will be denoted by* $\mathscr{E}^{\backprime}$. *The set* $\mathscr{E} \cup \mathfrak{W} \cup \mathfrak{S}$ *will be denoted by* $\mathscr{E}^{\frown}$.

Note that $\mathscr{E}' \cap \mathscr{E}^{\backprime} = \mathscr{E}$; $\mathscr{E}' \cup \mathscr{E}^{\backprime} = \mathscr{E}^{\frown}$.

Our first step toward extending the functions of **B** to the new points in $\mathscr{E}^\frown$ is to introduce a notion of $\mathscr{E}$-vicinity of an ideal point. Let $\mathfrak{C}$ be any m. p. c. and let $f \in \mathfrak{C}$, $\varepsilon > 0$. Then the set $\mathfrak{U}(f, \varepsilon) = \{x : f(x) < \varepsilon, x \in \mathscr{E}\}$ is called an *$\mathscr{E}$-vicinity* of $\mathfrak{C}$. Note that since f is singular, $\mathfrak{U}(f, \varepsilon)$ is not empty. If $\mathfrak{U}_1 = \mathfrak{U}(f_1, \varepsilon_1)$ and $\mathfrak{U}_2 = \mathfrak{U}(f_2, \varepsilon_2)$ are given, then, setting $\varepsilon = \inf(\varepsilon_1, \varepsilon_2)$, $f = f_1 + f_2$, and $\mathfrak{U} = \mathfrak{U}(f, \varepsilon)$, we have $\mathfrak{U} \subset \mathfrak{U}_1 \cap \mathfrak{U}_2$. Thus the intersection of a finite number of $\mathscr{E}$-vicinities contains an $\mathscr{E}$-vicinity.

Theorem 2. *Let $\mathfrak{C}$ be any* m. p. c. *Let $f \in$* B. *Then there exists a unique real number $\lambda(\lambda = \lambda(f, \mathfrak{C}))$ with the property: For every $\varepsilon > 0$ there exists an $\mathscr{E}$-vicinity $\mathfrak{U}$ such that $x \in \mathfrak{U}$ implies $|f(x) - \lambda| < \varepsilon$. The function $|f|$ is in $\mathfrak{C}$ if and only if $\lambda = 0$.*

Proof. For an arbitrary $\delta > 0$ and g in $\mathfrak{C}$, let $\mathfrak{U} = \mathfrak{U}(g, \delta)$. For $f \in \mathbf{B}$, let $M_{\mathfrak{U}} = \{\alpha : x \in \mathfrak{U}$ and $f(x) = \alpha\}$. The sets $M_{\mathfrak{U}}$ have the finite intersection property. Since for each $\mathfrak{U}$, $M_{\mathfrak{U}}$ lies in the closed interval $\{\alpha : |\alpha| \leqq \|f\|\}$, there is a point λ common to the closure of all sets $M_{\mathfrak{U}}$. The function $|f - \lambda e|$ is singular and the above argument demonstrates the singularity of $|f - \lambda e| + g$ where $g \in \mathfrak{C}$. Since $\mathfrak{C}$ is maximal, $|f - \lambda e| \in \mathfrak{C}$. If λ and μ belong to the closure of all sets $M_{\mathfrak{U}}$ then since $|\lambda - \mu| e \leqq |f - \lambda e| + |f - \mu e| \in \mathfrak{C}$, it is clear that $\lambda = \mu$. Now let $\varepsilon > 0$ be given. Let $\mathfrak{U} = \mathfrak{U}(|f - \lambda e|, \varepsilon)$. Then $x \in \mathfrak{U}$ implies $|f(x) - \lambda| < \varepsilon$.

Suppose $f \in \mathfrak{C}$ and let the number associated to f be λ. Then the previous paragraph shows that $|f - \lambda e| \in \mathfrak{C}$. Since $|f - 0e| \in \mathfrak{C}$ and since the number λ is unique, we have $\lambda = 0$. Now suppose that $f \in \mathbf{B}$ and that for the associated number λ, we have $\lambda = 0$. Then, as above, $|f - \lambda e| \in \mathfrak{C}$, that is, $|f| \in \mathfrak{C}$. This completes the proof.

Theorem 3. *Let $\mathfrak{C}$ be a* m. p. c. *and let $\Psi = \Psi_{\mathfrak{C}}$ be the mapping: $f \to \lambda$ of the preceding theorem. Then $\Psi_{\mathfrak{C}}$ is an algebra and lattice homomorphism. Thus if $\psi_{\mathfrak{C}} f = \lambda$ and $\psi_{\mathfrak{C}} g = \mu$, then*

(1) $\psi_{\mathfrak{C}} \alpha f = \alpha \lambda$;

(2) $\psi_{\mathfrak{C}}(f + g) = \lambda + \mu$;

(3) $\psi_{\mathfrak{C}}(f \cdot g) = \lambda \cdot \mu$;

(4) $\psi_{\mathfrak{C}} |f| = |\lambda|$.

If in an $\mathscr{E}$-vicinity $\mathfrak{U}$, $f(x)$ is close to λ and if in $\mathfrak{V}$, $g(x)$ is close to μ, then in $\mathfrak{U} \cap \mathfrak{V}$, $f + g$ is close to $\lambda + \mu$ and $f \cdot g$ is close to $\lambda \cdot \mu$. Similarly for the remaining cases.

Given $f \in \mathbf{B}$, we shall extend it to be a function $f^\frown$ over $\mathscr{E}^\frown$ in the following manner: if $x \in \mathfrak{W} \cup \mathfrak{S}$, that is, if $x \in \mathscr{E}^\frown - \mathscr{E}$, we set $f^\frown(x) = \lambda$, where the number λ is defined in the theorem 2. If $x \in \mathscr{E}$, we set $f^\frown(x) = f(x)$. By virtue of theorem 3, the algebraic and lattice-theoretic operations on the functions f commute with the extension operation "$^\frown$". That is, $(f+g)^\frown(x) = f^\frown(x) + g^\frown(x) = (f^\frown + g^\frown)(x)$, hence $(f+g)^\frown = f^\frown + g^\frown$; etc.

The totality of functions $f^\frown$ is called $\mathbf{B}^\frown$. It is clear from theorem 2 that $\sup_{x \in \mathscr{E}^\frown} |f^\frown(x)| = \sup_{x \in \mathscr{E}} |f(x)|$, hence we may introduce the supremum norm in $\mathbf{B}^\frown$ and we have $\|f^\frown\| = \|f\|$. This (and the preceding paragraph) shows that the mapping

$f \to f^\frown$ is an algebraic, lattice-theoretic, and metric isomorphism of $\mathbf{B}$ onto $\mathbf{B}^\frown$. In particular the spaces have isomorphic adjoint (or dual) spaces.

If $f^\frown \in \mathbf{B}^\frown$, and if $\mathscr{E}^\sim$ is a subset of $\mathscr{E}^\frown$ for which $\mathscr{E}^\sim \supset \mathscr{E}$, we shall denote by $f^\sim$ the restriction of $f^\frown$ to $\mathscr{E}^\sim$. The totality of functions $f^\sim$ will be denoted by $\mathbf{B}^\sim$. As before we see that the mapping $f \to f^\sim$ is an algebraic, lattice-theoretic, and metric isomorphism of $\mathbf{B}$ onto $\mathbf{B}^\sim$. If, in particular, $\mathscr{E}^\sim = \mathscr{E}'$ ($\mathscr{E}^\sim = \mathscr{E}`$), we shall be concerned with functions $f' \in \mathbf{B}'$ ($f` \in \mathbf{B}`$).

We shall introduce a topology in $\mathscr{E}^\frown$. Consider the family of zero-sets of the functions of $\mathbf{B}^\frown$: $\{\mathfrak{Z}(f^\frown): f^\frown \in \mathbf{B}^\frown\}$. Note that if $x \notin \mathfrak{Z}(f^\frown)$ and $x \notin \mathfrak{Z}(g^\frown)$, then $x \notin \mathfrak{Z}(f^\frown \cdot g^\frown)$. Thus the zero-sets of $\mathbf{B}^\frown$ are the base for the closed sets of a topology. It is this topology which is introduced into $\mathscr{E}^\frown$. It is easy to see that this topology is precisely the weak topology on $\mathscr{E}^\frown$; that is, it is the coarsest topology which renders continuous all the functions in $\mathbf{B}^\frown$. The topology thus introduced in $\mathscr{E}^\frown$ will be called the *β-topology*. The β-topology on $\mathscr{E}^\frown$ is a separeted or Hausdorff topology. To show this, it suffices to prove (since the β-topology is defined by means of the functions $f^\frown \in \mathbf{B}^\frown$) that $\mathbf{B}^\frown$ distinguish points in $\mathscr{E}^\frown$. If $x_0 \in \mathscr{E}$, it is easy to see that the totality of functions $f \geqq 0$ such that $f(x_0) = 0$ is a fixed m. p. c. If $x_0 \in \in \mathscr{E}^\frown - \mathscr{E}$ then by theorem 2, the totality of functions $f \geqq 0$ such that $f^\frown(x_0) = 0$ is a free m. p. c. Now, let $x, y \in \mathscr{E}^\frown$, $x \neq y$. Then if $x, y \in \mathscr{E}$, there exists by hypothesis an $f \in \mathbf{B}$ such that $f(x) \neq f(y)$. If $x \in \mathscr{E}$ and $y \in \mathscr{E}^\frown - \mathscr{E}$, then the two maximal positive cones in question cannot be the same since one is fixed and the other free. Thus, once more, there exists $f \in \mathbf{B}$ such that $f^\frown(x) \neq f^\frown(y)$. If $x, y \in \mathscr{E}^\frown - \mathscr{E}$, then $x \neq y$ implies that there is an $f \in \mathbf{B}$ such that $f^\frown(x) = 0$ (see theorem 2) and $f^\frown(y) \neq 0$. Thus in all cases, we see that the functions of $\mathbf{B}^\frown$ distinguish points and hence the β-topology on $\mathscr{E}$ is separated. One of our principal purposes will be to introduce later another characteristic topology in $\mathscr{E}^\frown$. In order to avoid confusion, we shall speak of sets as being β-open or β-closed rather than merely open or closed. If $\mathscr{E} \subset \mathscr{E}^\sim \subset \mathscr{E}^\frown$, the relative topology on $\mathscr{E}^\sim$ is the weak topology on $\mathscr{E}^\sim$ generated by the functions of $\mathbf{B}^\sim$.

Notice that the $\mathscr{E}$-vicinities of a m. p. c. $\mathfrak{C}$ are the traces on $\mathscr{E}$ of β-open neighborhoods of the point $x_\mathfrak{C}$ associated to $\mathfrak{C}$. This shows that $\mathscr{E}$ is β-dense in $\mathscr{E}^\frown$ since for each m. p. c. $\mathfrak{C}$, the $\mathscr{E}$-vicinities of $\mathfrak{C}$ are not empty.

4. Compactness properties

We establish briefly the β-compactness properties of $\mathscr{E}^\frown$.

Theorem 4. *The space $\mathscr{E}^\frown$ with the β-topology is compact. The algebra $\mathbf{B}^\frown$ is the algebra of all β-continuous functions on $\mathscr{E}^\frown$.*

Proof. We show that if $\mathfrak{F}$ is any family of closed sets having the finite intersection property, there exists a point $x \in \mathscr{E}^\frown$ such that x belongs to each $\mathscr{F}$ in $\mathfrak{F}$. First of all, we replace each $\mathscr{F}$ by a class of zero-sets whose intersection is $\mathscr{F}$. This gives a family $\mathfrak{F}'$ of zero-sets having the finite intersection property.

Let $\mathfrak{D}$ denote a set of functions $f \in \mathbf{B}$ such that (1) $f^\frown \geqq 0$; (2) for each $f \in \mathfrak{D}$, $\mathfrak{Z}(f^\frown)$ is a set in $\mathfrak{F}'$; (3) for each $\mathscr{F} \in \mathfrak{F}'$, there is an $f \in \mathfrak{D}$ such that $\mathfrak{Z}(f^\frown) = \mathscr{F}$. Let $\mathfrak{C}$ denote the set of all finite linear combinations $f = \alpha_1 f_1 + \ldots + \alpha_n f_n$, $f_i \in \mathfrak{D}$,

$\alpha_i \geqq 0$. Then $\mathfrak{C}$ is a p. c. since $f \geqq 0$ and since by the finite intersection property, f is singular. Now $\mathfrak{C}$ is included in a m. p. c. $\mathfrak{C}'$. Let $x_{\mathfrak{C}'}$ be the point in $\mathscr{E}^\frown$ associated with $\mathfrak{C}'$. Then $f \in \mathfrak{C} \Rightarrow f \in \mathfrak{C}' \Rightarrow f^\frown(x_{\mathfrak{C}'}) = 0$. Thus $x_{\mathfrak{C}'} \in \mathscr{F}$ for each $\mathscr{F} \in \mathfrak{F}$.

We have seen earlier that the β-topology on $\mathscr{E}^\frown$ is separated. Thus the above paragraph shows that $\mathscr{E}^\frown$ is β-compact. Now, note that $\mathbf{B}^\frown$ separates points of $\mathscr{E}^\frown$ and is uniformly closed. Thus by the Stone—Weierstrass theorem, $\mathbf{B}^\frown$ is the set of all β-continuous functions on $\mathscr{E}^\frown$.

Definition 2. *The space $\mathscr{E}^\sim \supset \mathscr{E}$ is β-countably compact if any countable covering of $\mathscr{E}^\sim$ by co-zero sets has a finite subcover.*

Note that the usual definition involves open sets and not co-zero sets.

Theorem 5. *The space $\mathscr{E}^`$ is β-countably compact.*

We have seen in section 1 that the denumerable intersection of zero-sets is a zero-set. Let $\mathfrak{F}$ be any denumerable family of zero-sets in $\mathscr{E}^`$, $\mathfrak{F} = \{\mathscr{F}_1, \mathscr{F}_2, \ldots\}$, which has the finite intersection property. Assume for a moment that $\mathscr{F} = \emptyset$ where $\mathscr{F} = \bigcap_{n=1}^{\infty} \mathscr{F}_n$. Then by a now standard construction there exists a positive singular function f such that $f \rhd 0$. Clearly, f is in some s. m. p. c. $\mathfrak{C}$ and hence $x_{\mathfrak{C}} \in \mathscr{E}^`$ by definition. Also $f^`(x_{\mathfrak{C}}) = 0$, a contradiction. Hence $\mathscr{F} \neq \emptyset$ and the theorem is proved.

Definition 3. *A space $\mathscr{E}^\sim \supset \mathscr{E}$ is said to be β-realcompact if and only if given any point $x \in \mathscr{E}^\frown - \mathscr{E}^\sim$, there exists a function $f \in \mathbf{B}$ such that $f^\sim \rhd 0$ and $f^\frown(x) = 0$.*

Thus, if $\mathscr{E}^\sim$ is β-realcompact, for $x \in \mathscr{E}^\frown - \mathscr{E}^\sim$, there exists a function which is β-continuous on $\mathscr{E}^\sim$ (of the form $f^{\sim -1}$) which cannot be extended continuously to any set containing x.

Theorem 6. *The space $\mathscr{E}'$ is β-realcompact. It is the smallest β-realcompact space containing $\mathscr{E}$.*

Proof. We have seen in § 2 that if $\mathfrak{C}$ is a m. p. c., if $f_n \in \mathfrak{C}$, $n = 1, 2, \ldots$, and if $f = \sum_{n=1}^{\infty} f_n$ converges uniformly, then $f \in \mathfrak{C}$. Moreover, if $\mathfrak{C}$ is a w. m. p. c. then $\mathscr{Z}(f) \neq \emptyset$. Since $\mathscr{Z}(f) = \bigcap_{n=1}^{\infty} \mathscr{Z}(f_n)$, this implies that for *any* sequence $\{f_n\}$ in a w. m. p. c., $\bigcap_{n=1}^{\infty} \mathscr{Z}(f_n) \neq \emptyset$.

We show that $\mathscr{E}'$ is β-realcompact. Let $x_0 \in \mathscr{E}^\frown - \mathscr{E}'$. Then x_0 arises from a s. m. p. c., hence there exists a function $f \in \mathrm{B}$ such that $f \rhd 0$ and $f^\frown(x_0) = 0$. It can easily be seen that $f' \rhd 0$, that is, that $x \in \mathscr{E}' - \mathscr{E}$ implies $f'(x) > 0$. For if $\mathfrak{C}_x$ denotes the w. m. p. c. corresponding to x, then $f'(x) = 0$ implies that $\mathscr{Z}(f) \neq \emptyset$. Thus $f \rhd 0$ implies $f' \rhd 0$.

We show now that if $\mathscr{E}^\sim$ is such that $\mathscr{E}^\sim \supset \mathscr{E}$ and if $\mathscr{E}^\sim$ is β-realcompact then $\mathscr{E}^\sim \supset \mathscr{E}'$. Since $\mathscr{E}^\sim$ is β-realcompact, there exists for each $x \in \mathscr{E}^\frown - \mathscr{E}^\sim$ an $f \in \mathrm{B}$ such that $f^\sim \rhd 0$ while $f^\frown(x) = 0$. Clearly, $f^\sim \rhd 0$ implies $f \rhd 0$; and the argument in the previous paragraph shows that $f \rhd 0$ implies $f' \rhd 0$. Invoking definition 3, we see that $\mathscr{E}^\sim \supset \mathscr{E}'$.

Theorem 7. *A space $\mathcal{E}$ is β-compact if and only if it is β-countably compact and β-realcompact.*

Proof: If $\mathcal{E}$ is β-compact, then all m. p. c. are fixed and hence $\mathcal{W}=\emptyset$ and $\mathcal{S}=\emptyset$. This implies that the sets $\mathcal{E}, \mathcal{E}', \mathcal{E}^{\backprime}, \mathcal{E}^{\frown}$ are all identical. By theorem 5, $\mathcal{E}$ is β-countably compact. By theorem 6, $\mathcal{E}$ is β-realcompact.

Now suppose $\mathcal{E}$ is β-realcompact and β-countably compact. Then by theorem 6, $\mathcal{E}=\mathcal{E}'$, that is, $\mathcal{W}=\emptyset$. We have seen in the proof of theorem 5 that a space is β-countably compact if and only there does not exist a function $f\in\mathbf{B}$ such that $f\rhd 0$; in other words, if and only if $\mathcal{S}=\emptyset$. Thus our hypotheses allow us to write $\mathcal{E}^{\frown}= =\mathcal{E}\cup\mathcal{W}\cup\mathcal{S}=\mathcal{E}$ and hence $\mathcal{E}$ is β-compact.

5. Baire functions and the ι-topology

In this section we introduce the ι-topology and indicate its relation to the Baire functions. Precise characterization of the sets $\mathcal{E}'$ and $\mathcal{E}^{\backprime}$ in terms of the ι-topology are given.

The zero-sets in $\mathbf{B}$ (or in $\mathbf{B}^{\sim}$) have the property that the intersection of two zero-sets is a zero-set. This fact permits us to introduce:

Definition 4. *The ι-topology on $\mathcal{E}^{\frown}$ is that topology having for a base of its open sets the zero-sets of $\mathbf{B}^{\frown}$.*

The relative ι-topology in $\mathcal{E}^{\sim}, \mathcal{E}^{\frown}\supset\mathcal{E}^{\sim}\supset\mathcal{E}$, thus has as a base of its open sets the zero-sets of $\mathbf{B}^{\sim}$. The ι-topology was introduced in [5], p. 476. Proof of some of the statements of this section can be found in that work.

Definition 5. *The class of bounded Baire functions on $\mathcal{E}^{\frown}$ generated by $\mathbf{B}^{\frown}$ is the smallest class of bounded functions containing $\mathbf{B}^{\frown}$ and closed under the operation of taking pointwise limits. It will be denoted by* $\mathbf{I}$.

Thus, if $\{\varphi_n\}$ is a sequence of functions in $\mathbf{I}$ and if for each $x\in\mathcal{E}^{\frown}$, $\varphi_n(x)\uparrow\varphi(x)$ (or $\varphi_n(x)\downarrow\varphi(x)$) where φ is a bounded function, then $\varphi\in\mathbf{I}$. Notice that $\mathbf{I}$ is an algebra and a lattice and, if we set $\|\varphi\|=\sup\limits_{x\in\mathcal{E}^{\frown}}|\varphi(x)|$, then $\mathbf{I}$ is a Banach space which contains $\mathbf{B}^{\frown}$.

Theorem 8. *The ι-topology on $\mathcal{E}^{\frown}$ is finer than the β-topology. It is the coarsest topology in which the functions of $\mathbf{I}$ are continuous. The denumerable intersection of ι-open sets is ι-open. The ι-topology is the coarsest topology containing the β-topology in which the set $\mathcal{E}^{\frown}$ is a P-space. The ι-topology is strictly finer than the β-topology if and only if $\mathcal{E}^{\frown}$ contains infinitely many points.*

Proof: If $\mathcal{M}$ is β-open and $x\in\mathcal{M}$, then, using the fact that $\mathcal{E}^{\frown}$ is β-compact and hence completely regular, and that each continuous function on $\mathcal{E}^{\frown}$ is in $\mathbf{B}^{\frown}$ (theorem 4), there exists a zero set $\mathcal{Z}(f^{\frown}), f\in\mathbf{B}$, containing x and contained in $\mathcal{M}$. Thus $\mathcal{M}$ is ι-open. Hence the ι-topology is finer than the β-topology. The proof of the second sentence of the theorem is to be found in [5], pp. 477—479. Since the denumerable intersection of zero-sets arising from $\mathbf{B}^{\frown}$ is a zero-set, the denumerable intersection of ι-open sets is ι-open. This shows also that $\mathcal{E}^{\frown}$ in the

ι-topology is a P-space — see [1], 4J (4), p. 63. If $\mathscr{E}^\frown$ is a P-space in some topology finer than the β-topology, this topology will contain denumerable intersections of β-open sets, hence will contain all zero-sets arising from $\mathbf{B}^\frown$. Thus the topology will contain the ι-topology. The proof of the last statement in the theorem follows after the next paragraph.

If $\varphi \in \mathbf{I}$, then the set $\mathfrak{Z}(\varphi)$ will be called a *Baire* set. It is easy to verify that if $\psi \in \mathbf{I}$ and α is a real number, the set $\{x: \psi(x) \leqq \alpha\}$ is a Baire set. One may also see that the family of Baire sets is closed under complementation and the formation of denumerable unions and intersections. The class of Baire sets is the smallest class closed under these operations and containing the zero-sets arising from $\mathbf{B}^\frown$. Clearly all Baire sets are ι-open and ι-closed. Since the β-open cozero sets in $\mathscr{E}^\frown$ constitute a base for the ι-closed sets, and since these same sets are Baire sets, the ι-closure of any set is the intersection of all Baire sets which contain it.

If $\mathscr{E}$ is finite, so is $\mathscr{E}^\frown$ and hence $\mathbf{B} = \mathbf{I}$. Assume now that $\mathscr{E}$ has infinitely many points. We shall construct a Baire function φ on $\mathscr{E}^\frown$, hence a ι-continuous function, which assumes values arbitrarily close to 1 but which does not assume the value 1. Thus φ cannot be β-continuous since $\mathscr{E}^\frown$ is β-compact and since the range of each β-continuous function is a closed set. This will prove that the β-topology and the ι-topology are distinct, hence that the latter is strictly finer than the former.

We construct φ. If $x, y \in \mathscr{E}^\frown$, there exists a β-continuous function $f^\frown$ in $\mathbf{B}^\frown$ such that $f^\frown(x) = 0$ and $f^\frown(y) \neq 0$. The sets $\mathfrak{M}_1 = Z(f^\frown)$ and $\mathfrak{M}_2 = \mathscr{E}^\frown - \mathfrak{M}_1$ are Baire sets and at least one of them contains infinitely many points. Assume $\mathfrak{M}_2$ is infinite and let ψ_1 be the characteristic function of $\mathfrak{M}_1$. Note that ψ_1 is a Baire function. We now split $\mathfrak{M}_2$ into two non-empty Baire sets, of which the second $\mathfrak{M}_3$ is infinite and we let ψ_2 be the non-zero characteristic function of the first. The splitting is accomplished as follows: $\mathfrak{M}_2$, being a Baire set, is the union of zero sets from $\mathbf{B}^\frown$. Furthermore, if $x, y \in \mathfrak{M}_2$, $x \neq y$, there exists a zero set containing x and not y. Continuing in this way, we construct a sequence $\{\psi_n\}$ of non-zero Baire functions satisfying $\psi_n \cdot \psi_m = 0$ for $n \neq m$. Let $\varphi = \sum_{n=1}^{\infty} (1 - 2^{-n})\varphi_n$. Then φ has the properties asserted above. This concludes the proof.

Theorem 9. *The set $\mathscr{E}'$ is precisely the ι-closure of $\mathscr{E}$. Thus $\mathscr{E}'$ is the intersection of all Baire sets containing $\mathscr{E}$. Also, $\mathfrak{W}$ is the set of ι-limit points of $\mathscr{E}$ lying in $\mathscr{E}^\frown - \mathscr{E}$. The operations of forming the β-realcompactification and of forming the ι-closure are identical.*

Proof. Suppose $x \in \mathscr{E}^\frown - \mathscr{E}$ and suppose that every zero-set containing x meets $\mathscr{E}$. Then $x \in \mathfrak{W}$. Thus the ι-closure of $\mathscr{E}$ lies in $\mathscr{E}'$. If now $x \in \mathfrak{W}$, each zero-set containing x intersects $\mathscr{E}$. This shows that the ι-closure of $\mathscr{E}$ is $\mathscr{E}'$. The last statement in the theorem follows from theorem 6.

Theorem 10. *The set $\mathfrak{S}$ in $\mathscr{E}^\frown$ is the ι-interior of the complement of $\mathscr{E}$. The only Baire set containing $\mathscr{E}'$ is $\mathscr{E}^\frown$. Let $\varphi \in \mathbf{I}$ and let φ' represent the restriction of φ to $\mathscr{E}'$. Then the map: $\varphi \to \varphi'$ is an algebraic isomorphism from $\mathbf{I}$ onto the set of all Baire functions defined on $\mathscr{E}'$ and generated by the functions of $\mathbf{B}'$.*

Proof. The ι-interior of the complement of $\mathscr{E}$ equals the complement of the ι-closure of $\mathscr{E}$. The preceding theorem shows that this is $\mathscr{E}^\frown - \mathscr{E}' = \mathfrak{S}$. Any Baire set containing $\mathscr{E}^{\backprime}$ contains $\mathscr{E}$ and hence contains $\mathscr{E}'$ by the preceding theorem. Thus the Baire set contains $\mathscr{E}' \cup \mathscr{E}^{\backprime} = \mathscr{E}^{\frown}$.

The map $\varphi \to \varphi^{\backprime}$ is obviously a homomorphism. Suppose $\varphi^{\backprime} = 0$. Thus $\mathscr{Z}(\varphi) \supset \mathscr{E}^{\backprime}$ and by the preceding paragraph, $\mathscr{Z}(\varphi) = \mathscr{E}^{\frown}$, that is, $\varphi = 0$. Thus the map is an isomorphism.

We show that the map is onto. If $f^{\backprime}$ is a Baire function of class zero on $\mathscr{E}^{\backprime}$ (hence continuous), there exists a unique function $f^{\frown} \in \mathbf{B}^{\frown}$ such that the restriction of $f^{\frown}$ to $\mathscr{E}^{\backprime}$ is $f^{\backprime}$. If $x_0 \in \mathscr{E}^{\frown} - \mathscr{E}^{\backprime}$, $x_0 \in {}^{c}\mathscr{W}$; hence for $f \in \mathbf{B}$, the set $\{x : f^{\frown}(x) = f^{\frown}(x_0)\}$ is a zero-set which intersects $\mathscr{E}$ and hence $\mathscr{E}^{\backprime}$. If $\{f_n\}$ is any sequence from $\mathbf{B}$, there is a zero-set $\mathscr{Z}$ intersecting $\mathscr{E}$ on which $f_n^{\frown}(x) = f_n^{\frown}(x_0)$, $n = 1, 2, \ldots$. Now let $\{f_n^{\backprime}\}$ denote a sequence of Baire functions of class 0 defined on $\mathscr{E}^{\backprime}$ and such that $f_n^{\backprime} \uparrow \psi$ pointwise. Then the associated sequence $\{f_n^{\frown}\}$ converges to a Baire function φ defined on $\mathscr{E}^{\frown}$ and $\varphi^{\backprime} = \psi$; in fact $\varphi^{\backprime}$ is constant on the zero-set $\mathscr{Z}$. Thus all Baire functions of order 1 on $\mathscr{E}^{\backprime}$ can be obtained by the mapping process of the theorem. The proof can now be extended to Baire functions of higher order, making use of the fact that for any sequence of such functions $\{\psi_n\}$, there exists a zero-set containing x_0 on which all functions are constant.

6. Pseudocompact spaces

We define the concept of pseudocompactness and set down equivalents for it.

Definition 6. *A space $\mathscr{E}$ is β-pseudocompact if every function $f \in \mathbf{B}$ such that $f(x) > 0$ for all $x \in \mathscr{E}$ is regular.*

In other words, there exists no function f such that $f \rhd 0$; hence $\mathscr{S} = \varnothing$.

Theorem 11. *The following statements are equivalent:*

(1) *$\mathscr{E} = \mathscr{E}^{\backprime}$ (or equivalently $\mathscr{E}$ is ι-dense in $\mathscr{E}^{\frown}$).*
(2) *$\mathscr{E}$ is β-pseudocompact.*
(3) *$\mathscr{E}$ is β-countably compact (for zero-sets).*
(4) *$\mathscr{E}$ intersects each non-empty zero set $\mathscr{Z}(f^{\frown})$, $f \in \mathbf{B}$.*
(5) *Each function in $\mathbf{B}$ assumes its $\mathscr{E}^{\frown}$ maximum and its $\mathscr{E}^{\frown}$ minimum in $\mathscr{E}$.*
(6) *For each $f \in \mathbf{B}$, the range of f is a closed set.*
(7) *If $f_n \in \mathbf{B}$, $n = 1, 2, \ldots$, and for each $x \in \mathscr{E}$, $f_n(x) \downarrow 0$, then $f_n \downarrow 0$ uniformly on $\mathscr{E}$.*
(8) *Each positive linear functional F over $\mathbf{B}$ is a positive Daniell integral.*
(9) *The map which restricts to $\mathscr{E}$ each Baire function on $\mathscr{E}^{\frown}$: $\varphi \to \varphi/\mathscr{E}$, is an isomorphism.*

Proof. It is trivial that (1) $\Rightarrow$ (2). The intersection of denumerably many zero-sets in $\mathscr{E}$ having the finite intersection property is empty if and only if there exists a positive singular f such that $f \rhd 0$. Thus (2) $\Rightarrow$ (3). Let $f \in \mathbf{B}$, let $\mathscr{Z}(f^{\frown}) \neq \varnothing$ and suppose $f \geqq 0$. Then $\mathscr{Z}(f) = \bigcap_{n=1}^{\infty} \{x : x \in \mathscr{E}, f(x) \leqq 1/n\}$. The zero-sets in this intersection are non-empty and have the finite intersection property. Thus (3) $\Rightarrow$(4). If $\max_{x \in \mathscr{E}^{\frown}} f^{\frown}(x) = \alpha$, then $\alpha e - f$ is singular. Thus (4) $\Rightarrow$ (5.) If β is a limit point of the range of f, then $|f - \beta e|$ is singular. Hence (5) $\Rightarrow$ (6).

Assume (6). Let $\{f_n\}$ be a sequence of functions such that $f_n(x)\downarrow 0$, $x\in\mathscr{E}$. Then the sequence of extended functions, $\{f_n^\frown\}$, converges monotonely for each $x\in\mathscr{E}^\frown$. In order to show uniform convergence it will be sufficient to show (since $\mathscr{E}^\frown$ is β-compact) that $x_0\in\mathscr{E}^\frown-\mathscr{E}$ implies $f_n^\frown(x_0)\downarrow 0$. This will be done by proving that there exists an y in $\mathscr{E}$ such that for each n, $f_n^\frown(x_0)=f_n(y)$. Write $\alpha_n=f_n^\frown(x_0)$. Then $|\alpha_n e-f_n|$ is singular. For a suitable sequence of constants $\{\beta_n\}$, $\beta_n>0$ (e. g., $\beta_n=2^{-n}\|\alpha_n e-f_n\|^{-1}$) $f=\sum_{n=1}^{\infty}\beta_n|\alpha_n e-f_n|$ converges uniformly hence $f\in\mathbf{B}$. Since $f^\frown(x_0)=0$, f is singular. By (6) there exists $y\in\mathscr{E}$ such that $f(y)=0$. This means that $f_n^\frown(x_0)=f_n(y)$ for all n and since by hypothesis $f_n(y)\downarrow 0$, we have $f_n^\frown(x_0)\downarrow 0$. Thus (6) $\Rightarrow$ (7).

A positive linear functional F is called a positive Daniell integral if it has the property: if $\{f_n\}$ is a sequence such that $f_n(x)\downarrow 0$ for all $x\in\mathscr{E}$, then $Ff_n\downarrow 0$. Now, let F be a positive linear functional over $\mathbf{B}$. Then F is bounded, with bound $\|F\|$. Let $\{f_n\}$ be any sequence from $\mathbf{B}$ such that $f_n(x)\downarrow 0$ for all x in $\mathscr{E}$. Then (7) implies that $f_n\downarrow 0$ uniformly, that is, that $\|f_n\|\downarrow 0$. We have therefore $Ff_n\leqq\|F\|\ \|f_n\|\downarrow 0$. Hence (7) $\Rightarrow$ (8).

Suppose the mapping $\varphi\to\varphi/\mathscr{E}$ is not an isomorphism. Then there exists a Baire function φ and a point $x_0\in\mathscr{E}^\frown-\mathscr{E}$ such that $\varphi(x_0)=1$ and $\varphi(x)=0$ for $x\in\mathscr{E}$. Clearly $x_0\in\mathbb{S}$. Hence there exists a positive singular function f such that $f\rhd 0$ and $f^\frown(x_0)=0$. Assume $f\leqq e$ and define $f_n = e-(nf\wedge e)$. Then $x\in\mathscr{E}$ implies $f_n(x)\downarrow 0$. Let F be a positive linear functional defined by $Fg=g^\frown(x_0)$, $g\in\mathbf{B}$. Then $Ff_n=1$. Thus F is not a positive Daniell integral. This means that (8) $\Rightarrow$ (9).

If the mapping: $\varphi\to\varphi/\mathscr{E}$ is an isomorphism, the argument of the preceding paragraph shows that $\mathbb{S}=\varnothing$, hence that $\mathscr{E}=\mathscr{E}^\frown$. Thus (9) $\Rightarrow$ (1). The theorem is proved.

For the space of all functions continuous and bounded on $\mathscr{E}$, the equivalence of (2), (5), (7), and (8) was shown by Glicksberg [2]; the relations (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (5) have been obtained by Hewitt [4]; the relation (2) $\Leftrightarrow$ (3) appears in Gillman and Jerison [1], p. 79.

7. Functionals and their associated measures

As indicated before, a positive Daniell integral is a positive linear functional F defined over a vector lattice of functions, which is continuous with respect to monotone pointwise convergence of functions. That is, $f_n(x)\downarrow 0$ for each x implies $Ff_n\downarrow 0$. If $\mathbf{B}$ is a Banach space of functions of the type we have been considering, each positive functional F is the sum of uniquely defined positive functionals G and H, $F=G+H$, where G is a positive Daniell integral and H will be called a positive anti-integral. For the complete discussion of this decomposition we refer the reader to [3]. (As one would expect, a Daniell integral is defined to be the difference of two positive Daniell integrals.) In the present work, there are two structures of interest with respect to Daniell integration: the algebra $\mathbf{B}$ of functions defined over $\mathscr{E}$ and the algebra $\mathbf{I}$ of Baire functions defined over $\mathscr{E}^\frown$. We systematize below the principal facts concerning the Daniell integrals over these algebras. For the algebra $\mathbf{B}^\frown$ of functions defined over $\mathscr{E}^\frown$, each linear functional is a Daniell integral. (We remind the reader that the set of all bounded linear functionals over B is denoted by $\mathbf{B}^*$.)

The mapping described in the next theorem is well known:

Theorem 12. *Let* **I** *denote the Banach algebra of bounded Baire functions over* $\mathscr{E}^\frown$. *Let* $\mathbf{M}^*$ *denote the closed linear manifold of Daniell integrals over* **I**, *thus* $\mathbf{M}^* \subset \mathbf{I}^*$. *If* $F \in \mathbf{M}^*$, *let* $F/\mathbf{B}^\frown$ *denote the restriction of* F *to the functions of* $\mathbf{B}^\frown$. *Then the map* $F \to F/\mathbf{B}^\frown$ *is a vector-lattice isomorphism. The range of the map is all of* $(\mathbf{B}^\frown)^*$. *The inverse mapping is the (canonical) imbedding of* $(\mathbf{B}^\frown)^*$ *into* $\mathbf{I}^*$. *The latter mapping covers* $\mathbf{I}^*$ *if and only if* $\mathscr{E}$ *has a finite number of points.*

The fact that the map is a vector-lattice homomorphism is obvious. We show that the range is all of $\mathbf{B}^*$. (Since $\mathbf{B}^*$ and $(\mathbf{B}^\frown)^*$ are isometrically isomorphic, we use the simpler notation $\mathbf{B}^*$.) Let $\Lambda \in \mathbf{B}^*$. Then there is associated to Λ a measure λ defined for all Baire sets in $\mathscr{E}^\frown$ such that for $f^\frown \in \mathbf{B}^\frown$, $\Lambda f^\frown = \int f^\frown d\lambda$. Furthermore, the functional Λ may be extended to a larger class of functions — which includes the Baire functions. Denote the extended functional by F. Then F is defined over **I** and since, as an integral, it satisfies LEBESGUE's dominated convergence theorem, it is a Daniell integral. Clearly, $F/\mathbf{B}^\frown = \Lambda$. This proves the "onto" property. To show that the homomorphism is an isomorphism, assume that the Daniell integral $F \geqq 0$ in $\mathbf{I}^*$ is zero over $\mathbf{B}^\frown$. If φ is a Baire function of the first class and $f_n^\frown \uparrow \varphi$, $f_n^\frown \in \mathbf{B}^\frown$, then the Daniell property shows that $F\varphi = 0$; similarly for functions of higher order. Thus $F\varphi = 0$ for all Baire functions φ and the map of the theorem is an isomorphism.

If $\mathscr{E}$ is finite, then $\mathscr{E} = \mathscr{E}^\frown$ and $\mathbf{B} = \mathbf{B}^\frown = \mathbf{I}$. Thus the canonical mapping of $(\mathbf{B}^\frown)^*$ into $\mathbf{I}^*$ is surjective (onto). If $\mathscr{E}$ is infinite, then there exists a singular Baire function $\xi \geqq 0$ such that $\xi(x) > 0$ for each $x \in \mathscr{E}^\frown$; in other words $\xi \rhd 0$. For example, the function $\xi = e^\frown - \varphi$, where φ is the function constructed in the last paragraph of the proof of theorem 8, has this property. This implies that $\mathscr{E}^\frown$ is not ι-countably compact and hence by theorem 11 (8), there exists a linear functional over **I** which is not a Daniell integral, hence which does not belong to $\mathbf{M}^*$. This means that the canonical mapping of $(\mathbf{B}^\frown)^*$ into $\mathbf{I}^*$ is not surjective.

In the remainder of this section, F represents a linear functional over **B**. Assume $F \geqq 0$. With respect to the base set $\mathscr{E}$, two extreme cases arise: F is a Daniell integral or F is an anti-integral. On the other hand, F considered as a linear functional over $\mathbf{B}^\frown$ has associated to it a countably additive measure defined for the Baire sets of $\mathscr{E}^\frown$. We shall say that the measure of F is concentrated on a set $\mathscr{A} \subset \mathscr{E}^\frown$ if the F-measure of any Baire set lying in $\mathscr{E}^\frown - \mathscr{A}$ is zero. The question arises to as the extent to which the F-measure of a Daniell integral is concentrated "close" to $\mathscr{E}$. The proper definition of "close" is given by the ι-topology. We shall show that the measure is concentrated on the ι-closure of $\mathscr{E}$, that is, on $\mathscr{E}^\iota$. Similarly, we shall show that F is an anti-integral if and only if its F-measure is concentrated in a Baire set whose complement contains $\mathscr{E}$.

Theorem 13. *Let* $F \geqq 0$ *be a linear functional over* **B**. *Then* F *is a Daniell integral if and only if the* F*-measure of each Baire set in* $\mathbb{S} = \mathscr{E}^\frown - \mathscr{E}'$ *is zero. Thus the measure of* F *is concentrated on each Baire set containing* $\mathscr{E}$.

Proof. Suppose that the F-measure of each Baire set in $\mathscr{E}^\frown - \mathscr{E}^\iota$ is zero. Let $f_n \in \mathbf{B}$, $f_n(x) \downarrow 0$, $x \in \mathscr{E}$. Then the set $\mathfrak{M} = \{x : f_n^\frown(x) \downarrow 0,\ x \in \mathscr{E}^\frown\}$ is a Baire set including

$\mathscr{E}$, hence including $\mathscr{E}'$. Its complement is a Baire set which lies in $\mathfrak{S}$. Since the F-measure of this complement is zero, an application of LEBESGUE's convergence theorem and the fact that $\mathscr{E}^{\frown}=\mathfrak{M}\cup(\mathscr{E}^{\frown}-\mathfrak{M})$ shows that $Ff_n\downarrow 0$. Thus F is a Daniell integral.

Now, assume that the linear functional F is a Daniell integral. Let μ be the measure associated to F and let φ represent any bounded Baire function. Then F may be extended to all of **I** by means of the formula $F\varphi=\int\varphi d\mu$. According to the theory of the Daniell integral, if $\varphi_1(x)=\varphi_2(x)$ for each $x\in\mathscr{E}$, $F\varphi_1=F\varphi_2$. Now let $\mathfrak{M}$ represent any Baire set lying in $\mathfrak{S}$ and let $\chi_{\mathfrak{M}}$ be its characteristic function. Then $\chi_{\mathfrak{M}}(x)=0$ for x in $\mathscr{E}$ and hence $F\chi_{\mathfrak{M}}=0$. Thus $\mathfrak{M}$ has F-measure equal to zero. This proves the theorem.

Let $\mathbf{M}^*$ denote the closed linear manifold of Daniell integrals; and let $\mathbf{N}^*$ denote the closed linear manifold of anti-integrals. We have indicated that $\mathbf{B}^*$ is the direct sum of these two manifolds: $\mathbf{B}^* = \mathbf{M}^*\oplus\mathbf{N}^*$.

Theorem 14. *Let $F\geqq 0$ be in $\mathbf{B}^*$. Then F is an anti-integral — $F\in\mathbf{N}^*$ — if and only if there exists a Baire set $\mathfrak{M}$ lying in $\mathfrak{S}=\mathscr{E}^{\frown}-\mathscr{E}'$ such that the measure of F is concentrated on $\mathfrak{M}$.*

Proof. Let $F\geqq 0$ be a linear functional over **B** whose measure is concentrated on a Baire set $\mathfrak{M}\subset\mathfrak{S}$. Let $F=G+H$ where $G\in\mathbf{M}^*$ and $H\in\mathbf{N}^*$. Since $F\geqq G$, the measure of G is concentrated on $\mathfrak{M}$ and the G-measure of $\mathscr{E}^{\frown}-\mathfrak{M}$ is zero. Since G is a Daniell integral, the G-measure of $\mathfrak{M}$ is zero by theorem 13. Thus $G=0$ and $F=H$, that is, $F\in\mathbf{N}^*$.

Assume next that $F\in\mathbf{N}^*$. Let us suppose first that every zero-set from $\mathbf{B}^{\frown}$ which lies in $\mathfrak{S}$ is of F-measure zero. Let $f_n\in\mathbf{B}$, $n=1,2,\ldots$, and suppose $f_n(x)\downarrow 0$, $x\in\mathscr{E}$. Let $\mathscr{A}_r=\{x:f_n^{\frown}(x)\geqq 2^{-r}\text{ for all }n\}$. Then $\mathscr{A}_r$ is the intersection of denumerably many zero sets hence is a zero-set. Note that $\mathscr{A}_r\subset\mathscr{E}^{\frown}-\mathscr{E}'$, thus the F-measure of $\mathscr{A}_r$ is zero. Now, if we write $\mathscr{A}=\{x:f_n^{\frown}(x)\not\downarrow 0,\ x\in\mathscr{E}^{\frown}\}$, then $\mathscr{A}=\bigcup_{r=1}^{\infty}\mathscr{A}_r$. Thus $\mathscr{A}$ is a Baire set lying in $\mathscr{E}^{\frown}-\mathscr{E}$ and the F-measure of $\mathscr{A}$ is zero. By LEBESGUE's theorem, $Ff_n\downarrow 0$. Thus F is a Daniell integral. Since $F\in\mathbf{N}^*$, $F=0$. We conclude that if $F\neq 0$ and $F\in\mathbf{N}^*$, there exist zero-sets in $\mathscr{E}^{\frown}-\mathscr{E}'$ of non-zero F-measure.

Suppose $\mathfrak{Z}$ is such a zero-set and suppose the F-measure of $\mathfrak{Z}$ is α. Then since $F>0$, we have $0<\alpha\leqq Fe$. Let λ_1 be the supremum of all values α so obtained. Let $\mathfrak{Z}_1$ be any zero-set whose F-measurable exceeds $\lambda_1/2$. Let χ_1 be the characteristic function of $\mathfrak{Z}_1$ and define F_1 by $F_1f=F\chi_1 f$, $f\in\mathbf{B}$. (In this context, we write indiscriminately Ff and $Ff^{\frown}$.) Then $F_1\in\mathbf{B}^*$ and since $F\geqq F_1>0$, $F_1\in\mathbf{N}^*$. Note that the measure of F_1 is concentrated on $\mathfrak{Z}_1$ which is a Baire set whose complement contains $\mathscr{E}$. If $F-F_1=0$, the theorem is proved.

Suppose $F-F_1\neq 0$; write $K_1=F-F_1$. We have $K_1>0$ and $K_1\in\mathbf{N}^*$. Note that the measure of K_1 is concentrated on the complement of $\mathfrak{Z}_1$. Let λ_2 be the supremum of all values α, where α represents the K_1-measure of an arbitrary zero-set lying in $\mathscr{E}^{\frown}-\mathscr{E}'$. Then $0<\lambda_2<2^{-1}\lambda_1$. Let $\mathfrak{Z}_2$ be any zero-set in $\mathfrak{S}-\mathfrak{Z}_1$ whose K_1-measure exceeds $\lambda_2/2$. Let χ_2 denote the characteristic function of $\mathfrak{Z}_2$ and define

F_2 by $F_2f=K_1\chi_2 f$, $f\in\mathbf{B}$. Then $F_2\in\mathbf{B}^*$, and since $K_1\geqq F_2>0$, $F_2\in\mathbf{N}^*$. Write $K_2 = K_1-F_2 = F-(F_1+F_2)$. Note that the measure of K_2 is concentrated on the complement of $\mathscr{Z}_1\cup\mathscr{Z}_2$. It is now clear how to construct the objects F_n, K_n, $\mathscr{Z}_n$, χ_n, and λ_n, $n=3, 4, \ldots$. If for any n, $K_n=0$ the theorem is proved. Otherwise we have for all n, $K_n\in\mathbf{N}^*$, $K_n = F-(F_1+\ldots+F_n)$ and the supremum λ_n of the K_n-measure of zero-sets in $\mathscr{E}^\frown-\mathscr{E}'$ satisfies $\lambda_n<2^{-(n-1)}\lambda_1$, hence $\lambda_n\to 0$. Also the measure of K_n is concentrated on the complement of $\mathscr{Z}_1\cup\ldots\cup\mathscr{Z}_n$. Let $K = F-\sum_{n=1}^{\infty} F_n$. Clearly $K\in\mathbf{N}^*$, $K_n\geqq K\geqq 0$. Since the K-measure of any zero set in $\mathscr{E}^\frown-\mathscr{E}'$ is inferior to λ_n, it is zero. Thus by the argument given earlier, $K=0$ and $F=\sum_{n=1}^{\infty} F_n$. Note that the measure of F is concentrated on the set $\bigcup_{n=1}^{\infty}\mathscr{Z}_n$ which is a Baire set and lies in $\mathscr{E}^\frown-\mathscr{E}'$. This concludes the proof.

9. Realcompactness and the Baire funtions

Given the set $\mathscr{E}^\frown$ and the algebra $\mathbf{I}$ of bounded Baire functions over $\mathscr{E}^\frown$, one may consider compactification questions of $\mathscr{E}^\frown$ with respect to $\mathbf{I}$ (just as at the outset we compactified $\mathscr{E}$ with respect to $\mathbf{B}$). Natural questions are: is $\mathscr{E}^\frown$ ι-realcompact, ι-countably compact, ι-compact. It has already been shown that $\mathscr{E}^\frown$ is not ι-countably compact, hence not ι-compact (theorem 12) if it is infinite. We show now that $\mathscr{E}^\frown$ is ι-realcompact. This will be achived by showing that there are no free weak maximal positive cones in $\mathbf{I}$ (see theorem 6).

Theorem 15. *The algebra of bounded Baire functions over $\mathscr{E}^\frown$ possesses no free weak maximal positive cones. Thus the space $\mathscr{E}^\frown$ is ι-realcompact.*

Proof. Suppose $\mathfrak{C}$ is a free w. m. p. c. in $\mathbf{I}$. Then it is easy to see that $\mathfrak{C}\cap\mathbf{B}^\frown$ is a m. p. c. in $\mathbf{B}^\frown$. (Proof. Since $\mathfrak{C}$ is a m. p. c., then $\varphi\in\mathbf{I}$ implies by the proof of theorem 2 that there exists a unique number λ such that $|\varphi-\lambda e^\frown|$ is in $\mathfrak{C}$. It is easy to see that $\mathfrak{C}\cap\mathbf{B}^\frown$ is a p. c. in $\mathbf{B}^\frown$. Suppose it is not maximal. Then there exists a m. p. c. $\mathfrak{D}$ in $\mathbf{B}^\frown$ properly including $\mathfrak{C}\cap\mathbf{B}^\frown$. Thus there is in $\mathfrak{D}$ a function $f^\frown\in\mathbf{B}^\frown$ with $f^\frown\notin\mathfrak{C}\cap\mathbf{B}^\frown$. Now for some $\lambda\neq 0$, $|f^\frown-\lambda e^\frown|$ is in $\mathfrak{C}$ hence in $\mathfrak{C}\cap\mathbf{B}^\frown$, hence in $\mathfrak{D}$. Since $|f^\frown-\lambda e^\frown|$ and $f^\frown$ are in the m. p. c. $\mathfrak{D}$, $|\lambda|e^\frown$ is in $\mathfrak{D}$ which is absurd. Hence $\mathfrak{C}\cap\mathbf{B}^\frown$ is a m. p. c.) Since $\mathscr{E}^\frown$ is β-compact, there exist a point $x_0\in\mathscr{E}^\frown$ such that $\mathfrak{C}\cap\mathbf{B}^\frown$ consists precisely of the positive singular functions $f^\frown\in\mathbf{B}^\frown$ such $f^\frown(x_0)=0$.

Since $\mathfrak{C}$ is free, there is a Baire function φ in $\mathfrak{C}$ such that $\varphi(x_0)>0$. The set $\{x: \varphi(x)=\varphi(x_0)\}$ is a Baire set, hence contains a zero set $\mathscr{Z}$ such that $x_0\in\mathscr{Z}$. Let $f^\frown\geqq 0$ be so chosen that $\mathscr{Z}(f^\frown)=\mathscr{Z}$. Then $f^\frown\in\mathfrak{C}\cap\mathbf{B}^\frown$. Thus $f^\frown+\varphi\in\mathfrak{C}$ and also $(f^\frown+\varphi)\rhd 0$ on $\mathscr{E}^\frown$ hence $\mathfrak{C}$ is not weak. This contradiction shows that $\mathfrak{C}$ does not exist. Hence $\mathscr{E}^\frown$ is τ-realcompact.

Bibliography

[1] L. GILLMAN and M. JERISON, *Rings of continuous functions* (New York, 1960).
[2] IRVING GLICKSBERG, The representation of functionals by integrals *Duke Math. J.*, **19** (1952), 253–261.
[3] HUGH GORDON and E. R. LORCH, The projection of a linear functional on the manifold of integrals, *Canadian J. Math.*, **9** (1957), 465–474.
[4] EDWIN HEWITT, Rings of real-valued continuous functions. I, *Trans. Amer. Math. Soc.*, **64** (1948), 45–99.
[5] E. R. LORCH, L'intégration dans les espaces généraux, *Bull. Soc. Math. France*, **88** (1960), 469–497.

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