A new proof of Plancherel's theorem for locally compact abelian groups

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§ 1. Introduction

Let G be a locally compact Abelian [LCA] group, with character group X. The famous theorem of PLANCHEREL—WEIL—KREĬN asserts that the Fourier transformation is a unitary mapping of $\mathfrak{L}_2(G)$ onto $\mathfrak{L}_2(X)$, the usual Fourier transformation being extended by continuity from the subspace $\mathfrak{L}_1(G)\cap\mathfrak{L}_2(G)$. The proofs given by KREĬN [see for example [2], § 31, N° 4] and WEIL [[4], pp. 113—118] use a number of delicate theorems of functional analysis. It seems worth while to give a completely elementary proof. Our argument is modelled on the beautiful proof given by F. RIESZ [3] for the classical case in which G is the additive group of real numbers. To apply RIESZ's idea, we need an analogue of the sequence of functions $\exp\left(\frac{-1}{2n^2}x^2\right)$ $(n=1, 2, \ldots)$ for an arbitrary LCA group containing a compact open subgroup. In § 2, we construct these functions quite explicitly. The proof of PLANCHEREL's theorem is then short, and is given in § 3. We also show that the "reverse" Fourier transformation is the inverse of the direct one.

Our notation is the following. For $f\in\mathfrak{L}_1(G)$ and $\psi\in X$, we write

$$Tf(\psi)=\int_G f(t)\overline{\psi(t)}\,dt.$$

That is, we write the Fourier transform of f as Tf. For $g\in\mathfrak{L}_1(X)$ and $s\in G$, we write

$$T^*g(s)=\int_X g(\psi)\psi(s)\,d\psi.$$

For functions $f_1, f_2\in\mathfrak{L}_1(G)$, we write f_1*f_2 for the convolution product of f_1 and f_2: $f_1*f_2(s)=\int_G f_1(st^{-1})f_2(t)\,dt$. All integrals are with respect to Haar measure. For an integer $a>1$, let R^a denote a-dimensional real Euclidean space. We write elements of R^a as $\mathbf{u}=(u_1, u_2, \ldots, u_a)$, etc. For a finite set B, let $\nu(B)$ denote the number of elements in the set B. The characteristic function of a set B is denoted by the symbol ξ_B [it will be clear from the context what the domain of ξ_B is].

We need the following facts about LCA groups, all of which are classical. [Complete proofs are found, for example, in [1].]

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(1. 1) A LCA group G is topologically isomorphic with a direct product $R^a \times H$ where R is the additive group of real numbers, a is a nonnegative integer, and H is a LCA group containing a compact open subgroup J.

(1. 2) Let H be as in (1. 1) and let Υ be the character group of H. The annihilator A of J in Υ is a compact open subgroup of Υ.

(1. 3) The Pontryagin-van Kampen duality theorem holds for H and Υ. That is, every continuous character of Υ has the form $\chi \to \chi(x)$ for some $x \in H$. Furthermore, J is the annihilator of A in H.

§ 2. Construction of auxiliary functions

We here make some preliminary constructions. The key to our whole argument is the following elementary fact.

(2. 1) Lemma. *Let S be a countable Abelian group. There is a sequence $\{P_n\}_{n=1}^{\infty}$ of finite subsets of S such that*

$$P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots, \quad P_n = P_n^{-1}, \quad \bigcup_{n=1}^{\infty} P_n = S, \quad and$$

(i)
$$\lim_{n\to\infty} \frac{\nu((xP_n) \cap P_n)}{\nu(P_n)} = 1$$

for all $x \in S$.[2])

Proof. (I) Suppose that A is any finite subset of S and that $x \in S$. Write $A_n = \bigcup_{k=0}^{n} x^k A$ $(n=0, 1, 2, \ldots)$ and $B_n = A_n \cap A'_{n-1}$ $(n=1, 2, 3, \ldots)$. Then it is evident that $B_n \subset x B_{n-1}$ $(n=2, 3, 4, \ldots)$, so that

(1)
$$\nu(A) \geqq \nu(B_1) \geqq \nu(B_2) \geqq \cdots \geqq \nu(B_n) \geqq \cdots,$$

and

(2)
$$\frac{\nu((xA_n) \cap A'_n)}{\nu(A_n)} = \frac{\nu(B_{n+1})}{\nu(A) + \nu(B_1) + \cdots + \nu(B_n)}$$

for $n = 1, 2, \ldots$. If $B_{n+1} \neq \emptyset$, (2) and (1) show that

(3)
$$\frac{\nu((xA_n) \cap A_n)}{\nu(A_n)} \leqq \frac{1}{n+1},$$

and (3) follows trivially from (2) if $B_{n+1} = \emptyset$.

(II) Now let U and V be any finite subsets of S such that $e \in U \cap V$ [e is the identity of S], $V = V^{-1}$, and let ε be a positive real number. Then we can find a finite subset P of S such that $P = P^{-1}$, $V \subset P$, and

(4)
$$\frac{\nu((UP) \cap P')}{\nu(P)} < \varepsilon.$$

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[2]) This lemma is closely related to although it is not a special case of Lemma (18.13) of [1].

To do this, write the set $U\cup U^{-1}$ as $\{t_1, t_2, \ldots, t_r\}$ and for $n=1, 2, 3, \ldots$, let $W_n=\bigcup\{t_1^{\alpha_1}t_2^{\alpha_2}\cdots t_r^{\alpha_r}V\}$, the union being taken over all ordered r-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ of nonnegative integers such that $\alpha_j\leqq n$ for $j=1, 2, \ldots, r$. It is plain that

$$UW_n\subset\bigcup_{j=1}^{r}t_jW_n. \tag{5}$$

Now for a fixed $j\in\{1, 2, \ldots, r\}$, we have $W_n=\bigcup_{k=0}^{n}t_j^kA$, where $A=\bigcup\{t_1^{\alpha_1}\cdots t_{j-1}^{\alpha_{j-1}}t_{j+1}^{\alpha_{j+1}}\cdots\cdots t_r^{\alpha_r}V\}$. From (3) we infer that

$$\frac{\nu((t_jW_n)\cap W_n')}{\nu(W_n)}\leqq\frac{1}{n+1} \tag{6}$$

for $n=1, 2, 3, \ldots$. Using (5) and (6), we have

$$\frac{\nu((UW_n)\cap W_n')}{\nu(W_n)}\leqq\sum_{j=1}^{r}\frac{\nu((t_jW_n)\cap W_n')}{\nu(W_n)}\leqq\frac{r}{n+1}.$$

Now take P as any W_n with $n>\frac{r}{\varepsilon}-1$.

(III) In completing the present proof, we may obviously suppose that S is infinite; and then we write $S=\{x_1, x_2, \ldots, x_n, \ldots\}$, where $x_1=e$ and the x_n's are all distinct. For $n=1, 2, 3, \ldots$, let $U_n=\{x_1, x_2, \ldots, x_n\}\cup\{x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$. We define the sets P_n by induction. Let $P_1=\{x_1\}$ and suppose that $P_2, P_3, \ldots, P_{n-1}$ have been defined. Use part (II) to find a set P_n such that $P_n=P_n^{-1}$, $P_n\supset P_{n-1}\cup U_n$, and

$$\frac{\nu((U_nP_n)\cap P_n')}{\nu(P_n)}<\frac{1}{n}.$$

Each $x\in S$ is some x_m; and so for $n\geqq m$, we have

$$\frac{\nu((xP_n)\cap P_n')}{\nu(P_n)}\leqq\frac{\nu((U_nP_n)\cap P_n')}{\nu(P_n)}<\frac{1}{n},$$

and

$$\frac{\nu((xP_n)\cap P_n)}{\nu(P_n)}=1-\frac{\nu((xP_n)\cap P_n')}{\nu(P_n)}>1-\frac{1}{n}.$$

This establishes (i).

(2.2) Theorem. *Let H be a LCA group containing a compact open subgroup J. Let Y be the character group H and let A be the annihilator of J in Y. Let Haar measures on H and Y be chosen so that both J and A have measure 1. Let Δ be any σ-compact subset of Y. There is then a sequence $\{w_n\}_{n=1}^{\infty}$ of functions on Y with the following properties.*

(i) *Each w_n is continuous and vanishes outside of a compact set, and $w_n(Y)\subset[0, 1]$.*

(ii) *The sequence $\{w_n\}_{n=1}^{\infty}$ is increasing and $\lim_{n\to\infty}w_n\geqq\xi_\Delta$.*

(iii) *Write $\varphi_n(x)=\int_Y w_n(\chi)\chi(x)\,d\chi$ for $x\in H$. Then we have $\varphi_n\geqq 0$ and $\int_H\varphi_n(x)\,dx=1$.*

(iv) *If* $\chi \in \Delta$, *then* $\lim_{n\to\infty} \int_H \varphi_n(x^{-1}y)\chi(y)dy = \chi(x)$ *for all* $x \in H$. *If* $\chi \in Y \cap \Delta'$, *then* $\int_H \varphi_n(x^{-1}y)\chi(y)dy = 0$ *for all positive integers* n.

Proof. As noted in (1. 2), A is a compact open subgroup of Y. With no loss of generality we suppose that Δ is a subgroup of Y that is a countably infinite union of distinct cosets of A; we write $\Delta = \bigcup_{k=1}^{\infty} \chi_k A$, where χ_1 is the character identically equal to 1 and $\chi_k\chi_l^{-1} \notin A$ if $k \neq l$. We apply (2. 1) to the countable group Δ/A. Let τ be the natural mapping of Δ onto Δ/A; let $\{P_n\}_{n=1}^{\infty}$ be the sets constructed for the group Δ/A in (2. 1); and let $B_n = \tau^{-1}(P_n)$ $(n = 1, 2, 3, \ldots)$. Clearly each set B_n is a finite union of cosets of A and is thus a compact open set; suppose that B_n is the union of r_n distinct cosets of A. Upon renumbering the characters $\{\chi_k\}_{k=1}^{\infty}$ if necessary, we may suppose that there is a sequence $1 = r_1 < r_2 < \cdots < r_n < \cdots$ of positive integers such that $B_n = \bigcup_{k=1}^{r_n} \chi_k A$ $(n = 1, 2, 3, \ldots)$.

We define w_n by

$$w_n = \frac{1}{r_n} \xi_{B_n} * \xi_{B_n} \qquad (n = 1, 2, 3, \ldots).$$

It is clear that $w_n(\chi)$ is equal to $\frac{1}{r_n}$ times the Haar measure of the set $(\chi^{-1}B_n) \cap B_n$. For $\chi \notin \Delta$, we have $(\chi^{-1}B_n) \cap B_n = \emptyset$. For $\alpha \in A$ and $m = 1, 2, 3, \ldots$, it is easy to see that

$$w_n(\chi_m \alpha) = \frac{\nu\big((\tau(\chi_m^{-1})P_n) \cap P_n\big)}{\nu(P_n)}.$$

This equality and (2. 1. i) imply that $\lim_{n\to\infty} w_n(\chi_m\alpha) = 1$. We have thus proved (i) and (ii).

Since J is the annihilator of A (1. 3), an easy computation shows that

$$\int_{B_n} \chi(x)\, d\chi = \sum_{k=1}^{r_n} \chi_k(x)\xi_J(x) \qquad (x \in H).$$

Since $B_n = B_n^{-1}$ [this follows from the equality $P_n = P_n^{-1}$ and the fact that A is a subgroup of Y], we have

$$\varphi_n(x) = \frac{1}{r_n}\left|\int_{B_n} \chi(x)\, d\chi\right|^2 = \frac{1}{r_n}\left|\sum_{k=1}^{r_n} \chi_k(x)\right|^2 \xi_J(x).$$

Thus φ_n is nonnegative. Since $\chi_k\chi_l^{-1} \in A$ only for $k = l$, we also have

$$\int_H \varphi_n(x)\, dx = \frac{1}{r_n} \sum_{k=1}^{r_n} \sum_{l=1}^{r_n} \int_J \chi_k(x)\chi_l^{-1}(x)\, dx = 1.$$

Thus we have proved (iii).

Now let χ be any character in Δ. Then $\chi=\chi_m\alpha$ for a unique positive integer m and a unique $\alpha\in\Delta$. We have

$$\int_H \varphi_n(x^{-1}y)\chi(y)\,dy = \int_H \varphi_n(y)\chi(xy)\,dy = \chi(x)\int_H \varphi_n(y)\chi(y)\,dy =$$

$$= \chi(x)\frac{1}{r_n}\sum_{k=1}^{r_n}\sum_{l=1}^{r_n}\int_J \chi_k(x)\chi_l^{-1}(x)\chi_m(x)\,dx.$$

The integral $\int_J \chi_k(x)\chi_l^{-1}(x)\chi_m(x)\,dx$ is 0 if $\chi_k\chi_l^{-1}\chi_m\notin \mathrm{A}$ and is 1 otherwise. The number of pairs (k, l) for which $\chi_k\chi_l^{-1}\chi_m\in\mathrm{A}$ is equal to $\nu\big((\tau(\chi_m)P_n)\cap P_n\big)$. Thus (2. 1) implies that $\lim_{n\to\infty}\int_H \varphi_n(x^{-1}y)\chi(y)\,dy=\chi(x)$. If $\chi\notin\Delta$, then $\chi_k\chi_l^{-1}\chi$ is in A for no choice of k and l; hence $\int_H \varphi_n(x^{-1}y)\chi(y)\,dy=0$ in this case. This establishes (iv) and completes the present proof.

(2. 3) Theorem. *Let H, J, Y, and A be as in* (2. 2). *Let g be a continuous function on H vanishing outside of a compact set F. Let Γ be a σ-compact subset of Y. Then there is an open σ-compact subgroup Δ of Y such that $\Delta\supset\Gamma$ and such that the functions φ_n constructed for Δ as in* (2. 2. iii) *have the property that*

$$\text{(i)}\qquad \lim_{n\to\infty}\int_H \varphi_n(x^{-1}y)g(y)\,dy = g(x)\qquad (x\in H).$$

Proof. Consider the open compact subset JF of H. The Stone—Weierstrass theorem implies that there is a countable subset Λ of Y such that complex linear combinations of characters in Λ approximate g arbitrarily in the uniform metric on JF. Let Δ be any σ-compact subgroup of Y that contains $\mathrm{A}\cup\Gamma\cup\Lambda$. Let ε be a positive real number and let

$$\text{(1)}\qquad \left|g(z)-\sum_{j=1}^{m}a_j\chi_j(z)\right|<\frac{\varepsilon}{3}$$

for all $z\in JF$, where the a_j are complex numbers and $\chi_j\in\Lambda$. We have $\int \varphi_n(x^{-1}y)g(y)\,dy=0$ for $x\notin JF$, so that (i) holds trivially for all $x\notin JF$. For n sufficiently large, (2. 2. iv) implies that

$$\text{(2)}\qquad \left|\int_H \varphi_n(x^{-1}y)\left(\sum_{j=1}^{m}a_j\chi_j(y)\right)dy-\sum_{j=1}^{m}a_j\chi_j(x)\right|<\frac{\varepsilon}{3}$$

for all $x \in JF$. Finally we have

(3) $$\left| \int_H \varphi_n(x^{-1}y)g(y)\,dy - \int_H \varphi_n(x^{-1}y)\left(\sum_{j=1}^m a_j\chi_j(y)\right)dy \right| =$$

$$= \left| \int_J \varphi_n(y)\left[g(xy) - \sum_{j=1}^m a_j\chi_j(xy)\right]dy \right| \leqq$$

$$\leqq \int_J \varphi_n(y)\,dy \cdot \max\left\{\left|g(xy) - \sum_{j=1}^m a_j\chi_j(xy)\right| : x \in JF,\ y \in J\right\} < \frac{\varepsilon}{3}.$$

Combining (1), (2), and (3), we obtain (i).

§ 3. Proof of Plancherel's theorem

Throughout this section, G is an arbitrary LCA group and X is its character group.

(3. 1) Theorem. *Let f be a function in $\mathfrak{L}_1(G) \cap \mathfrak{L}_2(G)$. Then Tf is in $\mathfrak{L}_2(\mathsf{X})$ and for an appropriate choice of Haar measure on G and X, we have*

(i) $$\|Tf\|_2 \leqq \|f\|_2.$$

Proof. Let G be represented as $R^a \times H$ as in (1. 1), so that X is represented as $R^a \times \mathsf{Y}$, where Y is the character group of H. A generic [continuous!] character of $R^a \times H$ has the form

$$(\mathbf{u}, x) \to \exp\left[i(u_1 v_1 + \cdots + u_a v_a)\right]\chi(x)$$

for some $\mathbf{u} \in R^a$ and $\chi \in \mathsf{Y}$. Let J be a compact open subgroup of H and A the annihilator of J in Y. We choose Haar measure in G to be the product of $(2\pi)^{-a/2}$ times Lebesgue measure on R^a and of the Haar measure on H assigning measure 1 to J. We choose Haar measure on X to be the product of $(2\pi)^{-a/2}$ times Lebesgue measure on R^a and of the Haar measure on Y assigning measure 1 to A.

The function Tf is continuous on X and vanishes at infinity, so that Tf vanishes outside of a set $R^a \times \Delta$, where Δ is a σ-compact subset of Y as in (2. 2). Let $\{w_n\}_{n=1}^{\infty}$ be a sequence of functions on Y as constructed in (2. 2) for this set Δ. Define the sequence $\{W_n\}_{n=1}^{\infty}$ of functions on $\mathsf{X} = R^a \times \mathsf{Y}$ by

(1) $$W_n(\mathbf{v}, \chi) = \exp\left[-\frac{1}{2n^2}(v_1^2 + \cdots + v_a^2)\right]w_n(\chi)$$

[with obvious modifications if the factor R^a or the factor Y is missing]. Defining Φ_n on $G = R^a \times H$ by

(2) $$\Phi_n(\mathbf{u}, x) = \int_{R^a \times \mathsf{Y}} W_n(\mathbf{v}, \chi)\exp\left[i(u_1 v_1 + \cdots + u_a v_a)\right]\chi(x)\,d(\mathbf{v}, x),$$

we have

$$(3)\qquad \Phi_n(\mathbf{u}, x) = (2\pi)^{-a/2} \prod_{k=1}^{a} \int_{-\infty}^{\infty} \exp\left[\frac{-1}{2n^2} v_k^2 + iu_k v_k\right] dv_k \int_{\mathsf{Y}} \chi(x) w_n(\chi)\, d\chi$$

$$= n^a \left(\prod_{k=1}^{a} \exp\left[-\frac{n^2}{2} u_k^2\right]\right) \varphi_n(x).$$

It is obvious from (3) that

$$\int_{R^a \times H} \Phi_n(\mathbf{u}, x)\, d(\mathbf{u}, x) = n^a \left(\prod_{k=1}^{a} (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp\left[\frac{-n^2}{2} u_k^2\right] du_k \int_H \varphi_n(x)\, dx \right) = 1.$$

Also each function Φ_n is plainly nonnegative.

For notational convenience, we now revert to one-variable notation in writing integrals over G and X. By Fubini's theorem [which evidently applies], we have

$$(4)\qquad \int_{\mathsf{X}} |Tf(\psi)|^2 W_n(\psi)\, d\psi = \int_{\mathsf{X}} \int_G f(t)\overline{\psi(t)}\, dt \int_G \overline{f(s)}\, \psi(s)\, ds\, W_n(\psi)\, d\psi$$

$$= \int_G \int_G \int_{\mathsf{X}} W_n(\psi)\, \psi(t^{-1}s)\, d\psi f(t) \overline{f(s)}\, dt\, ds$$

$$= \int_G \int_G \Phi_n(t^{-1}s) f(t) \overline{f(s)}\, dt\, ds.$$

Applying the Cauchy—Schwarz inequality to the last integral, and taking cognizance of the invariance of the Haar integral, we have

$$(5)\qquad \int_G \int_G \Phi_n(t^{-1}s) f(t) \overline{f(s)}\, dt\, ds \leqq$$

$$\leqq \left[\int_G \int_G \Phi_n(t^{-1}s) |f(t)|^2\, ds\, dt \right]^{\frac{1}{2}} \times \left[\int_G \int_G \Phi_n(t^{-1}s) |f(s)|^2\, ds\, dt \right]^{\frac{1}{2}} = \|f\|_2^2.$$

Combining (4) and (5) and taking the limit as $n \to \infty$, we obtain (i).

Theorem (3. 1) shows that the Fourier transformation T, which is linear on $\mathfrak{L}_1(G) \cap \mathfrak{L}_2(G)$, carries this space into $\mathfrak{L}_2(\mathsf{X})$ without increasing the $\mathfrak{L}_2$ norm. Therefore there is a unique, linear, norm nonincreasing mapping of $\mathfrak{L}_2(G)$ into $\mathfrak{L}_2(\mathsf{X})$ that extends T. We call this extended mapping T as well, and we note that if $\|f_n - f\|_2 \to 0$, where $f_n \in \mathfrak{L}_1(G) \cap \mathfrak{L}_2(G)$ and $f \in \mathfrak{L}_2(G)$, then $\|Tf_n - Tf\|_2 \to 0$.

(3. 2) Theorem. *Let g be a function in $\mathfrak{L}_1(\mathsf{X}) \cap \mathfrak{L}_2(\mathsf{X})$. Then T^*g is in $\mathfrak{L}_2(G)$, and if Haar measures on G and X are chosen as in (3. 1), we have*

$$\text{(i)}\qquad \|T^*g\|_2 \leqq \|g\|_2.$$

Proof. This assertion is proved just as (3. 1) was proved. Plainly the integral $\int_X g(\psi)\psi(x)d\psi$ behaves just like the integral $\int_G f(x)\overline{\psi(x)}dx$. Since the annihilator of A in G is J (1. 3) and since Haar measures in G and X have been chosen symmetrically, the proof of (3. 1) can be repeated *verbatim* to yield the present theorem.

Like T, the transformation T^* can be extended to a linear norm nonincreasing mapping of $\mathfrak{L}_2(X)$ into $\mathfrak{L}_2(G)$.

(3. 3) Theorem. *For $f\in\mathfrak{L}_2(G)$, we have*

$$\text{(i)} \qquad T^*Tf=f.$$

Proof. We lose no generality in proving the theorem for $f\in\mathfrak{L}_1(G)\cap\mathfrak{L}_2(G)$. At first we write G in the form $R^a\times H$ as in (1. 1), and we consider any function h on $R^a\times H$ of the form

$$h(\mathbf{u}, x) = g_1(u_1)g_2(u_2)\cdots g_a(u_a)g(x),$$

where $g_1, g_2, \ldots, g_a, g$ are continuous on $R, R, \ldots, R, H$ respectively, and vanish outside of compact sets. It is elementary and easy to show that

$$\lim_{n\to\infty}\left\{(2\pi)^{-1/2}n\int_{-\infty}^{\infty}\exp\left[-\frac{n^2}{2}(u-v)^2g(v)\right]dv\right\} = g(u)$$

for all $u\in R$ if g is bounded and uniformly continuous on R. We now construct a subset Δ of Y containing the set $\{\chi\in Y: Tf(\mathbf{v},\chi)\neq 0$ for some $\mathbf{v}\in R^a\}$ and having also the property that (2. 3. i) holds for the function g. Then we form the functions W_n and Φ_n as in (3. 1. 1) and (3. 1. 2) *for this choice of* Δ. We see immediately that

$$\text{(1)} \qquad \lim_{n\to\infty}\int_{R^a\times H}\Phi_n(-\mathbf{u}+\mathbf{v}, x^{-1}y)h(\mathbf{v}, y)\,d(\mathbf{v}, y) = h(\mathbf{u}, x)$$

for all $(\mathbf{u}, x)\in R^a\times H$. Reverting to one-variable notation for G and X, we write

$$\text{(2)} \qquad \int_G\int_X W_n(\psi)Tf(\psi)\psi(s)\,d\psi h(s)\,ds = \int_G\int_X W_n(\psi)\int_G f(t)\psi(t^{-1})\,dt\psi(s)\,d\psi h(s)\,ds =$$

$$= \int_G f(t)\int_G\int_X W_n(\psi)\psi(t^{-1}s)\,d\psi h(s)\,ds\,dt = \int_G f(t)\int_G\Phi_n(t^{-1}s)h(s)\,ds\,dt.$$

By (1), the integral $\int_G\Phi_n(t^{-1}s)h(s)ds$ converges [boundedly!] to $h(t)$ for all $t\in G$. The integral $\int_X W_n(\psi)Tf(\psi)\psi(s)d\psi$ is equal to $T^*(W_nTf)$. Since T^* is linear and norm nonincreasing (3. 2), and $\|W_nTf-Tf\|_2\to 0$, we have $\lim_{n\to\infty}T^*(W_nTf)=T^*(Tf)$

in the $\mathfrak{L}_2(G)$ metric. The equalities (2) then imply that

$$\int_G T^*Tf(s)h(s)\,ds = \int_G \lim_{n\to\infty} T^*(W_nTf)(s)h(s)\,ds = \tag{3}$$

$$= \lim_{n\to\infty} \int_G T^*(W_nTf)(s)h(s)\,ds = \int_G f(t)h(t)\,dt.$$

Linear combinations of functions h are dense in $\mathfrak{L}_2(G)$, and so (3) implies that $T^*Tf=f$. Since $\mathfrak{L}_1(G)\cap\mathfrak{L}_2(G)$ is dense in $\mathfrak{L}_2(G)$, this equality holds for all $f\in\mathfrak{L}_2(G)$.

(3. 4) Theorem. *The Fourier transformation T is a linear isometry of $\mathfrak{L}_2(G)$ onto $\mathfrak{L}_2(X)$, and T^*T and TT^* are the identity transformations on $\mathfrak{L}_2(G)$ and $\mathfrak{L}_2(X)$ respectively.*

Proof. For $g\in\mathfrak{L}_2(X)$, we have $T^*g\in\mathfrak{L}_2(G)$ by (3. 2). We apply (3. 3) [with the rôles of G and X interchanged] to infer that $TT^*g=g$. Hence T carries $\mathfrak{L}_2(G)$ *onto* $\mathfrak{L}_2(X)$. For $f\in\mathfrak{L}_2(G)$, we have

$$\|f\|_2 = \|T^*Tf\|_2 \leqq \|Tf\|_2 \leqq \|f\|_2.$$

Thus T preserves norms and so is an isometry.

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