On some combinatorial relations concerning the symmetric random walk

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Dedicated to the three inseparable friends P. Erdős, T. Gallai, and P. Turán at the occasion of all being close to 50

§ 1. Introduction and notations

1. In this paper we shall consider sequences $\vartheta=(\vartheta_1, \vartheta_2, \ldots, \vartheta_{2n})$ of n $(+1)$-s and n (-1)-s, each possible array being of equal probability $1\Big/\binom{2n}{n}$. Thus the partial sums $s_0=0, s_i = \vartheta_1+\vartheta_2+\ldots+\vartheta_i (i=1,2,\ldots,2n)$ generate a simple symmetric random walk, returning after $2n$ steps to the origin.

We use the following notations:

$\varkappa = \max\limits_{0\leq i\leq 2n} s_i$; $\varrho=\min\{i:s_i=\varkappa\}$ (index of the first maximum).

$\lambda-1$ is the number of the intersections, i. e. the number of i-s with $s_i=0$, $s_{i-1}s_{i+1} = -1$ (thus λ is the number of half-waves).

γ is the Galton-statistics (i. e. 2γ is the number of indices i for which either $s_i>0$, or $s_i=0, s_{i-1} = +1$).

The authors have found the following asymptotic relation [6], [2]:

$$\lim_{n\to\infty} P(\varkappa<y\sqrt{2n},\ \varrho<2nz) = \lim_{n\to\infty} P(\lambda<y\sqrt{2n},\ \gamma<nz) =$$

$$= \sqrt{\frac{2}{\pi}} \int_0^y \int_0^z \frac{u^2}{[v(1-v)]^{3/2}} e^{-\frac{u^2}{2v(1-v)}}\, du\, dv \qquad (y\geq 0;\ 1\geq z\geq 0).$$

In connection with this relation, E. SPARRE ANDERSEN raised the question[1]), whether there exists some equivalence principle for the finite case too.

In the following we give some equivalence theorems and prove among others that

$$P(\varkappa=l) = \tfrac{1}{2}\big(P(\lambda=l)+P(\lambda=l+1)\big) \qquad (l=0, 1, \ldots, n), \tag{1.1}$$

$$P(\varkappa=l,\ \varrho=r) = P(\lambda'=l,\ \pi=r), \tag{1.2}$$

[1]) At the occasion of the *Conference on Probability and Statistics* held in Oberwolfach, August 20—26, 1961.

where π denotes the number of positve terms in $(s_0, s_1, \dots, s_{2n})$, while λ' is the number of indices i for which $s_{i-1}=0$, $s_i=+1$. (1.2) implies

$$P(\varrho=r)=P(\pi=r),$$

which is a special case of a well-known result of SPARRE ANDERSEN [1]; it implies also the following result of MIHALEVIČ [5]:

$$P(\varkappa=l) = P(\lambda'=l).$$

Thus *we have a joint equivalence between* $(\varkappa, \varrho)$ *and* (λ', π). We would like to point out furthermore that *each of our theorems is proved by means of one-to-one correspondences between the sets of paths considered.* This indicates a combinatorial and geometrical background of these equivalences.

We also remark that in our constructions $\varkappa$ appears virtually more as the number of ladder indices (see FELLER [3]), than as the maximum, both coinciding for the special variables $\vartheta_i=\pm 1$.

2. We shall make use of the following further terminology and notations:

The polygonal line whose subsequent vertices have the coordinat es (i, s_i) $(i=0, 1, 2, \dots, j)$ is called the path $(s_0, s_1, \dots, s_j)$.

E_{2n} is a path $(s_0, s_1, \dots, s_{2n})$ with $s_{2n}=0$. A point $(2i, s_{2i})$ of the path E_{2n}, for which $s_{2i}=0$ and $s_{2i-1}s_{2i+1}=-1$, as well as the points $(0,0)$ and $(2n,0)$ of E_{2n}, are called intersection points or T-points. As defined above, $\lambda+1$ is the number of T-points.

By a T'-point we mean a point $(2i+1, 1)$ of the path E_{2n}, for which $s_{2i}=0$, $s_{2i+1}=+1$ (this kind of points was treated by MIHALEVIČ [5]). λ' is the number of T'-points.

E^l_{2n} is a path E_{2n} with $\lambda=l$,

$\langle i,j\rangle$ is a section of a path lying between the points (i, s_i) and (j, s_j), i. e. the sequence $(\vartheta_{i+1}, \vartheta_{i+2}, \dots, \vartheta_j)$.

k is called a strict ladder index (FELLER [3]); if $s_k>s_i$ for $i=0, 1, \dots, k-1$; k is called a strict backward ladder index if $s_k>s_i$ for $i=k, k+1, \dots, 2n$.

A^l_r is a path $(s_0, s_1, \dots, s_r)$, for which $s_0=0$, $s_1<l$, $s_2<l$, $\dots$, $s_{r-1}<l$, $s_r=l$, i. e. its l-th strict ladder index being r.

$N(\cdot)$ is the number of all possible paths whose type is given in the brackets $\left(\text{e. g. } N(E_{2n})=\binom{2n}{n}\right)$.

§ 2. Equivalence relations

1. The maxium and the number of waves. We shall prove the following

Theorem 2.1. $P(\varkappa=l) = \frac{1}{2}[P(\lambda=l)+P(\lambda=l+1)]$ $(l=0, 1, 2, \dots, n)$.

Proof. We consider a path E_{2n} with $\varkappa=l$. According to the index ϱ of the first maximum, we distinguish two different cases:

a) ϱ is the only position, for which the maximum takes place;

b) there are more than one maximum places.

In both cases we shall make use of the following

Lemma 2.1. $\frac{1}{2} N(E_{2n}^{l}) = N(A_{2n}^{2l})$.

This was proved in [2] by means of a one-to-one correspondence between the sets of paths.

In case a) we consider the sections $\langle 0, \varrho \rangle$ and $\langle \varrho, 2n \rangle$. Replacing in the second part the steps $(\vartheta_{\varrho+1}, \vartheta_{\varrho+2}, \ldots, \vartheta_{2n})$ by the steps $(-\vartheta_{2n}, -\vartheta_{2n-1}, \ldots, -\vartheta_{\varrho+2}, -\vartheta_{\varrho+1})$, we obtain a path A_{2n}^{2l}. According to Lemma 2.1 this path can be transformed into a path E_{2n}^{l} with $s_1 = +1$.

Obviously this procedure is invertible, by considering the l-th strict ladder index of the path A_{2n}^{2l}.

In case b) let us denote by $\bar{\varrho}$ the index of the last maximum. The path E_{2n} with $s_{\varrho} = s_{\bar{\varrho}} = l$ consists of the following three sections: $\langle 0, \varrho \rangle$, $\langle \varrho, \bar{\varrho} \rangle$, $\langle \bar{\varrho}, 2n \rangle$. We apply the following transformation: we replace in $\langle \varrho, \bar{\varrho} \rangle$ the steps $(\vartheta_{\varrho+1} = -1, \vartheta_{\varrho+2}, \ldots, \vartheta_{\bar{\varrho}})$ by $(\vartheta_{\varrho+2}, \ldots, \vartheta_{\bar{\varrho}}, +1)$ and in $\langle \bar{\varrho}, 2n \rangle$ the steps $(\vartheta_{\bar{\varrho}+1}, \vartheta_{\varrho+2}, \ldots, \vartheta_{2n})$ by the steps $(-\vartheta_{2n}, -\vartheta_{2n-1}, \ldots, -\vartheta_{\bar{\varrho}+2}, -\vartheta_{\bar{\varrho}+1})$. Thus we obtain a path A_{2n}^{2l+2}. According to Lemma 2.1 this path can be transformed into a path E_{2n}^{l+1} with $s_1 = +1$.

In order to invert this procedure we have only to find the l-th and $l+2$-th ladder indices of the path A_{2n}^{2l+1}. Cases a) and b) complete the proof of Theorem 2.1.

2. Two variate equivalences. We shall prove the following

Theorem 2.2. $P(\varkappa = l, \varrho = r) = P(\lambda' = l, \pi = r)$ $(l = 0, r = 0; l = 1, 2, \ldots, n,$ $r = l, l+2, \ldots, 2n-l)$.

Proof[2]. For $r = 0$, $l = 0$ the paths of both kinds coincide, we have to consider only the case $l \geqq 1$. Then each path with $(\lambda' = l, \pi = r)$ can be divided by the T'-points $(2i+1, 1)$ and the points $(2j, 0)$ with $s_{2j} = 0$ and $s_{2j-1} = +1$ into $2l$ or $2l+1$ sections, some of which are starting from $+1$ and ending in 0, all inner points being strictly positive (type α)), while the others are starting from 0, ending in $+1$, all inner points being non-positive (type β)).

The first section is always of type β); the last section is either of type α) or of type β), but in the latter case the last $(\vartheta_{2n+1} = +1)$ step is missing.

There are altogether l sections of type α) with total length r and l or $(l+1)$ sections of type β).

Let us now consider the sections of type α). We change all ϑ_i-s occurring in them into $-\vartheta_i$ and link together the new sections obtained by this procedure, maintaining their original order of succession. We now link together all sections of type β); denoting the steps of the section thus obtained by $(\vartheta'_{r+1}, \vartheta'_{r+2}, \ldots, \vartheta'_{2n})$ we transform them into $(-\vartheta'_{2n}, -\vartheta'_{2n-1}, \ldots, -\vartheta'_{r+2}, -\vartheta'_{r+1})$ and join the respective section to the first section obtained. As a result we obtain a path with $\varkappa = l$, $\varrho = r$.

The reverse procedure transforms each path $\{\varkappa = l, \varrho = r\}$ into the corresponding path $\{\lambda' = l, \pi = r\}$; this can be performed by considering the strict ladder indices in section $\langle 0, r \rangle$ and the strict backward ladder indices in section $\langle r, 2n \rangle$.

[2]) Similar construction is used by CH. HOBBY and R. PYKE [4].

In the following theorem we shall prove two equivalences according to whether the maximum is even or odd.

Theorem 2. 3.

$$P(\varkappa=s_{2r'}=2l)=\tfrac{1}{2}P(\lambda=2l,\gamma=r')+P(\lambda=2l+1,\gamma=r')+\tfrac{1}{2}P(\lambda=2l+2,\gamma=r')$$

and
$$P(\varkappa=s_{2r'+1}=2l-1)=$$
$$=P(\lambda=2l-1,\gamma=r';s_1=-1)+P(\lambda=2l,\gamma=r')+P(\lambda=2l+1,\gamma=r',s_1=+1).$$

Proof. We use the same procedure as in the proof of Lemma 2. 1.

The crucial point in the proof of Lemma 2. 1 was the division of a path A_{2n}^{2l} by means of its even strict ladder indices. The last step of each section between two consecutive ladder indices is always $(+1)$; omitting this and placing a (-1) before the section, we obtain a negative half wave.

Considering a path whose maximum $2l$ is taken on for the index $2r'$ let us denote by $2r$ $(2\bar{r})$ the first (last) index of maximum. The section $\langle 0, 2r\rangle$ is a path A_{2r}^{2l}, the section $\langle 2\bar{r}, 2n\rangle$ is an inverted path $A_{2(n-\bar{r})}^{2l}$. As described before, both sections can be divided into l parts and each part can be transformed into a negative half wave. The half waves generated by A_{2r}^{2l} will be turned into positive half waves by reflection. If $r=r'=\bar{r}$ (case ā)) there is no other section; if $r<\bar{r}$ but either $r'=r$ or $r'=\bar{r}$ (case b̄)) the section $\langle 2r, 2\bar{r}\rangle$ is a negative half wave itself. In this case if $r'=r$, then this half wave will remain negative, if $r'=r$, it will be turned into positive one. If $r<r'<\bar{r}$ (case c̄)) the sections $\langle 2r, 2r'\rangle$ and $\langle 2r', 2\bar{r}\rangle$ are half waves themselves. The former will be turned into a positive one, the latter will remain negative. What remains to be done is to connect these half waves, namely a positive after a negative one; in case ā) and c̄) beginning with a positive half wave, in case b̄) with a negative one if $r'=r$ and with a positive one if $r'=\bar{r}$.

Each of these procedures determines uniquely the inverse construction, leading to a one-to-one mapping of the sets of corresponding paths. For the second part of this theorem similar construction can be applied.

Summation over l of the relations in Theorem 2. 3 results in the following

Corollary 2. 2. $P(s_{2r+1}=\varkappa)=P(s_{2r}=\varkappa)=P(\gamma=r)$ *for* $r=0, 1, 2, \ldots, n$.

Another fact proved herewith is expressed in the

Corollary 2. 3.

$$P(\varkappa=2l,\varrho=2r)=\tfrac{1}{2}P(\lambda=2l,\gamma=r)+P(\lambda=2l+1,\gamma=r,\ s_1=+1).$$

and
$$P(\varkappa=2l-1,\varrho=2r+1)=$$
$$=P(\lambda=2l-1,\gamma=r,s_1=-1)+\tfrac{1}{2}P(\lambda=2l,\gamma=r).$$

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