On regular vector measures

By N. DINCULEANU in Bucharest (Roumania)

1. Introduction

Let T be a locally compact space, $\mathcal{C}$ a nonvoid class of subsets of T, X a Banach space and $\mathbf{m}: \mathcal{C} \to X$ a set function.

A set $A \in \mathcal{C}$ is said to be *regular* (with respect to $\mathbf{m}$) if for every $\varepsilon > 0$ there exist a compact set $K \subset A$ and an open set $G \supset A$ such that if $A' \in \mathcal{C}$ and $K \subset A' \subset G$, then $|\mathbf{m}(A) - \mathbf{m}(A')| < \varepsilon$. The set function $\mathbf{m}$ is called *regular* if every set $A \in \mathcal{C}$ is regular.

The class $\mathcal{C}$ is a *lattice* if $A \in \mathcal{C}$ and $B \in \mathcal{C}$ imply $A \cup B \in \mathcal{C}$ and $A \cap B \in \mathcal{C}$.

The class $\mathcal{C}$ is a *clan* if $A \in \mathcal{C}$ and $B \in \mathcal{C}$ imply $A \cup B \in \mathcal{C}$ and $A - B \in \mathcal{C}$.

We shall denote by $\mathcal{B}$ the clan of the Borel subsets of T which are relatively compact. We call (Borel) *measure on T* with values in X, every countably additive set function defined on $\mathcal{B}$ with values in X.

By the theorem of KAKUTANI [6] a positive regular Borel measure on T can be identified with a positive Radon measure on T [1].

We shall consider sometimes the following conditions on $\mathcal{C}$:

(i) *For each compact set $K \subset T$ and each open set $G \supset K$, there exists a set $A \in \mathcal{C}$ such that $K \subset A \subset G$.*

(ii) *For each set $A \in \mathcal{C}$ there exists a set $A' \in \mathcal{C}$ such that $A \subset \operatorname{Int} A'$.*

The following result is known ([1], chap IV, § 4, No. 10, and [5], § 53, § 54).

If $\mathcal{C}$ is a lattice satisfying the condition (i), *and if a positive (finite) set function μ defined on $\mathcal{C}$ is increasing, subadditive, additive and regular, then there exists a unique positive Radon measure μ_1 on T such that the sets $A \in \mathcal{C}$ are μ_1-integrable and $\mu_1(A) = \mu(A)$ for $A \in \mathcal{C}$.*

In particular, if $\mathcal{C}$ is a *clan* satisfying the condition (i) and if the *positive* set function μ is *additive and regular*, the above conclusion remains valid (because in this case μ is also increasing and subadditive).

In this paper, we extend this last result to the case $\mathcal{C}$ is a clan satisfying the conditions (i) and (ii) and $\mathbf{m}: \mathcal{C} \to X$ is additive, regular and of finite variation (theorem 3).

In case T is a compact metric space this extension was done by C. FOIAȘ and has been exposed in [7] (Chapter 25, § 5).

In case T is compact and X is the space of the complex numbers, see [4]. We remark that in [4] the definition of the regularity is different from that used in the present paper. However, these two definitions are equivalent, for instance, if $\mathcal{C}$ contains the compact subsets of T.

2. Regular set functions

In the sequel, we shall suppose that $\mathcal{C}$ is a *clan* and that the set function $\mathbf{m}: \mathcal{C} \to X$ is *additive*.

We say that a set $A \in \mathcal{C}$ is *regular on the left (on the right)* if for every $\varepsilon > 0$ there exists a compact set $K \subset A$ (an open set $G \supset A$) such that if $A' \in \mathcal{C}$ and $K \subset A' \subset A (A \subset A' \subset G)$ then $|\mathbf{m}(A) - \mathbf{m}(A')| < \varepsilon$.

All the compact sets $K \in \mathcal{C}$ are regular on the left, and all the open sets $G \in \mathcal{C}$ are regular on the right.

If all the sets $A \in \mathcal{C}$ are regular on the left (on the right) we say that the set function $\mathbf{m}$ is regular on the left (on the right).

It is clear that a regular set $A \in \mathcal{C}$ is regular on the left and on the right and we shall show that the converse is also true.

We first prove

Proposition 1. *A set $A \in \mathcal{C}$ is regular if and only if for every $\varepsilon > 0$ there exist a compact set $K \subset A$ and an open set $G \supset A$ such that if $B \in \mathcal{C}$ and $B \subset G - K$ then $|\mathbf{m}(B)| < \varepsilon$.*

A set $A \in \mathcal{C}$ is regular on the left (on the right) if and only if for every $\varepsilon > 0$ there exists a compact set $K \subset A$ (an open set $G \supset A$) such that if $B \in \mathcal{C}$ and $B \subset A - K$ $(B \subset G - A)$ then $|\mathbf{m}(B)| < \varepsilon$.

We shall prove only the part concerning the regularity. Suppose first that A is regular and let $\varepsilon > 0$. Take a compact set $K \subset A$ and an open set $G \supset A$ such that if $A' \in \mathcal{C}$ and $K \subset A' \subset G$ then $|\mathbf{m}(A) - \mathbf{m}(A')| < \frac{\varepsilon}{2}$.

If $B \in \mathcal{C}$ and $B \subset G - K$ then

$$B = (A \cup B) - (A - B), \quad A - B \subset A \cup B$$

and

$$K \subset A \cup B \subset G, \quad K \subset A - B \subset G,$$

therefore

$$|\mathbf{m}(B)| = |\mathbf{m}(A \cup B) - \mathbf{m}(A - B)| \leqq$$

$$\leqq |\mathbf{m}(A \cup B) - \mathbf{m}(A)| + |\mathbf{m}(A) - \mathbf{m}(A - B)| < \varepsilon,$$

hence A verifies the condition of the propositon.

Conversely, suppose that this condition is verified and let $\varepsilon > 0$. Take a compact set $K \subset A$ and an open set $G \supset A$ such that if $B \in \mathcal{C}$ and $B \subset G - K$ then $|\mathbf{m}(B)| < \frac{\varepsilon}{2}$.

If $A' \in \mathcal{C}$ and $K \subset A' \subset G$ then

$$A - A' \subset G - K, \quad A' - A \subset G - K$$

and

$$A' = (A \cap A') \cap (A' - A), \quad A - (A \cap A') = A - A',$$

therefore

$$|\mathbf{m}(A) - \mathbf{m}(A')| = |\mathbf{m}(A) - \mathbf{m}(A \cap A') - \mathbf{m}(A' - A)| =$$

$$= |\mathbf{m}(A - A') - \mathbf{m}(A' - A)| \leqq |\mathbf{m}(A - A')| + |\mathbf{m}(A' - A)| < \varepsilon,$$

hence A is regular.

Proposition 2. *A set $A \in \mathcal{C}$ is regular if and only if it is regular on the left and on the right.*

We have already noticed that if A is regular then it is regular on the left and on the right.

Suppose now that A is regular on the left and on the right and let $\varepsilon > 0$. Take a compact set $K \subset A$ and an open set $G \supset A$ such that if $B \in \mathcal{C}$ and $B \subset A - K$ or $B \subset G - A$ then $|\mathbf{m}(B)| < \frac{\varepsilon}{2}$ (proposition 1).

If $C \in \mathcal{C}$ and $C \subset G - K$, then

$$C = (C - A) \cup (C \cap A), \quad (C - A) \cap (C \cap A) = \emptyset$$

and

$$C - A \subset G - A, \quad C \cap A \subset A - K.$$

The sets $B_1 = C - A$ and $B_2 = C \cap A$ are in $\mathcal{C}$ therefore $|\mathbf{m}(B_1)| < \frac{\varepsilon}{2}$ and $|\mathbf{m}(B_2)| < \frac{\varepsilon}{2}$. It follows that

$$|\mathbf{m}(C)| = |\mathbf{m}(C - A) + \mathbf{m}(C \cap A)| \leqq |\mathbf{m}(C - A)| + |\mathbf{m}(C \cap A)| < \varepsilon,$$

hence, by proposition 1, A is regular.

Remark. If $\mathbf{m}$ is not additive, it is possible that there exist sets $A \in \mathcal{C}$, regular on the left and on the right, without being regular.

Theorem 1. *Suppose that $\mathcal{C}$ verifies the condition* (ii). *Then* $\mathbf{m}$ *is regular if and only if it is regular on the left.*

By proposition 2, we have only to prove that if $\mathbf{m}$ is regular on the left then $\mathbf{m}$ is regular on the right.

Suppose that all the sets $A \in \mathcal{C}$ are regular on the left. Let $A \in \mathcal{C}$ and $\varepsilon > 0$. Take a set $A' \in \mathcal{C}$ such that $A \subset \mathrm{Int} A'$. The set $A' - A$ is in $\mathcal{C}$ hence it is regular on the left: there exists a compact set $K \subset A' - A$ such that if $B \in \mathcal{C}$ and $B \subset (A' - A) - K$ then $|\mathbf{m}(B)| < \varepsilon$. Note $U = \mathrm{Int} A'$. The set $G = U - K$ is open,

$$A = U - (U - A) \subset U - K = G$$

and

$$G - A = (U - K) - A = (U - A) - K \subset (A' - A) - K,$$

therefore if $B \in \mathcal{C}$ and $B \subset G - A$, then $B \subset (A' - A) - K$ hence $|\mathbf{m}(B)| < \varepsilon$. It follows that A is regular on the right.

Remarks. 1. If all the sets $A \in \mathcal{C}$ are *relatively compact*, then condition (i) implies condition (ii).

Indeed, for every set $A \in \mathcal{C}$ we can choose a relatively compact open set $U \supset A$. If V is an arbitrary open set containing U, then by condition (i) there exists a set $A' \in \mathcal{C}$ such that $\overline{U} \subset A' \subset V$, hence $A \subset \mathrm{Int}\, A'$.

The condition (i) is verified, for instance, if the clan $\mathcal{C}$ contains a base of the topology of T. In particular, the condition (i) is verified if $\mathcal{C}$ contains all the compact subsets of T, or all the compact subsets of T which are G_δ ([5], § 50, theorem 4).

2. *Suppose that the sets $A \in \mathcal{C}$ are relatively compact and that $\mathcal{C}$ verifies the condition* (i). *Then* **m** *is regular if and only if it is regular on the right.*

For every set $A \in \mathcal{C}$ we take a set $A' \in \mathcal{C}$ such that $\bar{A} \subset A'$. From the right regularity of $A' - A$ we deduce that A is regular on the left.

It follows that if the sets $A \in \mathcal{C}$ are relatively compact and if $\mathcal{C}$ verifies the condition (i), then the three kinds of regularity are equivalent to each other.

3. Set functions with finite variation

For every set $A \in \mathcal{C}$ we define the variation $\mu(A)$ of **m** on A by the equality:

$$\mu(A) = \sup \sum_i |\mathbf{m}(A_i)|$$

where the supremum is taken for all the finite families (A_i) of disjoint sets $A_i \in \mathcal{C}$ contained in A.

The *variation* μ of **m** is a positive (finite or $+\infty$) and additive set function defined on $\mathcal{C}$, $\mu(\Phi) = 0$ and

$$|\mathbf{m}(A)| \leqq \mu(A) \quad \text{for} \quad A \in \mathcal{C}.$$

We say that **m** is *of finite variation* if $\mu(A) < +\infty$ for every $A \in \mathcal{C}$.

Proposition 3. *If* **m** *is of finite variation* μ, *and if* μ *is countably additive then* **m** *is countably additive.*

The proof is based on the relations

$$\left|\mathbf{m}\left(\bigcup_{i=1}^{\infty} A_i\right) - \sum_{i=1}^{n} \mathbf{m}(A_i)\right| = |\mathbf{m}(\bigcup_{i>n} A_i)| \leqq \mu(\bigcup_{i>n} A_i) = \sum_{i>n} \mu(A_i)$$

where (A_i) is a sequence of disjoint sets in $\mathcal{C}$ with $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$.

Proposition 4. *Suppose that* **m** *is of finite variation*. Then **m** *is regular on the left if and only if its variation* μ *is regular on the left.*

If μ is regular on the left, we deduce immediately that **m** is regular on the left, using the inequality $|\mathbf{m}(B)| \leqq \mu(B)$ and proposition 2.

Conversely, suppose that **m** is regular on the left. Let $A \in \mathcal{C}$ and $\varepsilon > 0$. Let $(A_i)_{1 \leqq i \leqq n}$ be a finite family of disjoint sets $A_i \in \mathcal{C}$, contained in A, such that

$$\mu(A) < \sum_{i=1}^{n} |\mathbf{m}(A_i)| + \frac{\varepsilon}{2}.$$

Because each set A_i is regular on the left (with respect to **m**) there exists a compact set $K_i \subset A_i$, such that if $A_i' \in \mathcal{C}$ and $K_i \subset A_i' \subset A_i$ then $|\mathbf{m}(A_i) - \mathbf{m}(A_i')| < \frac{\varepsilon}{2n}$.

The set $K = \bigcup_{i=1}^{n} K_i$ is compact and $K \subset A$. Let now $A' \in \mathcal{C}$ be such that $K \subset A' \subset A$. For each i, the set $A_i' = A' \cap A_i$ is in $\mathcal{C}$ and $K_i \subset A_i' \subset A_i$; the sets A_i' are disjoint, therefore

$$\mu(A') \geqq \sum_{i=1}^{n} |\mathbf{m}(A_i')|$$

hence

$$0 \leqq \mu(A) - \mu(A') \leqq \sum_{i=1}^{n} |\mathbf{m}(A_i)| + \frac{\varepsilon}{2} - \sum_{i=1}^{n} |\mathbf{m}(A_i')| =$$

$$= \sum_{i=1}^{n} [|\mathbf{m}(A_i)| - |\mathbf{m}(A_i')|] + \frac{\varepsilon}{2} \leqq \sum_{i=1}^{n} |\mathbf{m}(A_i) - \mathbf{m}(A_i')| + \frac{\varepsilon}{2} < \varepsilon.$$

It follows that A is regular on the left with respect to μ, hence μ is regular on the left.

Theorem 2. *Suppose that $\mathcal{C}$ verifies the condition* (ii) *and that* $\mathbf{m}$ *is of finite variation. Then* $\mathbf{m}$ *is regular if and only if its variation μ is regular.*

Using the inequality $|\mathbf{m}(B)| \leqq \mu(B)$ and the proposition 2, we see that if μ is regular, then $\mathbf{m}$ is regular.

Conversely, if $\mathbf{m}$ is regular, then $\mathbf{m}$ is regular on the left; by proposition 4, μ is regular on the left and by theorem 1, μ is regular.

Remark. The conclusion of the theorem 2 remains valid if $\mathcal{C}$ is a clan of relatively compact sets and if $\mathcal{C}$ verifies the conditions (i). In particular, we have

Corollary. *If* $\mathbf{m}$ *is a regular Borel measure with finite variation, then its variation is a positive regular Borel measure.*

Indeed, the measure $\mathbf{m}$ is defined on the clan $\mathcal{B}$ of the relatively compact Borel sets, which verifies the condition (ii).

4. Extension of a regular additive set function to a measure

Proposition 5. *Let μ be a positive Radon measure on T and suppose that the sets of $\mathcal{C}$ are μ-integrable and that $\mathcal{C}$ verifies the condition* (i). *Then for every μ-integrable set $E \subset T$ and every number $\varepsilon > 0$, there exists a set $A \in \mathcal{C}$ such that $\mu(E \Delta A) < \varepsilon$.*

Let $E \subset T$ be a μ-integrable set and let $\varepsilon > 0$.

There exist a compact set $K \subset E$ and a μ-integrable open set $G \supset E$ such that $\mu(G - K) < \frac{\varepsilon}{2}$ ([1], chap IV, § 4, no 6, theorem 4). Because $\mathcal{C}$ verifies the condition (i), there exists a set $A \in \mathcal{C}$ such that $K \subset A \subset G$. Then the sets $E - A$ and $A - E$ are μ-integrable and contained in $G - K$, therefore

$$\mu(E \Delta A) = \mu\big((E - A) \cup (A - E)\big) = \mu(E - A) + \mu(A - E) \leqq 2\mu(G - K) < \varepsilon.$$

Proposition 6. *Let μ be a positive Radon measure on T and suppose that the sets of $\mathcal{C}$ are μ-integrable and that $\mathcal{C}$ verifies the condition* (i). *Then the set $\mathcal{E}(\mathcal{C})$ of the step functions of the form $\sum_{i=1}^{n} \varphi_{A_i} \alpha_i$ with $A_i \in \mathcal{C}$, is dense in $\mathfrak{L}^1(\mu)$.*

We know that the set $\mathscr{E}(\mathscr{B})$ of the step functions of the form $\sum_{i=1}^{n} \varphi_{E_i}\alpha_i$, where E_i are relatively compact Borel sets, is dense in $\mathfrak{L}^1(\mu)$. Then it is sufficient to prove that $\mathscr{E}(\mathcal{C})$ is dense in $\mathscr{E}(\mathscr{B})$ for the topology of $\mathfrak{L}^1(\mu)$.

Let $f=\sum_{i=1}^{n} \varphi_{E_i}\alpha_i$ be a function in $\mathscr{E}(\mathscr{B})$ with $\alpha_i \neq 0$ for every i, and let $\varepsilon > 0$. For each i there exists a set $A_i \in \mathcal{C}$ such that

$$\mu(E_i \Delta A_i) < \frac{\varepsilon}{n|\alpha_i|}.$$

The function $g = \sum_{i=1}^{n} \varphi_{A_i}\alpha_i$ is in $\mathscr{E}(\mathcal{C})$ and

$$|f-g| = \left| \sum_{i=1}^{n} (\varphi_{E_i} - \varphi_{A_i})\alpha_i \right| \leqq \sum_{i=1}^{n} |\varphi_{E_i} - \varphi_{A_i}|\,|\alpha_i| = \sum_{i=1}^{n} \varphi_{E_i \Delta A_i}\alpha_i,$$

therefore $\int |f-g|\, d\mu \leqq \sum_{i=1}^{n} \mu(E_i \Delta A_i)|\alpha_i| < \varepsilon.$

It follows that $\mathscr{E}(\mathcal{C})$ is dense in $\mathscr{E}(\mathscr{B})$, hence $\mathscr{E}(\mathcal{C})$ is dense in $\mathfrak{L}^1(\mu)$.

Remark. The propositions 5 and 6 are valid for an arbitrary class $\mathcal{C}$ verifying the condition (i). We can take $\mathcal{C}$ to be, for instance: the class of the compact sets; the class of the compact sets which are G_δ ([5], § 50, theorem 4); the class of the relatively compact open sets; the class of the relatively compact open sets which are F_σ ([5], § 50, theorem 4).

If $\mathbf{m}_1 : \mathscr{B} \to X$ is a regular Borel measure with finite variation μ_1, the μ_1-integrable real functions are called $\mathbf{m}_1$-integrable and we put $\mathfrak{L}^1(\mathbf{m}_1) = \mathfrak{L}^1(\mu_1)$. For every $\mathbf{m}_1$-integrable function $f \in \mathfrak{L}^1(\mathbf{m}_1)$ it is defined the integral $\int f d\mathbf{m}_1$ (see [2], [3] and [7]).

For every $\mathbf{m}_1$-integrable set $A \subset T$ (with $\varphi_A \in \mathfrak{L}^1(\mathbf{m}_2)$) we put $\mathbf{m}_1(A) = \int \varphi_A d\mathbf{m}_1$.

Theorem 3. *Suppose that* $\mathcal{C}$ *verifies the conditions* (i) *and* (ii) *and that* $\mathbf{m}$ *is regular with finite variation* μ. *Then there exists a unique regular Borel measure* $\mathbf{m}_1$ *with finite variation* μ_1 *such that the sets* $A \in \mathcal{C}$ *are* $\mathbf{m}_1$*-integrable and* $\mathbf{m}_1(A) = \mathbf{m}(A)$ *for* $A \in \mathcal{C}$. *In this case we have* $\mu_1(A) = \mu(A)$ *for* $A \in \mathcal{C}$.

The variation μ of $\mathbf{m}$ is a positive and additive set function defined on the clan $\mathcal{C}$. Because $\mathcal{C}$ verifies the condition (ii), μ is regular (theorem 2). Because $\mathcal{C}$ verifies the condition (i), there exists a positive Radon measure ν on T such that the sets $A \in \mathcal{C}$ are ν-integrable and $\nu(A) = \mu(A)$ for $A \in \mathcal{C}$ (see Introduction). Then $|\mathbf{m}(A)| \leqq \nu(A)$ for $A \in \mathcal{C}$. For each step function $f = \sum_i \varphi_{A_i}\alpha_i \in \mathscr{E}(\mathcal{C})$ put

$$U(f) = \sum_i \mathbf{m}(A_i)\alpha_i.$$

The definition of $U(f)$ is independent of the form in which f can be written as a step function.

The mapping $f \to U(f)$ of $\mathscr{E}(\mathcal{C})$ in X is linear; it is also continuous for the topology of $\mathfrak{L}^1(\nu)$, because, taking the sets A_i disjoint, we have

$$\|U(f)\|_1 \leqq \sum_i |\mathbf{m}(A_i)|\,|\alpha_i| \leqq \sum_i \nu(A_i)|\alpha_i| = \int |f|\, d\nu.$$

Because $\mathcal{C}$ verifies the condition (i), $\mathscr{E}(\mathcal{C})$ is dense in $\mathfrak{L}^1(\nu)$ (proposition 6), therefore U can be uniquely extended to a continuous linear mapping $U_1 : \mathfrak{L}^1(\nu) \to X$ and we have

$$\|U_1(f)\|_1 \leqq \int |f|\, d\nu \quad \text{for} \quad f \in \mathfrak{L}^1(\nu).$$

For every relatively compact Borel set $A \in \mathscr{B}$ put

$$\mathbf{m}_1(A) = U_1(\varphi_A).$$

It is clear that $\mathbf{m}_1$ is additive on $\mathscr{B}$ and that

$$|\mathbf{m}_1(A)| \leqq \nu(A) \quad \text{for} \quad A \in \mathscr{B}.$$

From this inequality we deduce that $\mathbf{m}_1$ is countably additive, regular and of finite variation μ_1, therefore $\mathbf{m}_1$ *is a regular Borel measure with finite variation.* It follows that μ_1 is a positive regular Borel measure and that

$$\mu_1(A) \leqq \nu(A) \quad \text{for} \quad A \in \mathscr{B}$$

i. e. $\mu_1 \leqq \nu$. Then $\mathfrak{L}^1(\nu) \subset \mathfrak{L}^1(\mu_1) = \mathfrak{L}^1(\mathbf{m}_1)$. Because the sets $A \in \mathcal{C}$ are ν-integrable, we deduce that these sets are $\mathbf{m}_1$*-integrable.*

For every function $f \in \mathfrak{L}^1(\nu)$ we have

$$\left| \int f\, d\mathbf{m}_1 \right| \leqq \int |f|\, d\mu_1 \leqq \int |f|\, d\nu$$

therefore the linear mapping $f \to \int f d\mathbf{m}_1$ of $\mathfrak{L}^1(\nu)$ into X is continuous.

On the other hand, for the step functions $f = \sum_i \varphi_{A_i}\alpha_i \in \mathscr{E}(\mathscr{B})$ we have

$$\int f\, d\mathbf{m}_1 = \sum_i \mathbf{m}_1(A_i)\alpha_i = U_1(f).$$

Because $\mathscr{E}(\mathscr{B})$ is dense in $\mathfrak{L}^1(\nu)$ and the continuous linear mappings $f \to \int f d\mathbf{m}_1$ and $U_1(f)$ of $\mathfrak{L}^1(\nu)$ into X coincide on $\mathscr{E}(\mathscr{B})$, we deduce that

$$\int f\, d\mathbf{m}_1 = U_1(f), \quad \text{for every} \quad f \in \mathfrak{L}^1(\nu).$$

In particular, for every set $A \in \mathcal{C}$ we have

$$\mathbf{m}_1(A) = \int \varphi_A\, d\mathbf{m}_1 = U_1(\varphi_A) = U(\varphi_A) = \mathbf{m}(A).$$

Because $\mu_1 \leqq \nu$, we have $\mu_1(A) \leqq \nu(A)$ for every ν-integrable set $A \subset T$. In particular, we have

$$\mu_1(A) \leqq \nu(A) = \mu(A) \quad \text{for} \quad A \in \mathcal{C}.$$

Conversely, if $A \in \mathcal{C}$ and if (A_i) is a finite family of disjoint sets of $\mathcal{C}$ contained in A, we have

$$\sum_i |\mathbf{m}(A_i)| = \sum_i |\mathbf{m}_1(A_i)| \leqq \sum_i \mu_1(A_i) = \mu_1(\bigcup_i A_i) \leqq \mu_1(A)$$

hence $\mu(A) \leqq \mu_1(A)$, therefore

$$\mu_1(A) = \mu(A) \quad \text{for} \quad A \in \mathcal{C}.$$

From the uniqueness of ν we deduce that $\mu_1 = \nu$. Let now $\mathbf{m}_2$ be a regular Borel measure with finite variation μ_2 such that the sets of $\mathcal{C}$ are $\mathbf{m}_2$-integrable and $\mathbf{m}_2(A) = \mathbf{m}(A)$ for $A \in \mathcal{C}$.

We have then $\mathbf{m}_1(A) = \mathbf{m}_2(A)$ for $A \in \mathcal{C}$, therefore

$$\int f\, d\mathbf{m}_1 = \int f\, d\mathbf{m}_2 \quad \text{for} \quad f \in \mathcal{E}(\mathcal{C}).$$

By proposition 6, the set $\mathcal{E}(\mathcal{C})$ is dense in the space $\mathfrak{L}^1(\mu_1 + \mu_2)$. Because $\mathfrak{L}^1(\mu_1 + \mu_2)$ is contained in $\mathfrak{L}^1(\mu_1)$ and in $\mathfrak{L}^1(\mu_2)$, the linear mappings $f \to \int f d\mathbf{m}_1$ and $f \to \int f d\mathbf{m}_2$ are defined and continuous on $\mathfrak{L}^1(\mu_1 + \mu_2)$ and coincide on the dense set $\mathcal{E}(\mathcal{C})$, therefore

$$\int f\, d\mathbf{m}_1 = \int f\, d\mathbf{m}_2 \quad \text{for} \quad f \in \mathfrak{L}^1(\mu_1 + \mu_2),$$

In particular

$$\mathbf{m}_1(A) = \mathbf{m}_2(A) \quad \text{for} \quad A \in \mathcal{B},$$

hence $\mathbf{m}_1 = \mathbf{m}_2$. This proves the uniqueness of $\mathbf{m}_1$ and completes the proof of the theorem.

Corollary 1. *If $\mathcal{C}$ is a clan verifying the conditions* (i) *and* (ii), *every additive and regular set function $\mathbf{m}: \mathcal{C} \to X$ with finite variation is countably additive.*

In particular, every additive and regular set function $\mathbf{m}: \mathcal{B} \to X$ with finite variation is a regular Borel measure with finite variation.

Corollary 2. *If $\mathcal{C}$ is a clan contained in $\mathcal{B}$ and containing a base of the topology of T, then every additive and regular set function $\mathbf{m}: \mathcal{C} \to X$ with finite variation can be extended to a regular Borel measure with finite variation.*

In particular, corollaries 1 and 2 are valid in each of the following cases: $\mathcal{C}$ is the clan generated by all the compact sets; $\mathcal{C}$ is the clan generated by all the compact sets which are G_δ; $\mathcal{C}$ is the clan of the Baire sets wich are relatively compact; T is totally disconnected and $\mathcal{C}$ is the clan of all compact-open subsets of T.

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