An application of the theory of regressive functions

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Let E be an arbitrary set of power $\aleph_\alpha$ and suppose that with every element x of E there is associated a non empty set $f(x)$ such that for any $x \in E$ the power of the set $f(x)$ is smaller than a given cardinal number $\aleph_\beta$ which is smaller than $\aleph_\alpha$. We say that the subset Γ of E has the property $T(\mathfrak{q}, \mathfrak{p})$, where $\mathfrak{q}$ and $\mathfrak{p}$ are two cardinal numbers such that $\mathfrak{p} \leqq \mathfrak{q} \leqq \aleph_\alpha$, if $\overline{\overline{\Gamma}} = \mathfrak{q}$ and

$$\overline{\overline{\bigcup_{x, y \in \Gamma} (f(x) \cap f(y))}} < \mathfrak{p}.$$

Consider the following

Proposition. *Under the above conditions E has a subset Γ with the property $T(\aleph_\alpha, \aleph_\alpha)$.*

This proposition was proved in [1] for $\aleph_\alpha$ not the sum of $\aleph_\beta$ or fewer cardinal numbers less than $\aleph_\alpha$, for $\aleph_\alpha$ of the form $\aleph_{\gamma+\omega}$ and — using the generalized continuum hypothesis — in the remaining case too.

We define the sequence $\{\gamma_n\}_{n<\omega}$ as follows:

$$\gamma_1 = \omega_\gamma, \quad \gamma_2 = \omega_{\gamma_1}, \quad \ldots, \quad \gamma_{n+1} = \omega_{\gamma_n}, \ldots.$$

We shall prove in this paper the following

Theorem. *If*

a) $\mathrm{cf}(\gamma) > 0$, $\alpha = \gamma_{n+1}$ *and* $\beta < \gamma_n$, *or* b) $\mathrm{cf}(\alpha) > 0$ *and* $\omega_\alpha = \alpha$,

then the above proposition is true.

We shall use the following notations. For any subset Γ of E let

$$\Pi_\Gamma = \bigcup_{\substack{x, y \in \Gamma \\ x \neq y}} (f(x) \cap f(y)).$$

For any cardinal number $\mathfrak{r}$ we denote by $\mathfrak{r}^+$ the cardinal number immediately following $\mathfrak{r}$. The symbols $\overline{\overline{S}}$ and $\overline{\gamma}$ denote the cardinal number of the set S and the ordinal number γ, respectively. For every ordinal number τ, $\aleph_{cf(\tau)}$ denotes the least cardinal number $\mathfrak{n}$ such that $\aleph_\tau$ can be expressed as the sum of $\mathfrak{n}$ cardinal numbers each $< \aleph_\tau$.

By the proof of the theorem we shall use the following

Theorem 1. *If $\aleph_\alpha$ is not the sum of $\aleph_\beta$ or fewer cardinal numbers less than $\aleph_\alpha$, then the above proposition is true.* (See [1], theorem 1.)

Theorem 2. *Let $\aleph_\alpha$ be a singular cardinal number, $\mathfrak{r}_0$ a cardinal number which is smaller than $\aleph_\alpha$ and $\{\aleph_{\nu_\xi}\}_{\xi<\omega_{cf(\alpha)}}$ a sequence of regular cardinal numbers such that $\aleph_{\nu_\sigma}>\aleph_{\nu_\tau}$ $(\sigma>\tau)$, $\max\{\aleph_{cf(\alpha)}, \aleph_\beta, \mathfrak{r}_0\}<\aleph_{\nu_\xi}<\aleph_\alpha$ and $\aleph_\alpha=\sum_{\xi<\omega_{cf(\alpha)}}\aleph_{\nu_\xi}$. If, for every $\xi<\omega_{cf(\alpha)}$, E_ξ is a subset of power $\geqq\aleph_{\nu_\xi}$ of E such that E_ξ has a subset E'_ξ with the property $T(\aleph_{\nu_\xi}, \mathfrak{r}_0)$, then E has a subset with the property $T(\aleph_\alpha, [\aleph_{cf(\alpha)}\cdot\mathfrak{r}_0]^+)$.* (See [1], theorem 4.)

Theorem 3. *Let ω_α be an initial number which is not confinal to ω and M a subset of $W(\omega_\alpha)=\{\eta:\eta<\omega_\alpha\}$. Suppose that to every element $\alpha\in M$ there corresponds an ordinal number $g(\gamma)$ such that $g(\gamma)<\gamma$ for $\gamma>0$ (and $g(0)=0$ for $0\in M$). If $W(\omega_\alpha)-M$ does not contain a closed subset confinal to $W(\omega_\alpha)$ (i. e. M is a stationary subset of $W(\omega_\alpha)$), then there exists an ordinal number $\pi<\omega_\alpha$ and a stationary subset N of M such that $g(\gamma)\leqq\pi$ for every $\gamma\in N$.* (See [2], theorem 2.)

Theorem 4. *Let ω_α be an initial number $>\omega$ and ϱ a regular ordinal number of the second kind with $\varrho<\omega_\alpha$. The set of all ordinal numbers $\lambda<\omega_\alpha$ of the second kind which are confinal to ϱ, is a stationary subset of $W(\omega_\alpha)$.* (See [3], theorem 8.)

Proof of the theorem. We are going to prove a). The proof of b) is quite similar and will be omitted. Let

$$x_0, x_1, \ldots, x_\omega, x_{\omega+1}, \ldots, x_\xi, \ldots \qquad (\xi<\omega_\alpha)$$

be a well-ordering of the type ω_α of the set E. By the hypothesis, $\beta<\gamma_n$. Hence $\beta+1<\gamma_n$ i. e. $\omega_{\beta+1}<\omega_{\gamma_n}=\gamma_{n+1}=\alpha$. Let now $M=\{\mu_\nu\}_{\nu<\alpha}$ be the set of all ordinal numbers of the second kind of $W(\alpha)$ which are confinal to $\omega_{\beta+1}$. By theorem 4 M is a stationary subset of $W(\alpha)$. Put

$$E_\nu=\{x_\eta: \eta<\omega_{\mu_\nu}\}.$$

Obviously $\overline{\overline{E_\nu}}=\aleph_{\mu_\nu}$. Since $\aleph_{\mu_\nu}$ is not the sum of $\aleph_\beta$ or fewer cardinal number less than $\aleph_{\mu_\nu}$ there exists, by theorem 1. a subset Γ_ν of E_ν with the property $T(\aleph_{\mu_\nu}, \aleph_{\mu_\nu})$. Hence, the power of the set Π_{Γ_ν} is smaller than $\aleph_{\mu_\nu}$.

Put $\overline{\overline{\Pi_{\Gamma_\nu}}}=\aleph_{\varrho_\nu}$ and $g(\mu_\nu)=\varrho_\nu$. Thus we have associated with every element μ_ν of M an ordinal number $g(\mu_\nu)$ such that $g(\mu_\nu)<\mu_\nu$ for every $\mu_\nu\in M$. By theorem 3 there exists an ordinal number $\pi<\alpha$ and a stationary subset M' of M such that $g(\mu_\nu)\leqq\pi$ for every $\mu_\nu\in M'$.

Let $\{\mu_{\nu_\eta}\}_{\eta<\omega_{cf(\gamma)}}$ be a subset of the type $\omega_{cf(\gamma)}$ of M' such that $\lim\mu_{\nu_\eta}=\alpha$.

Consider now an increasing sequence $\{\aleph_{\tau_{\nu_\eta}}\}_{\eta<\omega_{cf(\gamma)}}$ of regular cardinal numbers $<\aleph_\alpha$ such that for every $\eta<\omega_{cf(\gamma)}$,

$$\aleph_{\mu_{\nu_\eta}}<\aleph_{\tau_{\nu_\eta}}\leqq\aleph_{\mu_{\nu_{\eta+1}}}.$$

Let Γ'_{ν_n} be a subset of power $\aleph_{\tau_{\nu_n}}$ of $\Gamma_{\nu_{n+1}}$. It is obvious that $\bar{\bar{\Pi}}_{\Gamma'_{\nu_n}} \leqq \bar{\pi} = \mathfrak{r}_0$. Thus by theorem 2 there exists a subset of E with the property $T(\aleph_\alpha, [\aleph_{cf(\alpha)}\mathfrak{r}_0]^+)$.

The theorem is proved.

References

[1] G. Fodor, Some results concerning a problem in set theory, *Acta Sci. Math.*, **16** (1955), 232–240.

[2] G. Fodor, Eine Bemerkung zur Theorie der regressiven Funktionen, *Acta Sci. Math.*, **18** (1956), 139–142.

[3] W. Neumer, Verallgemeinerung eines Satzes von Alexandroff und Urysohn, *Math. Zeitschrift*, **54** (1951), 254–261.

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