Numerical ranges and normal dilations *)

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Each operator A on a Hilbert space $\mathfrak{H}$ induces a quadratic form Q_A; by definition

$$Q_A(x) = (Ax, x)$$

for every x in $\mathfrak{H}$. (In what follows all Hilbert spaces are complex and all operators are bounded.) The *numerical range* of A, in symbols $W(A)$, is the range of Q_A on the unit sphere; explicitly

$$W(A) = \{(Ax, x) : \|x\| = 1\}.$$

The Toeplitz—Hausdorff theorem says that the numerical range of every operator is a convex subset of the complex plane ([1], [3], [4]). It is disappointing that all the known proofs of this elegant statement are ugly. The methods are elementary, but the arguments are computational. The purpose of this paper is to give a new insight into the geometric structure of numerical ranges, which seems to be interesting in its own right, and which may some day lead to a clean conceptual proof of the Toeplitz—Hausdorff theorem.

Suppose that $\mathfrak{H}$ is a subspace (closed linear manifold) of a Hilbert space $\mathfrak{K}$, and suppose that A and B are operators on $\mathfrak{H}$ and on $\mathfrak{K}$ respectively. If $Q_A(x) = Q_B(x)$ for every vector x in $\mathfrak{H}$, then the operator A is called the *compression* of B to $\mathfrak{H}$, and B is called a *dilation* of A to $\mathfrak{K}$ (see [2]). Compression and dilation for operators are the same as restriction and extension for the corresponding quadratic forms. Usually the most convenient way to study a dilation of A to $\mathfrak{K}$ is to regard it as an operator matrix $\begin{pmatrix} A & X \\ Y & Z \end{pmatrix}$; where X maps $\mathfrak{H}^{\perp}$ into $\mathfrak{H}$, Y goes in the other direction, and Z operates on $\mathfrak{H}^{\perp}$. The easiest dilations are from $\mathfrak{H}$ to $2\mathfrak{H}$ (the direct sum of $\mathfrak{H}$ with itself); for such dilations all entries in the corresponding operator matrices may be regarded as operating on $\mathfrak{H}$.

There is a well known and easy argument that leads from the Toeplitz—Hausdorff theorem for two-by-two matrices to the most general version. In abbreviated form the argument is this: given any two unit vectors, restrict Q_A to their span, and apply the two-dimensional theorem to that restriction. The reason the argument works is that convexity is a condition on only two vectors at a time.

For normal matrices (two-by-two, or, for that matter, any size) the Toeplitz—Hausdorff theorem is an immediate consequence of diagonability (the spectral

*) Research supported in part by a grant from the National Science Foundation.

theorem). Indeed, since each normal matrix is unitarily equivalent to a diagonal one, it is sufficient to prove the theorem in case $A = \operatorname{diag}\langle\lambda_1, \ldots, \lambda_n\rangle$. If $x = \langle\xi_1, \ldots, \xi_n\rangle$ is a unit vector, then $(Ax, x) = \sum_{i=1}^{n} \lambda_i|\xi_i|^2$; since the ξ's vary over all n-tuples satisfying $\sum_{i=1}^{n} |\xi_i|^2 = 1$, it follows that the numerical range $W(A)$ is exactly the set of all convex linear combinations of the λ's. This proves more than just the convexity of $W(A)$; it proves that if A is a normal matrix, then $W(A)$ is the convex hull of the spectrum of A. The proof extends, with only trivial symbolic changes, to normal operators on infinite-dimensional spaces. Since, however, an integral is not a finite sum but a limit of finite sums, the conclusion is that $\overline{W(A)}$ (the closure of $W(A)$) is the convex hull of the spectrum of A.

The simplicity of the Toeplitz—Hausdorff theorem for normal matrices makes it natural to try to reduce the general case to the normal one. In principle such a reduction is possible; this is the statement of Theorem 1 below. Existing proofs do not, however, become simpler thereby; the exasperating fact is that the proof of Theorem 1 uses the Toeplitz—Hausdorff theorem.

Theorem 1. *The numerical range of every operator on a finite-dimensional Hilbert space $\mathfrak{H}$ is the intersection of the numerical ranges of its normal dilations to* $2\mathfrak{H}$.

Proof. If A is an operator on $\mathfrak{H}$ and if B is a dilation of A (normal or not), then Q_B is an extension of Q_A, and therefore $W(A) \subset W(B)$. It follows that

$$W(A) \subset \bigcap_{N \in \mathfrak{N}(A)} W(N),$$

where $\mathfrak{N}(A)$ is the set of all normal dilations of A to $2\mathfrak{H}$. It remains to prove the reverse inclusion. Since $W(A)$ is convex, it is sufficient to prove that to each closed half plane that includes $W(A)$ there corresponds an N in $\mathfrak{N}(A)$ such that $W(N)$ is included in the same half plane. (Observe that $W(A)$ is closed: it is a continuous image of the unit sphere.) By a translation and a rotation (i. e., by a substitution $A \to \alpha A + \beta$ with $|\alpha| = 1$) the desired assertion becomes this: if $W(A)$ is included in the closed right half plane, then so is $W(N)$ for some N in $\mathfrak{N}(A)$. To say of an operator that its numerical range is included in the closed right half plane is the same as to say that its real part (i. e., the arithmetic mean of it and its adjoint) is positive. In these terms the desired assertion is this: if $\operatorname{Re} A \geqq 0$, then $\operatorname{Re} N \geqq 0$ for some N in $\mathfrak{N}(A)$.

The proof of the last assertion is explicit: put $N = \begin{pmatrix} A & A^* \\ A^* & A \end{pmatrix}$. Since $N^* = \begin{pmatrix} A^* & A \\ A & A^* \end{pmatrix}$, it follows that

$$N^*N = \begin{pmatrix} A^*A + AA^* & A^{*2} + A^2 \\ A^2 + A^{*2} & AA^* + A^*A \end{pmatrix};$$

since this is symmetric in A and A^*, it follows that N is normal. It remains to prove that if $A + A^* \geqq 0$, then $N + N^* \geqq 0$. Since

$$N + N^* = \begin{pmatrix} A + A^* & A + A^* \\ A + A^* & A + A^* \end{pmatrix},$$

the problem reduces to proving that if T is positive, then so is $\begin{pmatrix} T & T \\ T & T \end{pmatrix}$, and this follows from the simple identity

$$\left(\begin{pmatrix} T & T \\ T & T \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\begin{pmatrix} Tx+Ty \\ Tx+Ty \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = (T(x+y), (x+y)).$$

The proof of the theorem is complete.

The normality of N and the positiveness of $N+N^*$ can be proved also by an amusing matrical computation. If $U=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, then

$$U^*\begin{pmatrix} A & A^* \\ A^* & A \end{pmatrix}U = \begin{pmatrix} A+A^* & O \\ O & A-A^* \end{pmatrix},$$

and, for every operator X on $\mathfrak{H}$,

$$U^*\begin{pmatrix} X & X \\ X & X \end{pmatrix}U = \begin{pmatrix} 2X & O \\ O & O \end{pmatrix};$$

in other words, N is unitarily equivalent to something obviously normal, and $N+N^*$ is unitarily equivalent to something obviously positive. These observations have the advantage that they clearly exhibit the spectra and the norms of N and $N+N^*$.

The proof of Theorem 1 used finite-dimensionality in one place only; that is what was needed to guarantee that the set under consideration (the numerical range of A) was closed. It is therefore a corollary of the proof that *the closure of the numerical range of every operator, on every Hilbert space, is the intersection of the closures of the numerical ranges of its normal dilations.* Whether or not the conclusion of Theorem 1, as is, is valid for infinite-dimensional spaces is an open question.

If A is a contraction (i. e., if $\|A\| \leq 1$), then A has not only normal but even unitary dilations; it is natural to ask whether Theorem 1 remains true if "normal" is replaced by "unitary". If $A=0$ (on, say, a one-dimansional space), the answer is yes. Indeed, if $|\lambda|=1$, then

$$U_\lambda = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$$

is a unitary dilation of A, and the intersection of all the $W(U_\lambda)$'s (in fact, the intersection of any two of them) is $\{0\}$, which is just what $W(A)$ is. Here is a more interesting example: write $A=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and

$$U_\lambda = \begin{pmatrix} 0 & 0 & \lambda \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The spectrum of U_λ consists of the three cube roots of λ, and, consequently, $W(U_\lambda)$ is an equilateral triangle (interior and boundary). The intersection of all the $W(U_\lambda)$'s is the disc with center 0 and radius $\frac{1}{2}$, which is just what $W(A)$ is. (The determination of this $W(A)$ is an amusing exercise; it was explicitly carried out by TOEPLITZ himself [5].) The experimental evidence is favorable, but the general result it indicates is not known; the following result about normal operators is a step in that direction.

Theorem 2. *The closure of the numerical range of every normal contraction on a finite-dimensional Hilbert space* $\mathfrak{H}$ *is the intersection of the closures of the numerical ranges of its unitary dilations to* $2\mathfrak{H}$.

Proof. Given A on $\mathfrak{H}$, with $\|A\| \leqq 1$, let $\mathfrak{U}(A)$ be the set of all unitary dilations of A to $2\mathfrak{H}$. As before, it is trivial that

$$\overline{W(A)} \subset \bigcap_{U \in \mathfrak{U}(A)} \overline{W(U)};$$

it remains to prove the reverse inclusion. Translations may push the norm of A beyond 1, but rotations are still permissible; it is therefore sufficient to prove that if $W(A)$ is included in a vertical half plane (i. e., one whose boundary is parallel to the imaginary axis), then there exists a U in $\mathfrak{U}(A)$ such that $W(U)$ is included in the same half plane. Equivalently, the desired assertion is this: if $\operatorname{Re} A \geqq \alpha$, then $\operatorname{Re} U \geqq \alpha$ for some U in $\mathfrak{U}(A)$.

The first step of the construction makes sense for any contraction, normal or not. Write S for the unique positive square root of $1 - AA^*$ and T for the unique positive square root of $1 - A^*A$. It is known (and easy to recompute) that if

$$U = \begin{pmatrix} A & -S \\ T & A^* \end{pmatrix},$$

then U is unitary. Since

$$U^* = \begin{pmatrix} A^* & T \\ -S & A \end{pmatrix},$$

the real part of U is given by

$$\operatorname{Re} U = \frac{1}{2}\begin{pmatrix} A+A^* & T-S \\ T-S & A+A^* \end{pmatrix}.$$

If A is normal, then $T = S$; it follows that

$$\left((\operatorname{Re} U)\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = ((\operatorname{Re} A)x, x) + ((\operatorname{Re} A)y, y).$$

If $\operatorname{Re} A \geqq \alpha$, and if $\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\| = 1$, then

$$\left((\operatorname{Re} U)\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) \geqq \alpha(\|x\|^2 + \|y\|^2) = \alpha.$$

The proof of the theorem is complete.

It is perhaps worth while to remark that more is true about U than was needed in the proof. It can be shown that the spectrum of U (for normal A) consists exactly of those complex numbers of modulus 1 whose real parts are in the spectrum of A. If A is not normal, then both this assertion and the weaker one about $\operatorname{Re} U$ may be false. Nevertheless, the conclusion of Theorem 2 is valid for many non-normal contractions and may be valid for all. $\left(\text{Recall the example } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.\right)$ One more remark along these lines is called for. A dilation of a dilation is a dilation; this may be thought to indicate that Theorems 1 and 2 could be combined to derive the conclusion of Theorem 2 for all contractions. The argument has a serious flaw, however: the normal dilations that Theorem 1 uses may not have norms less than or equal to 1, even if the operator to which Theorem 1 is being applied does, and this means that when Theorem 2 becomes needed it is not applicable. This does not mean that the proposed argument is worthless, but only that its use must be restricted to operators of small norm. An examination of the proof of Theorem 1 shows that the norms of the normal dilations of A that are introduced there need never exceed $3\|A\|$. Conclusion: *the numerical range of every operator A, with $\|A\| \leqq 1/3$, on a finite-dimensional Hilbert space $\mathfrak{H}$, is the intersection of the numerical ranges of its unitary dilations to* $4\mathfrak{H}$. If "numerical range" is changed to "closure of numerical range", the conclusion is valid for infinite-dimensional spaces also.

In conclusion it seems appropriate to mention a possible generalization of the preceding considerations that is interesting and non-trivial. Suppose that k is a positive integer and that A is an operator on a Hilbert space of dimension at least k. If P is a projection with $\mathrm{r}(P)=k$ ("r" stands for rank), then $\mathrm{r}(PAP)\leqq k$, and, consequently, it is possible to form $\mathrm{tr}(PAP)$ ("tr" stands for trace). Write

$$W_k(A)=\left\{\frac{1}{k}\,\mathrm{tr}\,(PAP)\colon\ \mathrm{r}\,(P)=k\right\}.$$

(The normalizing factor $1/k$ is not essential, but it serves to make some of the formulas more elegant and more familiar.) It is easy to verify that $W_1(A)$ is the same as the numerical range of A. Question 1: is $W_k(A)$ always convex? Question 2: is $W_k(A)$ the intersection of all $W_k(N)$'s for N in $\mathfrak{N}(A)$? Question 3: if $\|A\|\leqq 1$, is $W_k(A)$ the intersection of all $W_k(U)$'s for U in $\mathfrak{U}(A)$? None of the answers is known. Conjecturally they are all affirmative, but the proofs may be difficult.*)

References

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(Received November 7, 1962)

*) *Note added March 30, 1964.* All three questions have recently been answered by C. A. BERGER (Ph. D. thesis, Cornell University, 1963). The answers are yes, yes, and (for $k\geqq 2$) no. For $k=1$, on an infinite-dimensional space, E. DURSZT has shown that the answer to Question 3 is no; the finite-dimensional case is of interest, and remains open.