# Remarks on $n$-th roots of operators [1])

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Some years ago, P. HALMOS and the author showed, by means of general results on semi-continuity of spectral fine-structure, that the set of operators (i. e. bounded linear transformations) on a separable Hilbert space, which are invertible and fail to possess a root of order $n$ ($n=2, 3, \ldots$), has interior points in the uniform operator topology; [2]. Recently B. SZ.-NAGY has asked us if one could obtain quantitative results of that type, i. e. give estimates on how large can be the radius of an open sphere (in the operator norm) consisting of operators like mentioned above, as compared with the norm of the center. A similar question also arises without invertibility requirements. In either case, it is clear that an open sphere centered at $T$, and whose points have no $n$-th roots, must have a radius $\leqq \|T\|$; it is natural to ask (for both cases separately) if $\|T\|$ is actually reached for some $T$. The purpose of this note is to extract, by slightly changing the approach, such quantitative information implicitely contained in the methods of [2], and to answer at least in part, the questions stated above.

For a quantitative analogue of theorem 6 in [2], one must proceed in a slightly different manner, since the use of lemma 2, theorem 3, invoked there, would cut the sharpness of the estimates. Let $H$ be a complex Hilbert space, $B(H)$ the algebra of all operators on $H$. An adequate gauge for our purpose is defined as follows: For $T \in B(H)$, $\lambda$ complex, we set $N(\lambda, T)=\inf\{\|(T-\lambda I)x\|: \|x\|=1\}$, where $I$ denotes the identity operator. As in lemma 1, [2], one sees easily that $N(\lambda, T)$ is a lower semi-continuous function of $\lambda$, for fixed $T$. If $K$ is any compact subset of the complex plane, we set $N(K, T)$ for the minimum of $N(\lambda, T)$ on $K$. We denote by $\Sigma(T)$, $\Pi(T)$, the spectrum, and approximate point spectrum of $T$; and by $m(\lambda, T)$ the multiplicity function corresponding to $T$ ([2], p. 592).

Theorem. *Suppose $T \in B(H)$, and $K$ is a compact subset of $\Sigma(T)-\Pi(T)$ such that $0 \notin K$, $m(\lambda, T)=1$ for $\lambda \in K$, $\sqrt{K}$ is a connected set (or, equivalently, 0 does not belong to the unbounded component of the complement of $K$). Then $S \in B(H)$ fails to possess a root of order $n$ ($n=2, 3, \ldots$) whenever $\|S-T\|<N(K, T)$.*

Proof. Suppose $S \in B(H)$, $\|S-T\|<N(K, T)$, $\lambda \in K$. Since $\|(S-\lambda I)-(T-\lambda I)\| = \|S-T\|<N(K, T)$, it follows from lemma 5, [2], that $R^{\perp}(S-\lambda I) \cap R(T-\lambda I) = R(S-\lambda I) \cap R^{\perp}(T-\lambda I) = 0$, where "$R(\,)$" denotes "range of".

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Again, from $\|(S-\lambda I)-(T-\lambda I)\| < N(K, T)$, and since by definition $\|(T-\lambda I)x\| \geqq N(K, T)\ \|x\|$, we have as in lemma 1, [2], that $N(K, S) > 0$. This implies, on one hand, that the above ranges are closed, so that we may conclude $m(\lambda, S)=1$, for $\lambda \in K$. On the other hand, we conclude that $K$ is in the complement of $\Pi(S)$; but in turn, $m(\lambda, S)=1$, for $\lambda \in K$, implies in particular that $\lambda \in \Sigma(S)$; thus we have $K \subset \Sigma(S)-\Pi(S)$. The rest follows from lemma 6, [2], and remarks (b), p. 149, [1].

Notice that at difference from theorem 6, [2], mentioned earlier, in the previous theorem nothing is said (nor can be said under the present assumptions, in general), about the invertibility of the operators $S$, whether $T$ itself be invertible or not.

Corollary 1. *If $H$ is a separable Hilbert space, there exists an operator $U$ on $H$ (non-invertible, and) such that in the open sphere centered at $U$, of radius $\|U\|$, every operator fails to possess a root of order $n$ ($n=2, 3, \ldots$).*

Proof. Let $x_1, x_2, \ldots$ be an orthonormal basis for $H$. Let $U$ be the so-called shift operator, which sends $x_n$ into $x_{n+1}$, for $n=1, 2, \ldots$ The adjoint operator $U^*$ sends $x_{n+1}$ into $x_n$, for $n=1, 2, \ldots$, and $x_1$ into 0. Direct computation readily shows that for every complex $\lambda$, with $|\lambda|<1$, $U^*$ admits exactly one eigenvector (within scalar multiples), so that $m(\lambda, U)=1$, for $|\lambda|<1$. Again direct computation shows that $N(\lambda, U) \geqq 1-|\lambda|$, for $\lambda \in \Sigma(U)$. Let now $S$ denote an operator such that $\|S-U\|<1$, and let $K$ denote the circumpherence of a circle centered at 0, of radius less than $1-\|S-U\|$. Then clearly $N(K, U) > \|S-U\|$, and by the previous theorem, $S$ has no $n$-th root. Since $\|U\|=1$, all is proved.

If we require invertibility, the answer is incomplete:

Corollary 2. *Suppose $H$ is a separable Hilbert space. Then for any $R < \frac{1}{2}\sqrt{2}$, there exists an invertible operator $A_1$ on $H$, of norm 1, center of a sphere of radius $R$, every point of which fails to possess a root of order $n$ ($n=2, 3, \ldots$). Again there exists an invertible operator $A_2$ on $H$, of norm 1, center of an open sphere of radius $(2\sqrt{2}+1)^{-1}$, every point of which is invertible and fails to possess a root of order $n$ ($n=2, 3, \ldots$).*

Proof. Let $A$ be an analytic position operator, as defined in [1], p. 143, corresponding to a bounded domain of the complex plane; and let $C$ denote the boundary of $D$. From lemma 3, [1], follows that for $\lambda \in D$, $N(\lambda, A)=d(\lambda, C)/\sqrt{2}$, where "$d(,)$" denotes "distance between". From now on $D$ shall be an annulus centered at 0, with radii 1 and $r$, $0<r<1$. If $K$ is the circumpherence of the circle centered at 0, of radius $(1+r)/2$, then $N(K, A)=(1-r)/2\sqrt{2}$. On the other hand one checks easily that $\|A^{-1}\| \leqq 1/r$, so that $\|S-A\| < r \leqq (\|A^{-1}\|)^{-1}$, insures the invertibility of $S$, (for instance, [3], p. 118). Then an operator having the properties required for $A_1, A_2$, can clearly be obtained by choosing $r$ sufficiently small, $=(2\sqrt{2}+1)^{-1}$. [2])

At this point, several remarks seem pertinent. Let $B$ denote an arbitrary complex Banach algebra with identity. Denote by $G$ the group of all invertible elements of $B$; by $F$ the set of all elements of $B$ that fail to possess a root of order $n$ ($n=2, 3, \ldots$).

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[2]) We have used the fact that $\|A^{-1}\| \leqq 1/r$. Actually equality holds, as is seen at once from $\Sigma(A)=$closure of $D$ (see [1]), and the spectral mapping theorem which yields $1/r=$spectral radius of $A^{-1} \leqq \|A^{-1}\|$, (of course one has also $\|A\|=1$), so that $(2\sqrt{2}+1)^{-1}$ is actually the best value the above procedure will yield.

One can, in this setting, consider questions analoguous to those just discussed for the case $B=B(H)$, and since the case of commutative Banach algebras is particularly easy to handle, gain some insight on what may be true for $B(H)$. It is easy to verify, by considering $2\times 2$ matrix algebras, that in general (whether commutativity is assumed or not) $F$ is neither open nor closed in $B$. However, if $B$ is commutative $F_1 = F\cap G$ is open in $B$, and is moreover the union of connected components of $G$ (which is open). For suppose $a\in G$, then the connected component of $G$ containing $a$ is of the form $aG_0$, where $G_0$ is the connected component of $G$ that contains the identity (see [3], p. 119); furthermore, every element of $G_0$ possesses a logarithm, hence $n$-th roots (see [3], p. 286); and it follows that $a$ fails to possess $n$-th roots if and only if every element of the open component $aG_0$ does. Noticing that for $B(H)$ the only examples of elements in $F_1$ we know, are interior points of $F_1$ it is natural to ask whether $F_1$ is open in this case also.

We observe, finally, that it is easy to construct a commutative $B$, for which $F_1$ contains elements of norm 1, centers of maximal open spheres consisting of regular elements without $n$-th roots, whose radii take on any preassigned value $>0$, and $\leqq 1$. In fact, let $X$ denote the compact space formed by the circumpherence of the unit disk in the complex plane, and $C(X)$ the algebra of all complex continuous functions on $X$, with the usual "sup"-norm. Let $a_s$ be an element of $C(X)$ mapping $X$ into a simply closed Jordan curve surrounding the origin, so chosen that $\|a_s\|=1$, and that the distance from the origin to $a_s(X)$, be $s$. Then, winding number considerations show at once that $a_s$ can have no $n$-th root, while homotopy considerations show that the same must hold for all $b\in C(X)$, such that $\|a_s-b\|<s$. On the other hand there exists $\lambda$ complex, $|\lambda|$ as close to $s$ as desired, such that $(\lambda 1+a_s)(X)$ does not surround the origin, hence $\lambda 1+a_s$ having $n$-th roots. Thus $a_s$ is the center of a maximal sphere as described above, of radius $s$. The similar question for $B(H)$ remains open.

## References

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