Spectral operators, hermitian operators, and bounded groups[1])

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The theory of spectral operators on a Banach space, developped by N. DUNFORD and others, extends the classical spectral theory of hermitian and normal operators on a Hilbert space. In [12], on the other hand, there was introduced a notion of semi-inner-product compatible with any Banach space structure, that leads in particular to a natural concept of hermitian operator on a Banach space[2]). Because of general characteristics of both developments it would seem natural to think that there exists a strong connection between spectral and hermitian operators in general. That this is the case, was proved recently by E. BERKSON [3], who kindly communicated his results to us. In particular, he shows that any spectral operator of scalar type, [6], is of the form $R+iJ$, where $RJ=JR$, and there exists an equivalent renorming of the underlying space such that R and J become hermitian. The results of [3], satisfactory from a descriptive point of view, have however the inconvenient of involving a renorming, as just described. Our first step, in part *I*, is to derive a necessary and sufficient condition, in terms of the boundedness of certain associated groups, for a commutative family of operators to be "hermitian-equivalent", in the sense that there exists an equivalent renorming of the underlying space under which all members of the given family become hermitian. The result shows in particular that if one starts with a Hilbert space, and a commuting, hermitian-equivalent, family of operators, then an appropriate renorming can be found which gives again a Hilbert space, thus hermitian operators in the classical sense. One also derives easily that given a finite collection of families of operators on a same Banach space (everything commuting), that are separately hermitian-equivalent, then they are jointly (i. e., their union is) hermitian-equivalent. An immediate consequence is an extension (also a short new proof) of a result of WERMER on commuting spectral measures. We have occasion to use this extension, in part III.

Our main results are in part II, where we discuss infinite collections of commuting families (in particular single elements), which are separately hermitian-equivalent. Here a necessary and sufficient condition is given, in terms of closure properties, for such a collection to be hermitian-equivalent. We derive, in particular, that if the uniformly closed algebra generated by a boolean algebra of projections on a reflexive Banach space, consists of spectral operators of scalar type, then the boolean

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[2]) This concept turns out to coincide with one introduced differently in [16]. See [12], p. 39.

algebra is uniformly bounded. Thus, one obtains converses of well-known results of DUNFORD. This is of some general interest as it supports the idea that the quite restrictive hypothesis of uniform boundedness made by DUNFORD on the admissible spectral measures, is necessary for a satisfactory theory along his lines.

Remarks and counter-examples are collected in part III. The latter are given to clarify the situation concerning the hypothesis of [3]. They show for instance that the characterization of spectral operators of scalar type, proved for reflexive spaces, really fails if reflexivity is omitted. It is also seen that, even on reflexive spaces, products of commuting hermitian operators are not necessarily hermitian.

Terminology and notations. In what follows, the term Banach space always means complex Banach space. Operator means bounded linear transformation. If T is an operator on a Banach space, we put

$$\exp T = \sum_{n=0}^{\infty} T^n/n!.$$

Basic definitions, and properties concerning semi-inner-product spaces and hermitian operators, are reviewed in the next section, in order to make the paper more self-contained. The reader familiar with these notions may conseqently omit the first section of I.[3])

I. Hermitian operators and bounded groups

1. Hermitian operators on a Banach space. We collect here basic definitions and facts, almost all taken from [12] and [13].

Definition 1. A complex vector space X is called a semi-inner-product space, if to each pair of vectors $x, y \in X$ corresponds a complex number $[x, y]$, called their semi-inner-product, and the following holds:

$$(1) \qquad \begin{aligned} &[x, y] \text{ is linear in } x \\ &[x, x] \text{ is real} > 0, \text{ for } x \neq 0 \\ &|[x, y]|^2 \leqq [x, x]\,[y, y]. \end{aligned}$$

A semi-inner-product space can always be normed, setting $\|x\| = ([x, x])^{\frac{1}{2}}$, [12]. Conversely it is shown in [12] that every normed space X admits at least one (in general infinitely many) semi-inner-product structures compatible with its norm, in the sense that $\|x\|^2 = [x, x]$. If X is a Hilbert space, the usual inner-product is the only semi-inner-product compatible with the norm of X.

If an operator T on a semi-inner-product space X (T being bounded in the sense of the induced norm) has the property that $[Tx, x]$ is real valued for all $x \in X$, then this situation is unchanged by replacing the given semi-inner-product by another one inducing the same norm, [12] p. 37. Hence no ambiguity is introduced in the following

[3]) Reference [8] may be used for general functional analysis results and terminology used below without explanation.

Definition 2. An operator T on a Banach space is called hermitian if $[Tx, x]$ is real valued for all $x \in X$, and some semi-inner-product on X compatible with its norm.

If X is a Hilbert space, this coincides, naturally, with the usual concept. It is shown that an operator T is hermitian if and only if $\|I+itT\| = 1+o(t)$, t real, where I denotes the identity operator, [12]. We also mention the case of unbounded hermitian transformations, although we shall later deal, essentially, with bounded transformations only.

Definition 3. A linear transformation T, with domain D, on a semi-inner-product space X, is called hermitian if $[Tx, x]$ is real valued for all $x \in D$.

Unlike the case of operators, the hermitian character of a general linear transformation may be altered by replacing the given semi-inner-product by another inducing the same norm, even if T is closed and densely defined, [13] p. 688. However this cannot happen if T is the infinitesimal generator of a strongly continuous semi-group of operators. The latter happens for instance if T is spectral, [1], and [13], theorem 3. 2.

Definition 4. Let T be a linear transformation on a semi-inner-product space X, with domain D. Then the numerical range of T, $W(T)$, is the set of complex numbers defined by

$$W(T) = \{[Tx, x] : x \in D, [x, x] = 1\}. \tag{2}$$

2. Hermitian-equivalent operators. What is meant by a hermitian-equivalent family of operators, has been discussed in the introduction. One might notice that in the case of a single operator on a Hilbert space, hermitian-equivalent coincides with what is often called "symmetrizable".

Definition 5. Let F be any commutative family of operators on a Banach space. We denote by $S(F)$ the real-linear span of F. The exponential group assotiated to F, $G(F)$, is then defined as follows

$$G(F) = \{\exp iT \colon T \in S(F)\}. \tag{3}$$

Theorem 6. *Let F be a commutative family of operators on a Banach space X. Then F is hermitian-equivalent if and only if its associated exponential group is uniformly bounded. Furthermore, if X is a Hilbert space, and F is hermitian-equivalent, then the renorming may be chosen so as to obtain again a Hilbert space.*

Proof. First we show that an operator T on X is hermitian, if and only if $\|\exp itT\| \doteq 1$, for t real. In fact, if this condition is satisfied, then by theorem 3. 1 of [13] follows that both iT and $-iT$ are dissipative operators, [13] p. 680, hence $W(T)$ is real and T hermitian. Conversely, T hermitian implies that iT and $-iT$ are dissipative, and the conclusion follows using theorem 2. 1 of [13].

If F is a commutative, hermitian-equivalent, family of operators on X, denote by T' the operator into which an operator T on X is carried by the equivalent renorming; let F' denote the family corresponding in that way to F. Then there exists, naturally, a constant K such that $\|T\| \leqq K\|T'\|$ for $T \in F$. Since F' consists of hermitian operators the same holds for $S(F')$; thus, by what we just saw above, the exponential group $G(F')$ is uniformly bounded by 1. Hence $G(F)$ is bounded by K.

The converse uses the fact that given a uniformly bounded commutative group G of operators on a Banach space X, there exists an equivalent renorming of X, carrying G into a group of unitary, i. e. norm-preserving operators (of X onto X); if X is a Hilbert space, its renorming may be chosen so as to obtain again a Hilbert space. See Sz.-Nagy [15] and Dixmier [4].

Thus, in the situation that interests us specifically, i. e. if F is a commutative family of operators on X, for which $G(F)$ is uniformly bounded, then, after equivalent renorming, all elements of $G(F)$ will have norm 1. In particular for any $T \in F$, we have for T', in the new norm $\| \ \|'$,

$$\|\exp itT'\|' = 1 \qquad (t \text{ real}). \tag{4}$$

From what we saw earlier, (4) implies that T' is hermitian.

Finally notice that if X is a Hilbert space, and F is hermitian-equivalent, then by the first part of the argument we know that $G(F)$ is uniformly bounded, and under this circumstance we have seen that (although there may exist other renormings) there is always a renorming which gives again a Hilbert space, and is otherwise as desired. This completes the proof.

Corollary 7. *Suppose F_i $(i = 1, 2, \ldots, n)$ are finitely many commutative families of operators on a Banach space X, each of them hermitian-equivalent; then if $\bigcup_{i=1}^{n} F_i$ is commutative it is hermitian-equivalent. If X is a Hilbert space, then the corresponding renorming may be chosen so as to obtain again a Hilbert space.*

Proof. The hypothesis implies that every $G(F_i)$ is uniformly bounded, say by K_i, in virtue of the preceding theorem. Since everything commutes, $G\left(\bigcup_{i=1}^{n} F_i\right) = \prod_{i=1}^{n} G(F_i)$. Hence $G\left(\bigcup_{i=1}^{n} F_i\right)$ is uniformly bounded, by $\prod_{i=1}^{n} K_i$, and it follows that $\bigcup_{i=1}^{n} F_i$ is Hermitian-equivalent.

3. Spectral operators. We had mentioned in the introduction a recent result of E. Berkson [3] connecting spectral operators of scalar type, [6], [7], with hermitian operators. Theorems I and II below, are taken from [3]:

Theorem I.[4]) *Let S be a spectral operator of scalar type, on a Banach space X, and with spectral measure $E = \{E(\sigma)\}$. Then E is hermitian-equivalent. S admits the decomposition $S = R + iJ$, $RJ = JR$, and the family $\{R^n J^m\colon n, m = 0, 1, 2, \ldots\}$ is hermitian-equivalent.*

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[4]) It might be noticed that the result from [3] we mainly use (i. e. the one mentioned in the introduction, and used in part II), can be derived from results above as follows: Let Re λ, as a function on the spectrum σ of the scalar spectral operator S, be denoted by r. Set $R = \int_\sigma r\, dE(\lambda)$. Since for t real, $|\exp itr| = 1$ on σ, it follows from theorem 7, [6], that $\|\exp itR\| \leqq \nu(E)$, independently of t. A similar argument for $j = \operatorname{Im} \lambda$, and application of theorem 6, corollary 7, finish the proof.

Theorem II. *If X is reflexive, and S an operator on X, then the existence of a decomposition as in theorem I, is necessary and sufficient for S to be a spectral operator of scalar type.*

By means of theorem 6, readily derives from the above theorems results not depending on equivalent renormings. We explicit the following:

Theorem 8. *If S is a spectral operator of scalar type on a Banach space, then $S = R+iJ$, where $RJ=JR$, and $G(\{R^n J^m: n, m=0, 1, 2, \ldots\})$ is uniformly bounded.*

In [3] it was shown that a decomposition like that of theorem I is unique. As a simple application of preceding results we derive uniqueness under less assumptions.

Proposition 9. *If S is a spectral operator of scalar type on a Banach space X, and $S = R+iJ$, where $RJ=JR$, and R and J are hermitian-equivalent, then $R=\int \operatorname{Re} \lambda \, dE(\lambda)$, $J=\int \operatorname{Im} \lambda \, dE(\lambda)$, $E(\lambda)$ denoting the spectral measure corresponding to S.*

Proof. If we write R_0, J_0, for the integrals that appear in the statement of the proposition, it follows from theorem I that R_0+iJ_0 certainly supplies a decomposition of S with the same properties as $R+iJ$. Since R and J commute with S, Dunford's extension of the Fuglede theorem, and the definition of R_0 and J_0 imply that R, J, R_0, J_0 is a commutative family, which by corollary 7 is hermitian-equivalent. Consequently, after equivalent renorming, we get $R'-R_0'+i(J'-J_0')=0$, where $R'-R_0'$ and $J'-J_0'$ are hermitian. The latter facts readily imply that for $x \in X$, $[(R'-R_0')x, x]=0$ and $[(J'-J_0')x, x]=0$, and by theorem 5 of [12], we conclude $R'-R_0'=0$, $J'-J_0'=0$. Thus $R=R_0$, $J=J_0$.

Remark 10. If S is a spectral operator of scalar type on X, then $S = R+iJ$ as described in theorem I. If now, X is a Hilbert space, and since R and J are hermitian-equivalent, theorem 6 tells us that this equivalence can be realized so as to obtain again a Hilbert space. This implies by a well-known argument that there exists on the Hilbert space X an invertible hermitian operator A such that ASA^{-1} is a normal operator (in the usual sense, of course). This is a result of Wermer [17]. In fact an extended version of Wermer's results on commuting spectral measures follows easily along the same lines. We make this explicit in the next section.

4. Commuting spectral measures. Wermer's theorem. For sake of simplicity, we call an operator T on a Banach space *normal*, if $T = R+iJ$, where R and J are hermitian, and $RJ=JR$. The meaning of the term spectral measure, in this section, will be the same as in [17], except the space considered is an arbitrary Banach space, instead of a Hilbert space.

Theorem 11. *Let $\{E_1(\sigma)\}$ and $\{E_2(\eta)\}$ be two commuting spectral measures on the Banach space X, in the sense that*

$$E_1(\sigma)E_2(\eta)=E_2(\eta)E_1(\sigma) \text{ for all } \sigma, \eta. \tag{5}$$

Then $\{E_1(\sigma)\} \cup \{E_2(\eta)\}$ is hermitian-equivalent. If T_1 and T_2 are spectral operators on X, there exists an equivalent renorming of X, under which the scalar parts of T_1, T_2, go into normal operators, provided T_1 and T_2 commute. If X is a Hilbert space the renorming may chosen so as to obtain again a Hilbert space.

Proof. Theorem I, or better, the underlying argument (since here the spectral measures are not immediately given as attached to certain operators) implies that $\{E(\sigma)\}$ and $\{E(\eta)\}$ are each hermitian-equivalent; since they commute the rest follows from corollary 7, and the decomposition of the scalar parts into "real" and "imaginary" parts, as before.

The results of [17] follow immediately from the above theorem, when X is a Hilbert space.[5])

Corollary 12. *If S_1, S_2, are spectral operators of scalar type on a Banach space, then S_1+S_2 becomes normal under some appropriate equivalent renorming of the underlying space.*

We shall use this fact later to show that it is not always true that the product of hermitian operators is hermitian, even if the space is reflexive.

II. Closure properties and boundedness of boolean algebras of projections

5. Equivalence and closure. In a sense, corrollary 7 represents the elementary case of what we want to do here, namely to determine under what circumstance an infinite commutative collection of elements, or families, which are separately hermitian-equivalent, is hermitian-equivalent. An answer, in terms of a closure property is supplied by the following

Theorem 13. *A necessary and sufficient condition for a commutative family of operators on a Banach space to be hermitian-equivalent, is that the uniform closure of its real-linear span consists of (individually) hermitian-equivalent operators.*

Proof. Let X denote the Banach space, F the family considered above. Since real-linear combinations, and uniform limits, of hermitian operators are again hermitian, the necessity of the condition is immediately verified. Next, we prove the sufficiency.

For any operator T on X, we define

$$M(T)=\sup\{\|\exp itT\|:\ t \text{ real}\}.$$

In general $M(T)$ may be $+\infty$, but theorem 6 tells us that if T is hermitian-equivalent, then $M(T)$ is finite. Now, given any real number M_0, we denote by A the algebra of all operators on X, and define

$$B(M_0)=\{T\in A: M(T)\leqq M_0\}.$$

The first step will be to show, using merely the fact that F is a commutative subset of A, that the uniform closure of $F\cap B(M_0)$ is contained in $B(M_0)$. In fact, let us consider a sequence, with elements $T_n\in F\cap B(M_0)$ $(n=1,2,\ldots)$ and such that $\|T-T_n\|\to 0$, for some $T\in A$. We wish to show that $T\in B(M_0)$. By definition, one has for any $T\in A$ the expansion

$$\exp tT = \sum_{n=0}^{\infty} t^n T^n/n!,$$

[5]) A particular case was obtained earlier by LORCH [11]; see also [14] theorem 55, and [4].

however, one may also use TAYLOR's formula with integral remainder, and thus obtain, in our context,

(6)
$$\exp it(T-T_n) = \sum_{m=0}^{N} \frac{t^m}{m!}(i(T-T_n))^m + \\ + \int_0^t \frac{(t-s)^N}{N!}(i(T-T_n))^{N+1} \exp is(T-T_n)\, ds.$$

Naturally, T commutes with all elements of F, and consequently, for $|s| \leqq |t|$,

$$\|\exp is(T-T_n)\| \leqq \|\exp isT\| \, \|\exp(-isT_n)\| \leqq M_0 \sup\{\|\exp isT\| : |s| \leqq |t|\} = K_t$$

where K_t depends on t only. Hence, one obtains for the remainder term the estimate

(7)
$$\left\| \int_0^t \frac{(t-s)^N}{N!}(i(T-T_n))^{N+1} \exp is(T-T_n)\, ds \right\| \leqq K_t \frac{|t|^{N+1}}{N!} \|T-T_n\|^{N+1}.$$

Suppose that n is taken sufficiently large to insure that $\|T-T_n\| \leqq 1$. Then, for fixed t, N may be chosen, in virtue of (7), independently of n, so as to insure that the remainder term is in norm $<\varepsilon/2$, for any $\varepsilon > 0$ given a priori. Next, one may choose n large enough, so that

$$\left\| \sum_{m=1}^{N} t^m (i(T-T_n))^m/m! \right\| \leqq \sum_{m=1}^{N} |t|^m \|T-T_n\|^m/m! < \varepsilon/2.$$

With this choice of n, we have from (6) that $\|\exp it(T-T_n)\| < 1+\varepsilon$. Thus

$$\|\exp itT\| = \|\exp it(T-T_n) \exp itT_n\| < (1+\varepsilon) M_0.$$

Since ε is arbitrary, we have $\|\exp itT\| \leqq M_0$. Since t was arbitrary, we have proved that $T \in B(M_0)$.

Now let $Y = \overline{S(F)}$ denote the closure of the real-linear span, in the uniform topology. Y is commutative, since F is. Taking $M_0 = 1, 2, \ldots, n, \ldots$; set $Y_n = Y \cap B(n)$. Then by what we have proved above, Y_n is uniformly closed in Y. If it is assumed that Y consists of hermitian-equivalent elements, then $Y = \bigcup_{n=1}^{\infty} Y_n$, and by the Baire—Banach category theorem, there exists an n_0 such that Y_{n_0} contains a non-trivial sphere Z. Let T_0 denote the center of that sphere. Then every operator $T \in Y$ is of the form

(8)
$$T = -rT_0 + rT', \quad T' \in Z, \quad r \text{ real } \geqq 0.$$

To complete the proof, observe that clearly for any pair of commuting operators T_1, T_2, and any real number t, one has

(9)
$$M(tT_1) \leqq M(T_1), \quad M(T_1+T_2) \leqq M(T_1) \cdot M(T_2).$$

It follows from (8) and (9) that for any $T \in Y$, $M(T) \leqq M(T_0) n_0$. But this says in particular that $G(F)$ is uniformly bounded, hence, theorem 6, F is hermitian-equivalent.

6. The boundedness of boolean algebras of projections. In order to discuss, below, a necessary and sufficient condition for a boolean algebra of projections to be uniformly bounded, we need an important property of hermitian operators contained in the following

Lemma 14. *Suppose T is a hermitian operator on a Banach space X. Let $\|T\|_\infty$ denote the spectral norm of T, i. e.* $\sup\{|\lambda|: \lambda \in \text{spectrum}(T)\}$. *Then* $\|T\| \leqq 4\|T\|_\infty$.

Proof. Considering the equivalence of concepts established in [12] p. 39, Hilfssatz 3* of [17] tells us that under the above assumptions, if a and b denote the extremities ($a \leqq b$) of the smallest closed interval containing the (real valued) spectrum of T, we have

$$a = -\lim_{t\to+0}\{(\|I-tT\|-1)/t\}, \quad b = \lim_{t\to+0}\{(\|I+tT\|-1)/t\}.$$

It was shown in [12], lemma 12, that for any operator T one has

$$\sup\{\operatorname{Re}\lambda: \lambda \in W(T)\} = \lim_{t\to+0}\{(\|I+tT\|-1)/t\}$$

where $W(T)$ denotes the numerical range of T (see definition 4). Thus a and b are also the extremities of the smallest closed interval containing the (real valued) numerical range of the hermitian operator T. If $|W(T)| = \sup\{\|\lambda\|: \lambda \in W(T)\}$, we have, consequently, for a hermitian operator, $|W(T)| = \|T\|_\infty$. But again, it was shown in [12], theorem 5, that, for any operator, $\|T\| \leqq 4|W(T)|$ (for this, it is essential that the space be a complex Banach space). Hence we have, finally,

$$\|T\| \leqq 4\|T\|_\infty$$

Lemma 15. *If T is an operator on X, and admits the decomposition $T = R+iJ$, where $RJ=JR$, and R and J are hermitian-equivalent, then $J=O$ if and only if the spectrum of T is real valued.*

Proof. The "only if part" is obvious. Suppose the spectrum is real. Because of corollary 7 we may suppose without loss of generality that R and J are actually hermitian. Let A be any maximal commutative subalgebra of the algebra of all operators on X, containing R and J, hence T. Then the spectra of R, J, T, are the same as operators or as elements of A. Consider the Gelfand representation of A ([10], chap. IV). If m is any point of the maximal ideal space, we have $\hat{T}(m) = \hat{R}(m) + i\hat{J}(m)$. But $\hat{R}(m)$ and $\hat{J}(m)$ are necessarily real valued. Hence $J(m)$ is identically 0 and by lemma 14, $\|J\| = \|J\|_\infty = 0$.

We turn now our attention to commutative families of projections on arbitrary Banach spaces.

Theorem 16. *If every operator in the uniform closure of the real-linear span of a commutative family $E = \{E_\sigma\}$ of projections on a Banach space is hermitian-equivalent, then E is uniformly bounded.*

Proof. By theorem 13, E is hermitian equivalent. Under the corresponding renorming each $E_\sigma \in E$ goes into a hermitian projection E'_σ, and there is a constant K such that $\|E_\sigma\| \leqq K\|E'_\sigma\|$, for all σ. Since all spectra consist at most of the points

0 and 1, it folows from lemma 14 that $\|E'_\sigma\| \leqq 4$, uniformly. Hence $\|E_\sigma\| \leqq 4K$, for all $E_\sigma \in E$.

In what follows, we shall mean by the expression boolean algebra of projections on the Banach space X, a family of projections on X having all the algebraic properties of a spectral measure as defined by DUNFORD [6].

Theorem 17. *Let E be a boolean algebra of projections on a Banach space X. Then a necessary and sufficient condition for that the adjoint of every operator in the uniformly closed algebra generated by E, be spectral of scalar type, is that E be uniformly bounded.*

Proof. The "if part" is a well-known result of DUNFORD [6], section 4. We prove the converse. Most of the work has already been done above. By the hypothesis, the adjoint T^* of every operator in the uniform closure $\overline{S(E)}$ of $S(E)$ is spectral of scalar type, hence of the form $R+iJ$, where $RJ=JR$, and R and J are hermitian-equivalent, by theorem I. A priori R and J need not be adjoint of operators on X, but we shall show that J is necessarily O, so that $T^*=R$. In fact, it is clear that the spectrum of every operator in $S(E)$ is real, and because of the commutativity, the same will then hold for any $T \in \overline{S(E)}$. (This is in the litterature, and follows also easily from a Gelfand representation argument as above). Hence lemma 15 implies $J=O$. Hence T^* is hermitian-equivalent; thus $G(\{T^*\})$ is uniformly bounded, which implies in turn that $G(\{T\})$ is uniformly bounded (same bound); and finally, that T itself is hermitian-equivalent, always by theorem 6. Now it suffices to invoke theorem 16, to complete the proof.

Corollary 18. *Let E be a boolean algebra of projections on a reflexive Banach space X. Then a necessary and sufficient condition for that the uniformly closed algebra generated by E should consist of spectral operators of scalar type, is that E be uniformly bounded.*

III. Remarks and counter-examples

7. Remarks. One should notice that the results of section 6 imply that for a boolean algebra E, of projections on a Banach space, the following are all equivalent:

(i) The uniform closure of the real-linear span of E consists of hermitian-equivalent operators.

(ii) The uniform closure of the real-linear span of E consists of operators whose adjoint is spectral.

(iii) E is uniformly bounded.

Under the circumstances, one might be easily led to believe that every hermitian operator is spectral. This is false, however, as we shall see later. On general Banach spaces, hermitian-equivalent and scalar type spectral operators, preserve each a somewhat different subset of the properties they share on Hilbert space. Thus, the sum of hermitian-equivalent operators is hermitian-equivalent, while the sum of scalar type spectral operators is not necessarily of the same kind; on the other hand at least powers of a scalar spectral operators are again scalar spectral, while we shall see later that the similar statement for hermitian operators is false. The following may perhaps shed some further light on the connection between spectral and hermitian operators (see also theorem II):

Proposition 19. *A necessary and sufficient condition for that the real-algebra generated by an operator T of real spectrum, on some Banach space X, and the identity operator I, be hermitian-equivalent, is that there exist a constant K, such that for any polynomial $p(z)$ of the complex variable z, one has $\|p(T)\| \leqq K\|p(T)\|_\infty$, where $\|T\|_\infty$ denotes* $\sup\{|p(z)|: z \in \text{spectrum } T\}$.

Proof. The necessity of the condition follows from lemma 14, and the spectral mapping theorem. Let $q(z)$ be any polynomial with real coefficients, then $|\exp iq(z)| = 1$, on the spectrum of T. Approximating $\exp iq(z)$, uniformly on the spectrum of T, by partial sums of the expansion, we conclude that $\|\exp iq(T)\| \leqq K$. The rest follows from theorem 6.

Naturally the existence of a constant like K above, is fundamental in the spectral theory.

8. Examples. In [9], S. KAKUTANI gives an example of cummuting spectral operators of scalar type, T and T', on some Banach space, whose sum is not spectral. On the other hand, it is easy to check, by direct computation (based on the fact that T is hermitian if and only if $\|I+itT\| = 1+o(t)$, t real), that $T+T'$ is hermitian. Thus not every hermitian operator is spectral. In fact, a similar computation will show that $T^nT'^m$ is hermitian for $n, m = 0, 1, 2, \ldots$. Since T and T' commute it follows that $(T+T')^n$ is hermitian for $n = 1, 2, \ldots$ Hence setting $R = T+T'$, $J = O$, the hypotheses of theorem II are satisfied, except for the fact that the underlying space constructed in [9] is not reflexive. Since here $R+iJ$ is not spectral, it turns out that reflexivity is indeed not a superfluous hypothesis in theorem II.

One might still wonder if R hermitian implies R^n hermitian, for all positive integers n, in general, or at least on a reflexive space. However C. A. MCCARTHY has shown recently[6]), that there exist commuting spectral operators T and T' (which one may assume to have real spectra), on a *reflexive* Banach space X, and such that $T+T'$ is not spectral. His method does not allow the same direct computational approach as above, but corollary 12 tells us that $T+T' = R+iJ$, where R and J commute and are hermitian-equivalent. Lemma 15 implies that $J = O$, since the spectrum of $T+T'$ is real valued (T and T' commuting, and having real spectra). Since X is reflexive, and we know that $T+T' = R$ is not spectral, it follows from theorem II that R^n $(n = 1, 2, \ldots)$ are not all hermitian-equivalent. Since R is, we conclude the existence of a hermitian operator whose powers are not all hermitian.

References

[1] W. BADE, Unbounded spectral operators, *Pacific J. Math.*, **4** (1954), 373—392.
[2] ——— Weak and strong limits of spectral operators, *Pacific J. Math.*, **4** (1954), 393—414.
[3] E. BERKSON, A characterization of scalar type operators on reflexive Banach spaces. To appear in *Pacific J. Math.*
[4] J. DIXMIER, Les moyennes invariantes dans les semi-groupes et leurs applications, *Acta Sci. Math.*, **12** C (1950), 213—217.
[5] N. BOURBAKI, *Eléments de mathématiques*, livre V, *Espaces vectoriels topologiques* (Paris, 1953).
[6] N. DUNFORD, Spectral operators, *Pacific J. Math.*, **4** (1954), 321—354.

[6]) C. A. MCCARTHY, Commuting boolean algebras of projections, *Pacific J. Math.*, **11** (1961), 295—307.

[7] N. DUNFORD, A survey of the theory of spectral operators, *Bull. Amer. Math. Soc.*, **64** (1958), 217–274.
[8] E. HILLE and R. S. PHILLIPS, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publ., XXXI (Providence, 1957).
[9] S. KAKUTANI, An example concerning uniform boundedness of spectral measures, *Pacific J. Math.*, **4** (1954), 363–372.
[10] L. H. LOOMIS, *Abstract harmonic analysis* (New York, 1953).
[11] E. R. LORCH, Bicontinuous linear transformations in certain vector spaces, *Bull. Amer. Math. Soc.*, **45** (1939), 564–569.
[12] G. LUMER, Semi-inner-product spaces, *Trans. Amer. Math. Soc.*, **100** (1961), 29–43.
[13] G. LUMER and R. S. PHILLIPS, Dissipative operators in a Banach space, *Pacific J. Math.*, **11** (1961), 679–698.
[14] G. W. MACKEY, *Commutative Banach algebras* (Harvard lecture notes, edited by A. Blair, 1952).
[15] B. SZ.-NAGY, On uniformly bounded linear transformations in Hilbert space, *Acta Sci. Math.*, **11** (1947), 152–157.
[16] I. VIDAV, Eine metrische Kennzeichnung der selbstadjungierten Operatoren, *Math. Z.*, **66** (1956), 121–128.
[17] J. WERMER, Commuting spectral measures on Hilbert space, *Pacific J. Math.*, **4** (1954), 355–362.

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