Fourier transform in locally compact groups *)

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SEGAL [7] has proved that if G is a locally compact abelian group with the dual group $\hat{G}$ then the Fourier transform maps $L^1(G)$ onto $C_0(\hat{G})$ if and only if G is finite. He has mentioned in the same paper that the non-commutative analogue of this result is true but a proof of it has not appeared so far.

In this paper we give some simple proofs to these results. These proofs are based on the following interesting observations:

(1) An extremally disconnected locally compact group which is Hausdorff must be discrete.

(2) Any weakly sequentially complete C^*-algebra must be finite-dimensional.

The second observation is the non-commutative analogue of the well-known result that the space $C_0(S)$ where S is a locally compact Hausdorff space is weakly sequentially complete iff it is finite-dimensional. We also give here a proof to the conjecture of SEGAL in [7] about the Fourier transform of $L^p(G)$ where $1<p<2$ and G is a locally compact abelian group.

Notations. All the topological spaces occurring in this paper are assumed to be Hausdorff. The symbol $C_0(S)$ where S is a locally compact Hausdorff space will denote the Banach space of all complex-valued continuous functions on S vanishing at infinity. If S is compact then we may write $C(S)$ instead of $C_0(S)$. If G is a locally compact group then $M(G)$ will denote the Banach space of all bounded Borel measures with the variation norm. If (X, Σ, μ) is a measure space then $L^p(X)$ where $1 \leqq p \leqq \infty$ will denote the usual L^p-space and $\|f\|_p$ will denote the L^p-norm of a measurable function f defined on X. For terms not defined here see [1], [3], [4], [5], [6], [9].

Theorem 1. *An extremally disconnected locally compact group G must be discrete.*

Proof. By a known theorem on totally disconnected groups (see [5]) we get that G contains a compact open subgroup H. Then H is also extremally disconnected. We claim that H is discrete. If not we can find a decreasing sequence $H_1 \supset H_2 \supset \dots$ $\dots \supset H_n \supset \dots$ of compact, open, normal subgroups of H such that $\mu(H_n) \to 0$ as $n \to \infty$, where μ is a Haar measure of H. Then $H_0 = \bigcap_{n=1}^{\infty} H_n$ is a compact normal

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subgroup of H and $\mu(H_0)=0$. Then the factor group H/H_0 is compact and metrisable by a theorem of KAKUTANI and BIRKHOFF (see [5]). Since H is compact and extremally disconnected and since the canonical map from H onto H/H_0 is open and continuous we have that H/H_0 is also extremally disconnected. Since H/H_0 is compact and metrisable we get that H/H_0 is finite. But then H_0 must be open in H and hence $\mu(H_0)\neq 0$ which is a contradiction. So H must be discrete and being compact must be finite. So G must be discrete.

Corollary 1. *Let G be a locally compact abelian group with the character group $\hat{G}$. Then the Fourier transform T maps $L^1(G)$ onto $C_0(\hat{G})$ if and only if G is finite.*

Proof. First of all we notice that if T_1 is a one-to-one, continuous, linear map from a Banach space X_1 onto a Banach space X_2 then the adjoint map T_1^* of T_1 is also a one-to-one and onto map from X_2^* to X_1^*. Now coming to the Corollary, if T maps $L^1(G)$ onto $C_0(\hat{G})$ then T^* maps $M(\hat{G})$ onto $L^\infty(G)$. By an easy calculation it is seen that T^* coincides with the Fourier transform on $M(G)$. So the image of T^* consists of continuous functions on G. So every function in $L^\infty(G)$ is equivalent to a continuous function on G. Then G must be extremally disconnected and hence discrete. Moreover T^* maps $L^1(\hat{G})$ onto $C_0(G)$. So $\hat{G}$ also must be discrete. So both G and $\hat{G}$ must be finite. The converse is obvious.

Theorem 2. *A C^*-algebra A is weakly sequentially complete if and only if it is finite-dimensional.*

Proof. If A is finite-dimensional then it is weakly sequentially complete. Conversely, let A be weakly sequentially complete. Let $B\subset A$ be any commutative, closed and $*$-closed subalgebra of A. Then, by a theorem of GELFAND on Banach algebras, B is isomorphic to the space $C_0(X)$ where X is a locally compact Hausdorff space. Since A is weakly sequentially complete we have that B is also weakly sequentially complete and hence $C_0(X)$ is finite-dimensional. So every closed, $*$-closed, commutative subalgebra of A is finite-dimensional. In particular the closed subalgebra generated by the element xx^* is finite-dimensional for any $x\in A$. Hence every closed left (or right) ideal of A contains minimal self-adjoint idempotents. Moreover, any set of pairwise orthogonal minimal, self-adjoint idempotents of A is finite. Then let $e_1, e_2, \ldots, e_n$ be a maximal set of pairwise orthogonal minimal self-adjoint idempotents of A. Then $e_1+e_2+\ldots+e_n = e$ is a self-adjoint idempotent. We must have that $ex=x$ for all $x\in A$, since otherwise, the set $B = \overline{\{ex-x|x\in A\}}$ will be a proper closed right ideal of A and will contain a minimal self-adjoint idempotent $e_{n+1}\neq 0$ orthogonal to $e_1, e_2, \ldots, e_n$. Then $e = e_1+e_2+\ldots+e_n$ is an identity of A. Then the centre of A is a proper closed, $*$-closed subalgebra of A and hence is finite-dimensional. So there is a maximal set $f_1, f_2, \ldots, f_k$ of minimal, self-adjoint and pairwise orthogonal, central idempotents $f_1, f_2, \ldots, f_k$. Clearly $f_1A, f_2A, \ldots, f_kA$ are closed, $*$-closed mutually orthogonal minimal two-sided ideals of A and $A = f_1A\oplus f_2A\oplus\ldots\oplus f_kA$. Let us denote by I_i the ideal f_iA for $i=1, 2, \ldots, k$. Then I_i contains a maximal set of minimal, pairwise orthogonal self-adjoint idempotents $e_{11}^i, e_{22}^i, \ldots, e_{n_i n_i}^i$. Then by adopting the same line of reasoning as that used in the Wedderburn structure theorem for algebras we get that I_i is isomorphic to the matrix algebra of order $n_i\times n_i$ over the complex numbers and hence finite-dimensional. Since I_i is finite-dimensional we get that A is finite-dimensional.

Corollary 2. *Let G be a locally compact group with a left Haar measure μ. For any $f \in L^1(G)$, let T_f denote the operator in $L^2(G)$ given by $g \to f * g$ for any $g \in L^2(G)$. Let $|||f||| = \|T_f\|$ for any $f \in L^1(G)$. Then $L^1(G)$ is complete under the norm $|||\cdot|||$ if and only if G is finite.*

Proof. Let $\Delta(x)$ be the modular function of G. Let us put $f^*(x) = \overline{f(x^{-1})}\,\Delta(x^{-1})$ for any $f \in L^1(G)$. Then $L^1(G)$ is a Banach algebra with $*$ as involution. Now suppose $L^1(G)$ is complete under the norm $|||\cdot|||$. Then, from the fact that $|||f||| \leqq \|f\|_1$ for all $f \in L^1(G)$ we get that $\|\cdot\|_1$ and $|||\cdot|||$ are equivalent norms on $L^1(G)$. So $L^1(G)$ with the norm $|||\cdot|||$ and convolution as multiplication and $*$ as involution is a weakly sequentially complete C^*-algebra. So $L^1(G)$ must be finite-dimensional. So G must be finite.

Now we proceed to give a proof to the following conjecture of SEGAL in [7]: "If G is a locally compact abelian group with the character group $\hat{G}$ and $1 < p < 2$ and $1/p + 1/q = 1$ then the Fourier transform will map $L^p(G)$ onto $L^q(\hat{G})$ if and only if G is finite." From now onwards all groups will be assumed to be abelian.

Definition 1. Let G be a locally compact group with the character group $\hat{G}$. Let $1 < p < 2$ and $1/p + 1/q = 1$. The group G is said to have (P) if the Fourier transform maps $L^p(G)$ onto $L^q(\hat{G})$.

Definition 2. Let G be a locally compact group. Then $L(G)$ will denote the set of all continuous functions on G which vanish outside a compact set.

Lemma 1. *Let G be a locally compact group with the character group $\hat{G}$. Let $1 < p < 2$ and $1/p + 1/q = 1$. Then G has (P) if and only if there is a constant K such that $\|Tf\|_q \geqq K\|f\|_p$ for all $f \in L(G)$.* (Here T stands for the Fourier transform from $L^p(G)$ into $L^q(\hat{G})$).

Lemma 2. *Let G, $\hat{G}$, p and q be as in Lemma 1. Then G has (P) if and only if $\hat{G}$ has (P).*

Proof. Let G have (P). Let T: $L^p(G) \to L^q(\hat{G})$ be the Fourier transform from $L^p(G)$ into $L^q(\hat{G})$. Since T is one-to-one and onto, we have that T^* is also one-to-one and onto. But T^* is only the Fourier transform from $L^p(\hat{G})$ onto $L^q(G)$. So $\hat{G}$ has (P). Now the lemma follows by the duality theorem.

Lemma 3. *Let G_1 and G_2 be locally compact groups and $1 < p < 2$ and $1/p + 1/q = 1$. Let $G = G_1 \times G_2$. Then, if G has (P) then both G_1 and G_2 have (P).*

Proof. This follows from Lemma 1 by considering functions of the form $f(x)g(y)$ on $G_1 \times G_2$ where $f(x) \in L(G_1)$ and $g(y) \in L(G_2)$.

Lemma 4. *Let G be a locally compact group with (P). Then G contains a compact open subgroup.*

Proof. By a theorem in [9], G is of the form $R^n \times G_1$ where R^n is the n-dimensional Euclidean space with the usual addition and G_1 is a locally compact group with a compact open subgroup H. Now R does not have (P). (See [8].) So $n = 0$ from Lemma 3. Hence the result.

Lemma 5. *Let $1 < p < 2$. Let G be a locally compact group with (P) and $H \subset G$ a compact subgroup of G. Then G/H also has (P).*

Proof. Let ν be the normalized Haar neasure of H. Let μ be a Haar measure on G. Let 0 be a Haar measure on G/H such that $\int\limits_G f(x)\,d\mu(x) = \int\limits_{G/H}\Big[\int\limits_H f(tx)\,d\nu(t)\Big]\,d0(x)$ holds for all $f \in L(G)$. ($\tilde{x}$ stands for the image of x in G/H under the canonical map φ from $G \to G/H$.) Let ψ be the map from $L(G/H) \to L(G)$ defined by $\psi(\tilde{f}) = \tilde{f} \circ \varphi$ for all $\tilde{f} \in L(G/H)$. Then $\|\tilde{f}\|_p = \|\psi(\tilde{f})\|_p$ for all $\tilde{f} \in L(G/H)$. Let $\hat{G}$ be the character group of G. Let $H^{\perp} = \{\chi | \chi \in \hat{G};\ \chi(x) = 1 \text{ for all } x \in H\}$. Then, since H is compact, $H^{\perp}$ is open in $\hat{G}$. So the restriction of the Haar measure of $\hat{G}$ to $H^{\perp}$ is a Haar measure of $H^{\perp}$. Now $H^{\perp} = (G/H)\hat{}$ and $\int\limits_{G/H} \tilde{f}(\tilde{x})\chi(\tilde{x})\,d\tilde{x} = \int\limits_G (\psi(f))(x)\overline{\chi(x)}\,dx$ for any $\chi \in H^{\perp}$ and $\tilde{f} \in L(G/H)$. So the result follows from Lemma 1.

Lemma 6. *Let G be a compact group and $1 < p < 2$. Let G have (P). Then G is finite.*

Proof. The dual group $\hat{G}$ of G is discrete. So if $f \in l^q(\hat{G})$ where $1/p + 1/q = 1$ and $g \in l^2(\hat{G})$ then $gf \in l^2(\hat{G})$. Now $L^p(G) \subset L^1(G)$ since G is compact. So every element of $l^2(\hat{G})$ is a Fourier transform of a function in $L^1(G)$. Hence from Theorem A on page 271 of [2], we get that $l^q(\hat{G}) \subset l^2(\hat{G})$. But $q > 2$ since $p > 2$. So $l^q(\hat{G}) = l^2(\hat{G})$. Hence $\hat{G}$ is finite. Then G is finite.

Theorem 3. *Let G be a locally compact abelian group with dual group $\hat{G}$. Let $1 < p < 2$ and $1/p + 1/q = 1$. Then, the Fourier transform maps $L^p(G)$ onto $L^q(\hat{G})$ if and only if G is finite.*

Proof. Let the Fourier transform map $L^p(G)$ onto $L^q(\hat{G})$. Then G contains a compact open subgroup H. Then G/H has (P) by Lemma 5. Since G/H is discrete, we get from Lemma 6 and 2 that G/H is finite. But then G must be compact and hence, finite by Lemma 6. Now the converse is obvious.

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