On group homomorphic images of partially ordered semigroups

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Our present aim is to investigate the case when a partially ordered semigroup S can be mapped order-homomorphically onto a partially ordered group G. The corresponding question without partial order has been considered in a number of papers by P. DUBREIL and members of his school (see e. g. [2]). If S is lattice-ordered (or at least a union-semilattice), commutative and residuated, then the answer is given by M.-L. DUBREIL-JACOTIN's well-known generalization of ARTIN's equivalence relation (see [3]) and its recent extensions (see [4]). In order to get satisfactory statements in the more general partially ordered case, it seems to be necessary to make rather natural restrictions. First, we assume that every relation $\leqq$ in G is a consequence of one in S; then it is more natural to consider congruence relations in S with convex classes. Secondly, we wish to obtain from S a possible largest group G, and to this end we suppose — in accordance with DUBREIL-JACOTIN's point of view — that the identity of S is the largest element of its class.[1]) Our discussion leads directly to generalized residuals which are related to the usual residuals as ideals are related to generators of principal ideals.

This note was inspired by the lectures of Madame DUBREIL held at Tulane University, New Orleans, March 1962. She gave a survey of the different generalizations of ARTIN's equivalence relation in partially ordered groupoids and semigroups. In the cases she discussed the equivalence was defined either in terms of closure operator or residual.

1. Assume that S is a partially ordered semigroup, i. e., S is a semigroup and a partially ordered set such that, for all $a, b, c \in S$, $a \leqq b$ implies $ac \leqq bc$ and $ca \leqq cb$. For all $a, b \in S$ we define the *generalized right residual* of a by b as the set[2])

$$\langle a^{\cdot}.b\rangle = \{x \in S | xb \leqq a\},$$

and the *generalized left residual* of a by b as

$$\langle a.^{\cdot}b\rangle = \{x \in S | bx \leqq a\}.$$

We call S *generalized right (left) residuated*[3]) if the set $\langle a^{\cdot}.b\rangle$ (the set $\langle a.^{\cdot}b\rangle$) is

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[1]) Actually, we assume somewhat more, namely, that the identity e of S is greater than or equal to any element of S whose class is less than or equal to the class of e.

[2]) BIRKHOFF [1] uses the symbol $a:b$ for the right and the symbol $a::b$ for the left residual of a by b; cf. also [5].

[3]) If S is trivially ordered, then it is easy to see that it is generalized right residuated if and only if it is a group.

never empty. S is *generalized residuated* if it has this property both from the right and from the left. Clearly, a generalized residual contains along with $x \in S$ also all $y \in S$ satisfying $y \leqq x$.

If the right residual element $a^{\cdot}.b$ happens to exist for certain $a, b \in S$, then $\langle a^{\cdot}.b \rangle$ is just the set of all $x \in S$ with $x \leqq a^{\cdot}.b$. Thus a generalized right residuated semigroup S is right residuated if and only if each $\langle a^{\cdot}.b \rangle$ contains a maximum element.

The generalized residuals obey a number of formal rules which are immediate extensions of known rules of the residuals.[4]) Since we shall not need them, we omit their systematic discussion.

Next suppose that S contains a left (right) identity e. Let U be a subset of S. A $u \in U$ is said to be *left (right) multiplicatively maximal* in U if $v \in S$ and $vu \in U$ ($v \in S$ and $uv \in U$) imply $v \leqq e$. It is evident that if u is left multiplicatively maximal in $U = \langle a^{\cdot}.b \rangle$ and if $u' \in U$ satisfies $u < u'$, then u' is likewise left multiplicatively maximal in U.

Let S be a partially ordered semigroup with left identity e. We consider congruence relations θ of S. Thus, by definition, θ is an equivalence relation such that, for all $a, b, c \in S$,

1. $a \equiv b(\theta)$ implies $ca \equiv cb(\theta)$ and $ac \equiv bc(\theta)$;
2. $a \leqq c \leqq b$ and $a \equiv b(\theta)$ imply $a \equiv c(\theta)$.

We denote by $\theta(a)$ the class of $a \in S$ under θ.

The set of classes under θ need not form a partially ordered semigroup, the quotient semigroup S/θ, if multiplication is defined by $\theta(a)\,\theta(b) = \theta(ab)$ and order relation is defined by putting $\theta(a) \leqq \theta(b)$ if and only if there are $a' \in \theta(a)$, $b' \in \theta(b)$ such that $a' \leqq b'$ in S. In fact, the relation $\theta(a) \leqq \theta(b)$ now defined is in general not a partial order. But it is whenever S/θ is a group and θ is a Dubreil-Jacotin congruence in the following sense.

A congruence θ of S is said to be a *Dubreil-Jacotin congruence* if, in addition to 1—2., it satisfies

3. $\theta(a) \leqq \theta(e)$ implies $a \leqq e$.

In particular, e is then the maximum element in its class.

Now let θ be a Dubreil-Jacotin congruence of S such that S/θ is a group, and let $\theta(a) \leqq \theta(b) \leqq \theta(c)$. If $a \leqq b$, $b' \leqq c$ $(b' \in \theta(b))$, then some $c^* \in S$ satisfies $c^*c \equiv e(\theta)$. We have $c^*b' \leqq c^*c \leqq e$ whence $c^*b \leqq e$ too, and so $cc^*a \leqq cc^*b \leqq c$. Since $a \equiv cc^*a(\theta)$, we have $\theta(a) \leqq \theta(c)$, i. e. transitivity. Antisymmetry follows in the same way.

2. Let S be a partially ordered semigroup with identity e, and θ a Dubreil-Jacotin congruence on S such that S/θ is a group G, necessarily a partially ordered group. Then we can in turn conclude:

A. *S is a generalized residuated semigroup.*

In order to show that $\langle e^{\cdot}.a \rangle$ is not void for any $a \in S$, it suffices to take an $a' \in S$ with $a'a \equiv e(\theta)$. Then, by 3., we have $a'a \leqq e$, and thus $a' \in \langle e^{\cdot}.a \rangle$. Clearly, $ba' \in \langle b^{\cdot}.a \rangle$, and therefore S is generalized right residuated. Similarly, it is generalized left residuated.

[4]) Cf. e. g. Birkhoff [1] and Fuchs [5].

B. *For each $a \in S$, $\langle e^{\cdot}.a\rangle$ contains left multiplicatively maximal elements.*

If a' is chosen as in A, then it has the desired property. In fact, $va' \in \langle e^{\cdot}.a\rangle$, i. e., $va'a \leqq e$ implies $\theta(v)=\theta(v)\,\theta(a'a)=\theta(va'a)\leqq\theta(e)$, and so, by 3., we obtain $v \leqq e$. — The converse holds as well:

C. *If $a' \in \langle e^{\cdot}.a\rangle$ is left multiplicatively maximal, then it satisfies $a'a \equiv e(\theta)$.*

For, if a' is such an element, then $a'a \leqq e$ whence $\theta(a'a)\leqq\theta(e)$. Thus there is a $v \in S$ such that $\theta(v)\geqq\theta(e)$ and $\theta(v)\,\theta(a'a)=\theta(e)$. By 3. we have $va'a\leqq e$, $va' \in \langle e^{\cdot}.a\rangle$. Our hypothesis on a' implies $v\leqq e$, $\theta(v)\leqq\theta(e)$. Therefore $\theta(v)=\theta(e)$ and $a'a\equiv e(\theta)$.

D. *$a\equiv b(\theta)$ if and only if $\langle e^{\cdot}.a\rangle=\langle e^{\cdot}.b\rangle$.*

Assume first that $a\equiv b(\theta)$ and let $x\in\langle e^{\cdot}.a\rangle$. Then $xa\leqq e$, $\theta(x)\,\theta(a)\leqq\theta(e)$, and thus $\theta(x)\,\theta(b)\leqq\theta(e)$. Hence $xb\leqq e$ and $x\in\langle e^{\cdot}.b\rangle$. Changing the roles of a and b we are led to $\langle e^{\cdot}.a\rangle=\langle e^{\cdot}.b\rangle$. The converse follows from the fact that if the left multiplicatively maximal elements in $\langle e^{\cdot}.a\rangle$ and in $\langle e^{\cdot}.b\rangle$ are the same, then by C they belong both to the left inverse class of a and to the left inverse class of b. Therefore the classes of a and b coincide.

E. *$\langle e^{\cdot}.a\rangle=\langle e.^{\cdot}a\rangle$ holds for all $a\in S$.*

Assume $x\in\langle e^{\cdot}.a\rangle$, that is to say, $xa\leqq e$. Then $\theta(x)\,\theta(a)\leqq\theta(e)$, and in the group of classes conjugation with $\theta(x)$ yields $\theta(a)\,\theta(x)\leqq\theta(e)$. Hence $ax\leqq e$ and $x\in\langle e.^{\cdot}a\rangle$. Changing the roles of left and right, we obtain the assertion.

Let us formulate what we have proved so far.

THEOREM 1. *Let θ be a Dubreil-Jacotin congruence of the partially ordered semigroup S with identity e. If S/θ is a group, then in S the generalized left and right residuals exist and satisfy*

(i) *$\langle e^{\cdot}.a\rangle=\langle e.^{\cdot}a\rangle$ for all $a\in S$;*
(ii) *$\langle e^{\cdot}.a\rangle$ contains left multiplicatively maximal elements for all $a\in S$;*
(iii) *$\langle e^{\cdot}.a\rangle=\langle e^{\cdot}.b\rangle$ if and only if $a\equiv b$ (θ).*

3. We are going to show that the converse of this result holds true.

Theorem 2. *Let S be a partially ordered semigroup with identity e such that S is generalized residuated and* (i), (ii) *are fulfilled. If we define the relation θ by* (iii), *then θ is a Dubreil-Jacotin congruence of S and S/θ is a group.*

Assume the hypotheses of the theorem and define θ by (iii). Then θ is evidently an equivalence relation. Since $\langle e^{\cdot}.a\rangle=\langle e^{\cdot}.b\rangle$ implies $\langle e^{\cdot}.ca\rangle=\langle e^{\cdot}.cb\rangle$, we have $ca\equiv cb(\theta)$, and because of (i) similarly $ac\equiv bc(\theta)$ whenever $a\equiv b(\theta)$. Hence it follows that θ is a congruence with respect to multiplication. If $a\leqq c\leqq b$ then $\langle e^{\cdot}.b\rangle\subseteqq\subseteqq\langle e^{\cdot}.c\rangle\subseteqq\langle e^{\cdot}.a\rangle$, and therefore $a\equiv b(\theta)$ will imply $a\equiv c(\theta)$. In order to verify that it is a Dubreil-Jacotin congruence, let $\theta(x)\leqq\theta(e)$. This means that there exist $x', e'\in S$ such that $x'\equiv x(\theta)$, $e'\equiv e(\theta)$ and $x'\leqq e'$. Then $\langle e^{\cdot}.x\rangle=\langle e^{\cdot}.x'\rangle\supseteqq\langle e^{\cdot}.e'\rangle==\langle e^{\cdot}.e\rangle\ni e$, and so $x\leqq e$. Finally we have to show that S/θ is a group.[5] By (ii), there exists a left multiplicatively maximal element a' in $\langle e^{\cdot}.a\rangle$. Thus $a'a\leqq e$, and $va'a\leqq e$ holds if and only if $v\leqq e$. Consequently, $\langle e^{\cdot}.a'a\rangle$ is just the set of all $v\in S$

[5]) It is clear that $\theta(e)$ must be the identity of S/θ.

with $v \leqq e$, and hence it is equal to $\langle e^{\cdot}.e\rangle$. In view of the definition of θ, $a'a \equiv e(\theta)$ and the class $\theta(a)$ has the left inverse $\theta(a')$. This completes the proof of Theorem 2.

4. In order to clarify the meaning of condition (ii) in the residuated case, and at the same time to show that our result is actually a generalization of ARTIN's equivalence from the residuated case, let us now suppose that $\langle e^{\cdot}.a\rangle$ contains a maximum element $e^{\cdot}.a$. In this event, $\langle e^{\cdot}.a\rangle$ contains a left multiplicatively maximal element exactly if $e^{\cdot}.a$ is left multiplicatively maximal, and this is the case if and only if

$$\langle (e^{\cdot}.a)^{\cdot}.(e^{\cdot}.a)\rangle = \{x \in S \mid x \leqq e\}.$$

In fact, $x \in \langle (e^{\cdot}.a)^{\cdot}.(e^{\cdot}.a)\rangle$ is equivalent to $x(e^{\cdot}.a) \in \langle e^{\cdot}.a\rangle$, and so $e^{\cdot}.a$ is left multiplicatively maximal in $\langle e^{\cdot}.a\rangle$ if and only if the last inclusion implies $x \leqq e$. It follows that, in the residuated case, (ii) can be replaced by the condition

$$(e^{\cdot}.a)^{\cdot}.(e^{\cdot}.a) = e \qquad \text{for all } a \in S$$

which is known to be equivalent to integral closure.

References

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