On the class of subdirect powers of a finite algebra

By G. GRÄTZER in Budapest

1. Introduction. Let **K** be a class of algebras, $Sp(\mathbf{K})$ the class of all algebras which are subdirect products of algebras in **K**. If **K** consists of a single algebra $\mathfrak{A}$ then we put $Sp(\mathfrak{A})$ for $Sp(\{\mathfrak{A}\})$.

In case **K** is an elementary class in the wider sense[1]) ($\mathbf{K} \in EC_\Delta$), $Sp(\mathbf{K})$ was studied by LYNDON [4]. He proved that $Sp(\mathbf{K})$ is, in general, not an arithmetical class in the wider sense.

The simplest case of $\mathbf{K} \in EC_\Delta$ is when $\mathbf{K} = \{\mathfrak{A}\}$, where $\mathfrak{A}$ is a finite algebra of finite order. This special case is discussed, but not completely settled, in this paper.

I will define a property (U_N) which may or may not hold for a finite algebra $\mathfrak{A}$. The main result is that if $\mathfrak{A}$ is a finite algebra of finite order and (U_N) holds on $\mathfrak{A}$ then $Sp(\mathfrak{A})$ is an elementary class, $Sp(\mathfrak{A}) \in EC$.

A simple example will show that $Sp(\mathfrak{A}) \notin EC$ may happen.

Several known classes of algebras can be represented as $Sp(\mathfrak{A})$. I will list a few examples:

(i) $\mathfrak{A}$ is the two element Boolean algebra; $Sp(\mathfrak{A})$ is the class of Boolean algebras;

(ii) $\mathfrak{A}$ is the two element lattice; $Sp(\mathfrak{A})$ is the class of distributive lattices;

(iii) $\mathfrak{A}$ is the n element chain; $Sp(\mathfrak{A})$ can be characterized and is in EC;[2])

(iv) $\mathfrak{A}$ is a group of order p (p is a prime); $Sp(\mathfrak{A})$ is the class of elementary abelian p-groups;

(v) $\mathfrak{A}$ is the ring of integers modulo 2; $Sp(\mathfrak{A})$ is the class of Boolean rings.

References. Ad (i): [6] (STONE's theorem); ad (ii): [2] (BIRKHOFF's theorem); ad (iii): ANDERSON and BLAIR [1].

The content of the paper is the following: § 2 gives the notions and notations. The algebraic part of the paper is § 3 where theorem 3. 3 gives a necessary and sufficient condition for $\mathfrak{B} \in Sp(\mathfrak{A})$. Condition (U_N) is introduced in § 4 and in 4. 3 we show how to use it. In § 5 a first order formula is constructed. The main result is given in § 6 (theorem 6. 1). A generalization of 6. 1 is given in § 7 where some examples are given too. § 7 is concluded with a list of problems.

I want to express my gratitude to Prof. R. C. LYNDON who read the manuscript of this paper and made several valuable suggestions. The metamathematical proof of 3. 3 and Problem 5 are due to him and are included with his permission.

[1]) For the notions see § 2.

[2]) The present paper was inspired by this very interesting result of ANDERSON and BLAIR [1]

2. Notions and notations. An algebra $\mathfrak{A}$ is a sequence $\langle A, f_0, \ldots, f_\gamma, \ldots \rangle_{\gamma<\alpha}$ where A is a set and f_γ is an n_γ-ary operation on A, i. e. $f_\gamma \in A^{A^{n_\gamma}}$ and $0 \leqq n_\gamma < \omega$. The sequence $\langle n_\gamma \rangle_{\gamma<\alpha}$ is the *type* of $\mathfrak{A}$, α is the *order* of $\mathfrak{A}$. An algebra $\mathfrak{A}$ is finite if A is a finite set.

In this note we will consider algebras of finite order, $\alpha < \omega$. Further, a type $\langle n_\gamma \rangle_{\gamma<\alpha}$ is fixed and every algebra considered (except when otherwise specified) will be of this type.

With this type we associate, in the well-known way, a first order predicate logic with the identity symbol, without predicate variables, with the individual variables $x, y, z, \ldots$ and with the operation symbols $F_0, \ldots, F_{\alpha-1}$.

We hope that the reader is familiar with the following notions: formula, closed formula; the validity of a formula in $\mathfrak{A}$; elementary class in the wider sense (EC_Δ; i. e. the class **K** of all algebras satisfying a given set of closed formulae); elementary class EC (algebras satisfying a single closed formula). Detailed information on these can be found in [7], [8].

We will need the notion of a *partial algebra*. It is a sequence $\langle A, f_0, \ldots, f_\gamma, \ldots \rangle_{\gamma<\alpha}$ where f_γ is an n_γ-ary *partial operation* on A, i. e. $f_\gamma(a_1, \ldots, a_{n_\gamma})$ is either not defined or an element of A.

In building up the logic for partial algebras one has to consider a partial algebra as a relational system $\langle A, r_0, \ldots, r_{\alpha-1} \rangle$ where r_γ is a relation of rank $n_\gamma+1$, $r(a_1, \ldots, a_{n_\gamma}, a_{n_\gamma+1})$ if and only if $a_{n_\gamma+1} = f_\gamma(a_1, \ldots, a_{n_\gamma})$, and to use the relational symbols $R_0, \ldots, R_{\alpha-1}$ as non-logical predicate constants. These relations always satisfy the sentences

$$(x_1)\ldots(x_{n_\gamma})(y)(z)\big(R_\gamma(x_1, \ldots, x_{n_\gamma}, y) \wedge R_\gamma(x_1, \ldots, x_{n_\gamma}, z) \rightarrow y = z\big).$$

If $\mathfrak{A} = \langle A, f_0, \ldots, f_{\alpha-1} \rangle$ is an algebra, $B \subseteqq A$, then restricting the f_γ to B we get a partial algebra $\mathfrak{B}$, which is called a partial subalgebra of $\mathfrak{A}$.

A congruence relation Θ on a partial algebra $\mathfrak{B}$ is an equivalence relation which satisfies the substitution property whenever it makes sense, i. e. if $f_\gamma(a_1, \ldots, a_{n_\gamma})$, $f_\gamma(a_1', \ldots, a_{n_\gamma}')$ exist and $a_i \equiv a_i' \, (\Theta) \, (1 \leqq i \leqq n_\gamma)$ then $f_\gamma(a_1, \ldots, a_{n_\gamma}) \equiv f_\gamma(a_1', \ldots, a_{n_\gamma}') \, (\Theta)$; the operations on A/Θ are defined as follows: $f_\gamma(a_1|\Theta, \ldots, a_{n_\gamma}|\Theta) = b|\Theta$ if there exist elements $a_1', \ldots, a_{n_\gamma}', b'$ such that $a_i \equiv a_i'(\Theta)$ $(1 \leqq i \leqq n_\gamma)$, $b \equiv b'(\Theta)$ and $f_\gamma(a_1', \ldots, a_{n_\gamma}') = b'$.

3. Subdirect products. We fix the algebra $\mathfrak{A} = \langle A, f_0, \ldots, f_{\alpha-1} \rangle$, $\alpha < \omega$, of type $\langle n_0, \ldots, n_{\alpha-1} \rangle$, $0 \leqq n_\gamma < \omega$ $(0 \leqq \gamma < \alpha)$. We assume that $\mathfrak{A}$ is finite, with elements $a_1, \ldots, a_n$.

We would like to find a criterion for $\mathfrak{B} \in Sp(\mathfrak{A})$. The well-known result of Birkhoff [3] takes on the following form:

3. 1. *$\mathfrak{B} \in Sp(\mathfrak{A})$ if and only if for every $x, y \in B$ $(x \neq y)$ there exists a homomorphism φ of $\mathfrak{B}$ onto $\mathfrak{A}$ such that $x\varphi \neq y\varphi$.*

We will give a "local condition" for $\mathfrak{B} \in Sp(\mathfrak{A})$. To formulate it we need

3. 2. *Let φ be a homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$. We can find a $B_1 \subseteqq B$ such that B_1 has at most $f(n)$ elements and $\mathfrak{A}$ is the homomorphic image of $\mathfrak{B}$ under φ_{B_1}, where φ_{B_1} denotes the restriction of φ to B_1. Here $f(n)$ depends only on n (the number of elements in A) and not on $\mathfrak{B}$ and φ.*

Let $c \in B_1$ if and only if

$$c = f_\gamma(b_{i_1}, \ldots, b_{i_{n_\gamma}})$$

for some $0 \leqq \gamma < \alpha$, $1 \leqq i_j \leqq n$, where $b_1, \ldots, b_n$ are elements of B such that every element of A has an inverse image which is a b_i. The number of such c with a fixed γ is at most n^{n_γ}. Hence $f(n) = n^{n_0} + \ldots + n^{n_{\alpha-1}}$ is certainly effective in 3. 2.

This $f(n)$ is used in the following statement:

3. 3. *$\mathfrak{B} \in Sp(\mathfrak{A})$ if and only if for every $x, y \in B$ $(x \neq y)$ there exists a partial subalgebra $\mathfrak{B}_1$ of $\mathfrak{B}$ with $x, y \in B_1$, containing at most $f(n)+2$ elements, and a homomorphism φ with $\mathfrak{B}_1\varphi = \mathfrak{A}$ $(x\varphi \neq y\varphi)$ such that whenever $\mathfrak{B}_2$ is a finite partial subalgebra of $\mathfrak{B}$ containing $\mathfrak{B}_1$ then φ can be extended to a homomorphism $\bar{\varphi}$ of $\mathfrak{B}_2$ such that $\mathfrak{B}_2\bar{\varphi} = \mathfrak{A}$.*

The "only if" part is obvious. Indeed, if $\mathfrak{B} \in Sp(\mathfrak{A})$ and $x, y \in B$, $x \neq y$ then by 3. 1 there exists a homomorphism ψ with $\mathfrak{B}\psi = \mathfrak{A}$, $x\psi \neq y\psi$. By 3. 2 there exists a partial subalgebra $\mathfrak{B}_1'$ containing at most $f(n)$ elements such that $\mathfrak{B}_1'\psi_{\mathfrak{B}_1'} = \mathfrak{A}$. Let B_1 equal B_1' with x and y adjoined and φ be the restriction of ψ to B_1. Then $\mathfrak{B}_1\varphi = \mathfrak{A}$ is trivial as well as $x\varphi \neq y\varphi$. Further, if $B_1 \subset B_2$ then $\bar{\varphi}$ i. e. the extension of φ to $\mathfrak{B}_2$ is simply the restriction of ψ to B_2.

To prove the less obvious "if" part of 3. 3 suppose that the algebra $\mathfrak{B}$ satisfies the condition formulated in 3. 3. By 3. 1 we have to show that if $x, y \in B$ $(x \neq y)$ then we can produce a homomorphism ψ with $\mathfrak{B}\psi = \mathfrak{A}$ $(x\psi \neq y\psi)$. By assumption we can find $\mathfrak{B}_1$ and φ as specified in 3. 3. Consider the set P of all couples $\langle \mathfrak{C}, \chi \rangle$ satisfying the following conditions:

(a) $\mathfrak{C}$ is a partial subalgebra of $\mathfrak{B}$, containing $\mathfrak{B}_1$;

(b) χ is a homomorphism with $\mathfrak{C}\chi = A$;

(c) φ is the restriction of χ to $\mathfrak{B}_1$;

(d) if $\mathfrak{C}_1$ is a partial subalgebra of $\mathfrak{B}$ such that $C \subseteqq C_1$ and $C_1 \setminus C$ is *finite* then χ can be extended to a homomorphism $\bar{\chi}$ of $\mathfrak{C}_1$ with $C_1\chi = A$.

P is not empty since $\langle B_1, \varphi \rangle \in P$.

We define a partial ordering $\leqq$ on P as follows:

$$\langle \mathfrak{C}^1, \chi^1 \rangle \leqq \langle \mathfrak{C}^2, \chi^2 \rangle$$

if and only if the following two conditions are fulfilled:

(e) $\mathfrak{C}^1$ is contained in $\mathfrak{C}^2$;

(f) χ^1 is the restriction of χ^2 to $\mathfrak{C}^1$.

We want to prove that we can apply ZORN's lemma to the partially ordered set $\mathfrak{P} = \langle P, \leqq \rangle$. To this end consider a well-ordered ascending chain in $\mathfrak{P}$:

$$\langle \mathfrak{C}^0, \chi^0 \rangle \leqq \langle \mathfrak{C}^1, \chi^1 \rangle \leqq \ldots \leqq \langle \mathfrak{C}^\gamma, \chi^\gamma \rangle \leqq \ldots \qquad (\gamma < \beta).$$

Let $C = \bigcup_{\gamma < \beta} C^\gamma$. There exists a unique mapping χ of C which is an extension of all χ^γ $(\gamma < \beta)$. We prove

$$\langle \mathfrak{C}, \chi \rangle \in P.$$

Conditions (a)—(d) are to be verified. Conditions (a), (b) and (c) hold for $\langle \mathfrak{C}, \chi \rangle$ trivially.

To prove (d) let $\mathfrak{D}$ be a partial subalgebra of $\mathfrak{B}$, containing $\mathfrak{C}$ such that $D\setminus C$ is finite, say $\{d_1, \ldots, d_m\}$. Consider the partial algebras $\mathfrak{D}^\gamma$ with $D^\gamma = C^\gamma \cup \{d_1, \ldots, d_m\}$. By definition χ^γ can be extended to D^γ. Let $Q(\gamma)$ denote the set of all possible extensions, i. e. an $\varepsilon \in Q(\gamma)$ is a homomorphism of D^γ onto A. If $\gamma_1 < \gamma_2$ then $D^{\gamma_1} \subseteqq D^{\gamma_2}$ and the restriction of an $\varepsilon \in Q(\gamma_2)$ to D^{γ_1} gives an element of $Q(\gamma_1)$. Thus a natural mapping $p(\gamma_2, \gamma_1)$ of $Q(\gamma_2)$ into $Q(\gamma_1)$ is defined. Obviously, if $\gamma_1 < \gamma_2 < \gamma_3$ then $p(\gamma_3, \gamma_2)\cdot p(\gamma_2, \gamma_1) = p(\gamma_3, \gamma_1)$. $Q(\gamma)$ is finite, it cannot contain more than n^m elements. Hence we may apply STEENROD's theorem [5] to the system $\langle Q(\gamma)\rangle_{\gamma<\alpha}$ with the mappings p, and we get the existence of $\varepsilon_\gamma \in Q(\gamma)$ such that $\gamma_1 < \gamma_2$ implies $\varepsilon_{\gamma_2} p(\gamma_2, \gamma_1) = \varepsilon_{\gamma_1}$. That is ε_γ is a homomorphism of $\mathfrak{D}^\gamma$ onto $\mathfrak{A}$ and ε_{γ_1} is the restriction of ε_{γ_2} to $\mathfrak{D}^{\gamma_1}$ if $\gamma_1 < \gamma_2$. Thus the limit of the ε_γ is a uniquely defined relation ε on $D = \bigcup D^\gamma$. Obviously, ε is a homomorphism of $\mathfrak{D}$ onto $\mathfrak{A}$ and ε is an extension of χ, which thus verifies condition (d).

By ZORN's lemma there exists a maximal element

$$\langle \mathfrak{C}, \chi \rangle$$

in $\mathfrak{P}$. If $b \in B, b \notin C$ then χ can be extended to a homomorphism $\bar{\chi}$ of $\mathfrak{C}'$ with $C' = \{C, b\}$ and $\langle \mathfrak{C}', \bar{\chi} \rangle$ were an element greater than $\langle \mathfrak{C}, \chi \rangle$.

Thus $C = B$, and the homomorphism χ is the one we were looking for. The proof of 3. 3 is complete.

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In a letter dated July 14, 1963, R. C. LYNDON gave a metamathematical version of 3. 3. Since his proof is very interesting, I include his proof too.

Lemma. *Let $\mathfrak{A}$ be finite, $B_1 \subseteqq B$, and Φ_1 be any map from $\mathfrak{B}_1$ into $\mathfrak{A}$. If Φ_1 cannot be extended to a homomorphism Φ from $\mathfrak{B}$ into $\mathfrak{A}$, then there exists some finite $B_2 \subseteqq B$ such that Φ_1 does not agree on $B_1 \cap B_2$ with any homomorphism Φ_2 from $\mathfrak{B}_2$ into $\mathfrak{A}$.*

Proof. In an extended language with predicates P_{a_i}, $1 \leqq i \leqq n$, and names b for the elements of B, consider sentences as follows:

S_b expressing that exactly one $P_{a_i}(b)$ holds;

$$S_{\gamma, b_1, \ldots, b_{n_\gamma}, b} \equiv \bigwedge_{h_1, \ldots, h_{n_\gamma}} \Big[\bigwedge_{1 \leqq i \leqq n_\gamma} P_{a_{h_i}}(b_i) \rightarrow P_f(a_{h_1}, \ldots)(b) \Big].$$

For $B_0 \subseteqq B$, let $S(B_0)$ be the set of all sentences S_b for b in B_0, and all $S_{\gamma, b_1, \ldots, b_{n_\gamma}}$, for $b_1, \ldots, b_{n_\gamma}$, b in B_0 such that $f_\gamma(b_1, \ldots, b_{n_\gamma}) = b$. Now $S(B_0)$ holds if and only if there exists a homomorphism Φ from $\mathfrak{B}_0$ into $\mathfrak{A}$ such that $b\Phi = a_i$ if and only if $P_a(b)$. Let Φ_1 be given. For $B_0 \subseteqq B_1$, let $D(B_0)$ be the set of all sentences

$$D_b \equiv P_{b\varphi_1}(b), \quad \text{for } b \text{ in } B_0.$$

Now, Φ_1 has an extension to a homomorphism φ from $\mathfrak{B}$ into $\mathfrak{A}$ if and only if $S(B) \cup D(B_1)$ is consistent. If this set is inconsistent, some finite subset is, hence, for some finite B_2, the set $S(B_2) \cup D(B_2)$ is inconsistent. This gives the asserted conclusion.

4. The condition (U_N). $\mathfrak{A}$ as in § 3. In order to formulate condition (U_N) we need the notion "a system of equations over $\mathfrak{A}$".

Let us fix a finite list of "unknowns": $x_1, \ldots, x_m$. Let X denote set of unknowns. An equation is

$$f_\gamma(y_1, \ldots, y_{n_\gamma}) = y_0. \quad (0 \leqq \gamma < \alpha) \tag{4. 1}$$

where y_i $(0 \leqq i \leqq n_\gamma)$ is either an x_j or an element of A $(y_i \in X \cup A)$.

E. g. if $n_0 = 2$ then examples of equations are the following:

$$f_0(x_1, x_5) = x_2, \quad f_0(a_1, a_2) = a_3, \quad f_0(a_5, x_3) = x_1, \quad f_0(a_1, a_3) = x_1,$$

and so on.

A system of equations $\Gamma = \Gamma(x_1, \ldots, x_n)$ over $\mathfrak{A}$ is a finite set of equations of the form (4. 1). A solution of Γ is a mapping $\chi: X \to A$ $(x \to x\chi)$ such that if (4. 1) is an equation in Γ then the relation

$$f_\gamma(y_1\chi, \ldots, y_{n_\gamma}\chi) = y_0\chi$$

holds in $\mathfrak{A}$, where $y_i\chi = y_i$ if $y_i \in A$. Γ is unsolvable if it has no solution.

The system of equations $\Gamma' = \Gamma'(x_{i_1}, \ldots, x_{i_t})$, $t \leqq m$, is a *consequence* of $\Gamma = \Gamma(x_1, \ldots, x_m)$ if using the equations in Γ and the known relations in $\mathfrak{A}$ one can prove the equations in Γ.

E. g. if $\Gamma = \Gamma(x_1, x_2)$ consists of two equations:

$$f_0(x_1, x_2) = a_1, \quad f_1(a_3, a_5) = x_1 \qquad (n_0 = n_1 = 2)$$

and we know that $f_1(a_3, a_5) = a_2$, then $\Gamma' = \Gamma'(x_2)$ consisting of a single equation $f_0(a_2, x_2) = a_1$ is a consequence of Γ.

The algebra $\mathfrak{A}$ is said to satisfy condition (U_N) if there exists an integer N such that whenever $\Gamma = \Gamma(x_1, \ldots, x_n)$ is a system of equations over $\mathfrak{A}$, Γ is unsolvable, then there exists a consequence $\Gamma' = \Gamma'(x_{i_1}, \ldots, x_{i_t})$ of Γ which is unsolvable too, and for which $t \leqq N$.

We will list without proof what is the value of N for the algebras listed in § 1: (i) $N=2$, (ii) $N=2$, (iii) $N=n$, (iv) $N=0$, (v) $N=0$. That is every algebra listed here satisfies condition (U_N), for some N.

4. 1. *There exist algebras which satisfy condition* (U_N) *for no* N. [3])

Let the algebra be of type $\langle 1 \rangle$, let $n=2$, i. e. $\mathfrak{A}$ contains two elements a_1, a_2, and f_0 is defined by

$$f_0(a_1) = a_2, \quad f_0(a_2) = a_1.$$

Now suppose that this algebra $\mathfrak{A}$ satisfies condition (U_N) and consider the following system $\Gamma = \Gamma(x_1, \ldots, x_{2N+1})$ of equations:

$$f_0(x_1) = x_2, \quad f_0(x_2) = x_3, \ldots, f_0(x_{2N}) = x_{2N+1}, \quad f_0(x_{2N+1}) = x_1.$$

Γ has no solution for if χ were a solution and e. g. $x_1\chi = a_1$, then $x_2\chi = f_0(x_1\chi) = f_0(a_1) = a_2$, similarly, $x_3\chi = a_1, \ldots, x_{2N+1}\chi = a_1$, and then the last equation gives $f_0(a_1) = a_1$, a contradiction. At the same time any consequence of Γ containing less than $2N+1$ unknowns has a solution, thus $\mathfrak{A}$ does not satisfy (U_N).

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[3]) Essentially the same example was suggested by A. Hajnal.

Now we prove what is condition (U_N) good for:

4. 2. *Let $\mathfrak{A}$ be an algebra satisfying condition (U_N). Suppose that $\mathfrak{B}_2$ is a finite partial algebra, $\mathfrak{B}_1$ a partial subalgebra of $\mathfrak{B}_2$, and φ a homomorphism of $\mathfrak{B}_1$ onto $\mathfrak{A}$. The homomorphism φ can be extended to $\mathfrak{B}_2$ if and only if it can be extended to any partial subalgebra $\mathfrak{B}_3$ of $\mathfrak{B}_2$ such that $B_1 \subseteqq B_3 \subseteqq B_2$ and $B_3 \setminus B_2$ contains not more than N elements.*

The "only if" part is obvious, therefore we prove the "if" part. Suppose that the condition holds and let $b_1, \ldots, b_m$ be the elements of B_2 not in B_1.

We build up a system of equations $\Gamma = \Gamma(x_1, \ldots, x_m)$ as follows:

We consider the relations

$$f_\gamma(c_0, \ldots, c_{n_\gamma - 1}) = c_0$$

in $\mathfrak{B}_2$. We substitute a b_i by x_i and a c_i by $c_i\varphi \in A$. Thus we get an equation. Γ is the set of all such equations.

E. g. if

$$f_0(b_1, c_1) = c_2$$

holds in $\mathfrak{B}_2$ $(c_1, c_2 \in B_1; c_1\varphi = a_2, c_2\varphi = a_5)$, then the corresponding equation is

$$f_0(x_1, a_2) = a_5.$$

It is now obvious that φ can be extended to $\mathfrak{B}_2$ if and only if Γ has a solution. If Γ has no solution then Γ has a consequence $\Gamma' = \Gamma'(x_{i_1}, \ldots, x_{i_t})$, $t \leqq N$ such that Γ' has no solution. Obviously then if we consider $\mathfrak{B}_3$, which contains besides the elements of $\mathfrak{B}_1$, the elements $b_{i_1}, \ldots, b_{i_t}$, we will find that φ cannot be extended to $\mathfrak{B}_3$, contradicting the hypothesis.

Now 3. 3 and 4. 2 give the following result:

4. 3. *Let the algebra $\mathfrak{A}$ satisfy condition (U_N). Then $\mathfrak{B} \in Sp(\mathfrak{A})$ if and only if for every $x, y \in B$ $(x \neq y)$ there exists a partial subalgebra $\mathfrak{B}_1$ of $\mathfrak{B}$ with $x, y \in \mathfrak{B}_1$ containing at most $f(n)+2$ elements, and a homomorphism φ with $\mathfrak{B}_1\varphi = \mathfrak{A}$ $(x\varphi \neq y\varphi)$ such that whenever $\mathfrak{B}_2$ is a finite partial subalgebra of $\mathfrak{B}$ $(B_1 \subseteqq B_2 \subseteqq B)$, $B_2 \setminus B_1$ contains at most N elements, then φ can be extended to a homomorphism $\overline{\varphi}$ of $\mathfrak{B}_2$ and $\mathfrak{B}_2\overline{\varphi} = \mathfrak{A}$.*

5. A first order formula.

5. 1. *There exists a first order formula*

$$\Phi(x_1, \ldots, x_l)$$

free in the variables $x_1, \ldots, x_l$ expressing the following fact: if $\mathfrak{B}$ is a partial algebra, and $b_1, \ldots, b_l$ are the elements of $\mathfrak{B}$, then $\Phi(b_1, \ldots, b_l)$ holds if and only if $\mathfrak{B}$ has a homomorphism φ onto $\mathfrak{A}$ such that $b_1\varphi \neq b_2\varphi$.

The formula we are going to construct is of the form

$$(\exists y_1^1)(\exists y_2^1)\ldots(\exists y_l^1)\ldots(\exists y_1^n)\ldots(\exists y_l^n)(\Phi_1 \wedge \Phi_2 \wedge \ldots \wedge \Phi_5), \tag{5. 2}$$

where the Φ_i contain no quantifier. (n is the number of elements in $\mathfrak{A}$.)

The idea of the construction is the following: we want to describe a congruence relation Θ, such that $\mathfrak{B}|\Theta \cong \mathfrak{A}$; let the congruence classes be $\{y_1^1, \ldots, y_l^1\}, \ldots$ $\ldots, \{y_1^n, \ldots, y_l^n\}$. Then we have to guarantee that:

(i) this is a partition;
(ii) $\{y_1^1, \ldots, y_l^n\} = \{x_1, \ldots, x_n\}$;
(iii) $x_1 \not\equiv x_2(\Theta)$;
(iv) Θ is a congruence relation;
(v) $y_j^i \to a_i$ is a homomorphism.

Conditions (i)—(v) are taken care of by $\Phi_1, \ldots, \Phi_5$.

(i) Φ_1 expresses that $y_{j_1}^{i_1} = y_{j_2}^{i_2}$ implies $i_1 = i_2$, thus

$$\Phi_1\colon\ y_1^1 \neq y_1^2 \wedge y_1^1 \neq y_2^2 \wedge \ldots \wedge y_1^1 \neq y_l^n \wedge y_2^1 \neq y_1^2 \wedge \ldots \wedge y_l^{n-1} \neq y_l^n.$$

(ii) every x_i is a $y_{j_1}^{i_1}$ and conversely:

$$\Phi_2\colon\ (x_1 = y_1^1 \vee x_1 = y_2^1 \vee \ldots \vee x_1 = y_l^n) \wedge \ldots \wedge (x_l = y_1^1 \vee \ldots \vee x_l = y_l^n) \wedge$$
$$\wedge (y_1^1 = x_1 \vee y_1^1 = x_2 \vee \ldots \vee y_1^1 = x_l) \wedge \ldots \wedge (y_l^n = x_1 \vee \ldots \vee y_l^n = x_l).$$

(iii) $x_1 \not\equiv x_2(\Theta)$:

$$\Phi_3\colon\ (x_1 = y_1^1 \wedge x_2 = y_1^2) \vee (x_1 = y_1^1 \wedge x_2 = y_2^2) \vee \ldots \vee (x_1 = y_1^1 \wedge x_2 = y_l^n) \vee \ldots$$
$$\vee (x_1 = y_l^{n-1} \wedge x_2 = y_l^n) \vee (x_2 = y_1^1 \wedge x_1 = y_1^2) \vee \ldots \vee (x_2 = y_l^{n-1} \wedge x_1 = y_l^n).$$

(iv) We want to assert that if $b_1 \equiv b_1'(\Theta), \ldots, b_{n_\gamma} \equiv b_{n_\gamma}'(\Theta)$ and $b \not\equiv b'(\Theta)$ then not $f_\gamma(b_1, \ldots, b_{n_\gamma}) = b$ and $f_\gamma(b_1', \ldots, b_{n_\gamma}') = b'$. This is expressed by the conjunction of all formulas of the form

$$(f_\gamma(y_{j_1}^{h_1}, \ldots, y_{j_{n_\gamma}}^{h_{n_\gamma}}) = y_j^h \wedge f_\gamma(y_{k_1}^{h_1}, \ldots, y_{k_{n_\gamma}}^{h_{n_\gamma}}) = y_k^{h'})$$

for $h \neq h'$.

(v) Now we would like $y_j^i \to a_i$ to be a homomorphism. This means that if e. g. $f(a_1, \ldots, a_{n_1}) = a_3$ then $f_1(y_{j_1}^1, \ldots, y_{j_{n_1}}^{n_1}) = y_p^3$ holds for suitable $j_1, \ldots, j_{n_1}$ and p, further if $f_1(y_{j_1}^1, \ldots, y_{j_{n_1}}^{n_1})$ is defined then it takes as value some y_p^3.

In formula:

$$f_1(y_1^1, \ldots, y_1^{n_1}) = y_1^3 \vee \ldots \vee f_1(y_1^1, \ldots, y_1^{n_1}) = y_l^3 \vee f_1(y_1^1, \ldots, y_2^{n_1}) =$$
$$= y_1^3 \vee \ldots \vee f_1(y_l^1, \ldots, y_l^{n_1}) = y_l^3.$$

We take one such formula for each (in $\mathfrak{A}$ valid) expression of the type $f_\gamma(a_1, \ldots, a_{n_\gamma}) = a_\delta$ and Φ_5 is the conjunction of all such formulae.

6. The main theorem. Now it is easy to formulate and prove our main result:

6. 1. *Let $\mathfrak{A}$ be a finite algebra of finite order satisfying condition (U_N). Then $Sp(\mathfrak{A}) \in EC$. In other words, there exists a closed first order formula ψ such that ψ is satisfied by an algebra $\mathfrak{B}$ if and only if $\mathfrak{B}$ is a subdirect power of $\mathfrak{A}$.*

Let

$$(\exists y_1^1)\dots(\exists y_l^n)(\Phi_{l,n}(x_1, \dots, x_l))$$

denote the formula which we constructed in § 5. The formula ψ is the following

$$(x)(y)(\exists y_1)\dots(\exists y_{f(n)})(\exists y_1^1)\dots(\exists y_{f(n)+2}^1)\dots(\exists y_1^n)\dots$$
$$\dots(\exists y_{f(n)+2}^n)(z_1)\dots(z_N)(\exists u_1^1)\dots(\exists u_{2+f(n)+N}^1)\dots(\exists u_{2+f(n)+N}^n)$$
$$(x\neq y\to((\exists y_1^1)\dots(\exists y_{f(n)+2}^n)(\Phi_{f(n)+2,n}(x, y, y_1, \dots, y_{f(n)}))\wedge$$
$$\wedge(\exists u_1^1)\dots(\exists u_{2+f(n)+N}^n)(\Phi_{2+f(n)+N,n}(x, y, y_1, \dots, y_{f(n)}, z_1, \dots, z_N))\wedge$$
$$\wedge(y_1^1=u_1^1\vee y_1^1=u_2^1\vee\dots\vee y_1^1=u_{2+f(n)+N}^1)\wedge\dots$$
$$\dots\wedge(y_{f(n)+2}^n=u_1^n\vee\dots\vee y_{f(n)+2}^n=u_{2+f(n)+N}^n))),$$

where $f(n)$ was specified in 3. 2.

The proof of 6. 1 is an application of 4. 4 and 5. 1.

7. Generalization, counter examples, problems. A natural generalization of the main theorem is the following:

7. 1. *Let $\mathfrak{A}_1, \dots, \mathfrak{A}_m$ be finite algebras of finite order, each of which satisfies condition (U_N), $\mathbf{K}=\{\mathfrak{A}_1, \dots, \mathfrak{A}_m\}$. Then $Sp(\mathbf{K})\in EC$.*

Let $\Psi(x, y)$ denote the formula free in x and y what we get from the formula Ψ in the proof of 6. 1 after omitting the first two universal quantifiers. Let $\Psi_i(x, y)$ denote the corresponding formula for the algebra $\mathfrak{A}_i$. Consider the following formula:

$$\Psi\colon\ (x)(y)(x\neq y\to(\Psi_1(x, y)\vee\dots\vee\Psi_m(x, y))).$$

Using the same ideas as in the proofs of 3. 3, 4. 4 and 5. 1 we can verify that this Ψ is satisfied on $\mathfrak{B}$ if and only if $\mathfrak{B}\in Sp(\mathbf{K})$.

All theorems proved so far remain true if we replace "algebra" by "partial algebra". Even more is true. These theorems hold true for relational systems as well. Some changes are, of course, necessary (e. g. in the definition of (U_N)); the details are obvious.

7. 2. *There exists a finite algebra $\mathfrak{A}$ of finite order such that $Sp(\mathfrak{A})\notin EC$.*

Let $\mathfrak{A}$ be the algebra given in the proof of 4. 2. Let Σ be the following system of axioms:

$$(x)(x\neq f_0(x))$$
$$(x_1)(x_2)(x_3)((f_0(x_1)=x_2\wedge f_0(x_2)=x_3)\to f_0(x_3)\neq x_1)$$
$$\dots\dots\dots\dots$$
$$(x_1)(x_2)\dots(x_{2t_1})((f_0(x_1)=x_2\wedge f_0(x_2)=x_3\wedge\dots\wedge f_0(x_{2t})=x_{2t+1})\to f_0(x_{2t+1})\neq x_1)$$
$$\dots\dots\dots\dots$$

Using 3. 1 one can verify that $\mathfrak{B}\in Sp(\mathfrak{A})$ if and only if Σ is satisfied in $\mathfrak{B}$. Thus $Sp(\mathfrak{A})\in EC_\Delta$. At the same time no finite subsystem of Σ is equivalent to Σ, thus $Sp(\mathfrak{A})\notin EC$.

7. 3. *Let $\mathfrak{A}$ be an algebra of power $>\omega$. Then $Sp(\mathfrak{A}) \notin EC_\Delta$.*

Obviously, $\mathfrak{B} \in Sp(\mathfrak{A})$ implies that the power of $\mathfrak{B} > \omega$ (or $\mathfrak{B}$ is a one element algebra) which (by the theorem of SKOLEM and LÖWENHEIM) implies $Sp(\mathfrak{A}) \notin EC_\Delta$. If the power of $\mathfrak{A}$ equals ω then either $Sp(\mathfrak{A}) \in EC_\Delta$ or not.

7. 4. *Let $\mathfrak{A}$ be a countable field, then $Sp(\mathfrak{A}) \notin EC_\Delta$.*

Indeed, suppose $Sp(\mathfrak{A}) \in EC$ then by the completeness theorem of the first order functional calculus (or by using prime products) we get that there are in $Sp(\mathfrak{A})$ fields of arbitrarily large power. Since these are subdirectly irreducible, we get a contradiction.

A less trivial example is the following:

7. 5. *Consider $\omega+\omega^*+\omega$ as a lattice. Then $Sp(\omega+\omega^*+\omega) \notin EC_\Delta$.*

Indeed, if $Sp(\omega+\omega^*+\omega)$ where in EC_Δ then ω (being elementarily equivalent to $\omega+\omega^*+\omega$) would be in $Sp(\omega+\omega^*+\omega)$ obviously contradicting 3. 1, since ω has no homomorphism onto $\omega+\omega^*+\omega$.

7. 6. *Let $\mathfrak{A}_i$ be the field of p_i elements where $p_1, p_2, \ldots$ is the sequence of prime numbers. Let $\mathbf{K} = \{\mathfrak{A}_1, \mathfrak{A}_2, \ldots\}$. Then $Sp(\mathbf{K}) \notin EC_\Delta$.*

The same reasoning as in 7. 4.

Now I will list a few problems which arise most naturally.

Problem 1. Let $\mathfrak{A}$ be a finite algebra of finite order. Find necessary and sufficient conditions for $Sp(\mathfrak{A}) \in EC$.

Problem 2. The same for $Sp(\mathfrak{A}) \in EC_\Delta$.

Problem 3. What can be said about $Sp(\mathfrak{A})$ if (i) $\mathfrak{A}$ is finite but of infinite order; or (ii) $\mathfrak{A}$ is countable?

Find sufficient conditions for $Sp(\mathfrak{A}) \in EC_\Delta$ in these cases.

Problem 4. $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ be an infinite sequence of finite algebras of flnite order, each of which satisfies condition (U_N). Let $\mathbf{K} = \{\mathfrak{A}_1, \mathfrak{A}_2, \ldots\}$. Under what conditions is $Sp(\mathbf{K}) \in EC_\Delta$ true?

Problem 5. Does $Sp(\mathfrak{A})$ agree with some $\mathbf{S}$ in EC_Δ for all $\mathfrak{B}$ of large power?

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(Received June 26, 1963)