A note on finite graphs

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In this paper we consider only finite graphs without loops and multiple edges. If A and B are two graphs of this kind, we shall denote by $A \cup B$ the graph consisting of all vertices and edges of both A and B. Our aim is to prove the following.

Theorem. *Let G be a finite graph. Suppose that A, B and C are independent complete subgraphs of G such that every vertex of G belongs to exactly one of them. Suppose that the number of vertices in A, B and C is*

$$v(A)=a, \quad v(B)=b, \quad \textit{and} \quad v(C)=c,$$

respectively, further that the inequality

$$c \geqq a \geqq b$$

holds. If the number of edges connecting a vertex of C with a vertex of $A \cup B$ is at least $(a+b)c-2b+1$, then there exist two subgraphs D and E in G, both complete, having no vertex in common, such that

$$v(D)=c, \quad v(E) \geqq a+1.^{1)}$$

Proof. We introduce a partition for the vertices of $A \cup B$. We will say that a vertex of $A \cup B$ belongs to the first, second or third class in our partition, if it is joined with all, with all but one, or with at most $c-2$ vertices of C, respectively. Let us denote by x_A, y_A the number of vertices belonging to the first or the second class in A, respectively. Let x_B and y_B have the same meaning with respect to the graph B. We shall give a lower and an upper estimation of the number of edges between $A \cup B$ and C.

The lower estimation is given by

$$(a+b)c-2b+1,$$

as supposed in the theorem.

An upper one can be obtained in virtue of the partition of the vertices in $A \cup B$ considered above. This gives that the number of edges examined is not greater than

$$c(x_A+x_B)+(c-1)(y_A+y_B)+(c-2)\{(a-x_A-y_A)+(b-x_B-y_B)\}$$

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1) This theorem plays an important role in a method developed in our paper entitled „On the maximal number of independent complete polygons in a graph" appearing in the *Acta Math. Hung.*

From these results it follows the inequality

$$2(x_A+x_B)+y_A+y_B \geqq 2a+1$$

which means that the two inequalities

$$x_A+x_B+y_A \leqq a, \quad x_A+x_B+y_B \leqq a$$

can not hold simultaneously. We may suppose without loss of the generality, that the first of them is false. We have also

$$x_A+x_B+y_A \geqq a+1.$$

This inequality enables us to construct the subgraphs D and E with the properties required. We begin with the construction of D.

Let D consist of all vertices of A belonging to the first or the second class in our partition, further of $c-x_A-y_A$ vertices of C, each of them joined with all vertices in A belonging to the second class. The inequality

$$c-x_A-y_A \geqq c-a \geqq 0$$

shows, that after omitting at most $y_A \leqq a$ vertices from C, all the remaining vertices satisfy the condition required. It is easy to see that the graph D constructed in this way fulfills all the requirements of our theorem.

Now we have to construct the graph E. Let E consist of all vertices of B belonging to the first class in our partition, further from the remaining vertices of C. In virtue of the inequality

$$x_A+x_B+y_A \geqq a+1$$

the graph E constructed above fulfills all the requirements of the theorem. So our theorem is proved.

We show now that our theorem is best possible of its kind. Namely we show that *for all values of a, b and c satisfying the inequality*

$$c \geqq a \geqq b$$

there exists a graph G_0 having independent complete subgraphs A_0, B_0 and C_0, every vertex of G_0 belonging to exactly one of them, for which

$$v(A_0)=a, \quad v(B_0)=b, \quad v(C_0)=c,$$

further the number of edges connecting a vertex of C_0 with a vertex of $A_0 \cup B_0$ is exactly $(a+b)c-2b$, and G_0 does not include subgraphs D_0 and E_0 of the type mentioned in the theorem.

To show this, let
$A_0=\{P_1, \ldots, P_a\}$, $B_0=\{Q_1, \ldots, Q_b\}$, $C_0=\{R_1, \ldots, R_c\}$, and $G_0 = A_0 \cup B_0 \cup C_0$.

Suppose that A_0, B_0 and C_0 are complete graphs, not two of them having a vertex in common. Suppose further that for all values of i $(1 \leqq i \leqq b)$ the edges between the vertices P_i, Q_i and R_i are not drawn. In virtue of the inequality

$$c \geqq a \geqq b$$

the systems $\{P_i, Q_i, R_i\}$ exist for all values of i in question. Suppose finally, that all other edges in G_0 not mentioned above are drawn.

The graph G_0 constructed in this way does not include subgraphs D_0 and E_0 with the properties required in the theorem. Namely from the $3b$ vertices

$$P_i, Q_i, R_i \qquad (1 \leqq i \leqq b),$$

no more than $2b$ can be used in the construction for D_0 and E_0, and so finally D_0 and E_0 cannot contain together more than

$$c+a$$

vertices from G_0. This shows that our theorem cannot be improved for values a, b and c satisfying the inequality

$$c \geqq a \geqq b$$

by weakening the inequality for the number of edges connecting a vertex of C with a vertex of $A \cup B$, assumed in the theorem.

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