A criterion for Neumann regularity of normal semigroups

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A semigroup S is a non-empty set, which is closed under an associative multiplication. An element a of a semigroup S is said to be regular if there exists an element x in S such that $axa=a$. (See, e. g. GREEN [2].) A semigroup S is called regular if every element of S is regular.

The notion of regularity was introduced for the case of (associative) rings by J. VON NEUMANN [6]. L. KOVÁCS [4] characterized the regular ring A as a ring satisfying the relation

$$RL=R\cap L$$

for every right ideal R of A and for every left ideal L of A. K. ISÉKI [3] extended this result for the case of semigroups. The present author proved in [5], that the following statements concerning a semigroup S are equivalent:

1. S is regular;
2. $RL = R\cap L$ for every right ideal R of S and for every left ideal L of S;
3. $(a)_R(b)_L = (a)_R \cap (b)_L$,[1]) for all a, b in S;
4. $(a)_R(a)_L = (a)_R \cap (a)_L$, for each a in S.

This result is generalized for the case of general algebraic systems in [8].

A semigroup is said to be normal, if

$$xS=Sx$$

for every element x in S. (See [7].)

Our regularity criterion for normal semigroups reads as follows.

Theorem. *A normal semigroup is regular if and only if every left ideal of it is idempotent.*

Proof. Let S be a regular semigroup, and let L be a left ideal of S. We show that L is idempotent, i. e.

$$L^2=L.$$

It is trivial that $L^2\subseteqq L$. To show that $L\subseteqq L^2$, let a be an element of L. Since S is regular, there exists an element x in S, such that $a=axa$. Hence we have $a=a(xa)\in L^2$. Therefore $L\subseteqq L^2$, which implies $L^2=L$, i. e. the left ideal L is idempotent.

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[1]) $(a)_R$ denotes the principal right ideal of S, generated by a.

Conversely, let S be a normal semigroup, in which every left ideal is idempotent. Let a be an element of S. Then the principal left ideal of S generated by a is the following:

$$(a)_L = a \cup Sa.$$

By our hypothesis $(a)_L$ is idempotent, i. e.

$$(a \cup Sa)(a \cup Sa) = a \cup Sa.$$

This implies that $a \in (a \cup Sa)(a \cup Sa)$. Since S is normal, it follows that

$$(a \cup Sa)(a \cup Sa) = a^2 \cup aSa,$$

therefore $a = a^2$ or $a \in aSa$. This means that a is regular in both cases. Thus S is regular, which proves our theorem.

Remark. The example constructed by J. Calais [1] shows that there exists such a semigroup, in which every left ideal is idempotent and wich is not regular. Thus the idempotency of all the left ideals of a semigroup S does not imply the regularity of S. It is therefore very natural to raise the problem to *describe the class of all semigroups, in which every left ideal is idempotent.*

It is known that the regular semigroups and the quasi-regular semigroups (see [1]) do have this property.

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