Remark on a set of integers*

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1. Introduction. Let k be a fixed integer >1 and let $Q_k^*(x)$ denote the number of positive integers $n \leqq x$ with the property that the multiplicity of each prime divisor of n is not a multiple of k. In a previous paper [1] it was shown by purely elementary methods that

$$Q_k^*(x) = \alpha_k x + O(\sqrt[k]{x} \log x),$$

where

$$\alpha_k = \zeta(k) \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right),$$

the product ranging over the primes. In the same paper, making use of the properties of real Dirichlet series, the above formula was refined to

$$Q_k^*(x) = \alpha_k x + O(\sqrt[k]{x}). \tag{1}$$

It is the purpose of the present note to give a proof of (1) which eliminates the use of Dirichlet series. The proof is based on the almost trivial observation that

$$[x] = x + O(x^s), \qquad 0 \leqq s < 1, \tag{2}$$

where $[x]$ is the number of $n \leqq x$. This device has been of use in previous papers; in particular, we refer to [2].

2. Proof of (1). As in [1] we work with the function $Q_{k,r}^*(x)$ which denotes the number of $n \leqq x$ none of whose prime factors is of multiplicity jk where $j \leqq r$. We shall prove that

$$Q_{k,r}^*(x) = \alpha_{k,r} x + O(\sqrt[k]{x}), \tag{3}$$

uniformly in r, where

$$\alpha_{k,r} = \zeta(k) \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{k+1}} + \frac{1}{p^{kr+k}} - \frac{1}{p^{kr+k+1}}\right).$$

It suffices to prove (3) since (1) results in the limiting case as $r \to \infty$. Let $\varphi(x, n)$ denote the number of integers $\leqq x$ which are prime to n, $\varphi(n) = \varphi(n, n)$, and let $\sigma_t^*(n)$ denote the sum of the t-th powers of the square-free divisors of n. Let $\mu(n)$ denote the Möbius function. We prove first the

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Lemma. *For each s,* $0 \leqq s < 1$,

$$\varphi(x, n) = \left(\frac{\varphi(n)}{n}\right) x + O(x^s \sigma^*_{-s}(n)). \tag{4}$$

Proof. It is well known that

$$\varphi(x, n) = \sum_{d|n} \mu(d) \left[\frac{x}{d}\right],$$

so that by (2)

$$\varphi(x, n) = x \sum_{d|n} \frac{\mu(d)}{d} + O\left(x^s \sum_{d|n} \frac{\mu^2(d)}{d^s}\right).$$

The lemma is proved.

Let $\mu_r(n)$ denote the generalized Möbius function introduced in [1, § 3] and let $\sigma_t(n)$ have its usual meaning as the sum of the t-th power divisors of n. We proceed on the basis of the relation

$$Q^*_{k,r}(x) = \sum_{n \leqq \sqrt[k]{x}} \mu_r(n) \varphi\left(\frac{x}{n^k}, n\right)$$

proved in [1, (3.14)]. By (4) we obtain then (cf. [1, Lemma 3.5])

$$Q^*_{k,r}(x) = \alpha_{k,r} x + O\left(x \sum_{n > \sqrt[k]{x}} \frac{1}{n^k}\right) + O\left(x^s \sum_{n \leqq \sqrt[k]{x}} \frac{\sigma_{-s}(n)}{n^{sk}}\right),$$

$0 \leqq s < 1$, where the O-terms are uniform in r. Denote the sum in the second O-term by $R_{k,s}(x)$. Then

$$R_{k,s}(x) = \sum_{n \leqq \sqrt[k]{x}} n^{-ks} \sum_{d\delta = n} d^{-s} = \sum_{d \leqq \sqrt[k]{x}} d^{-s(k+1)} \sum_{\delta \leqq \sqrt[k]{x}/d} \delta^{-ks}.$$

Restricting s to the range $0 \leqq s < 1/k$, it follows that

$$R_{k,s}(x) = O\left(\sum_{d \leqq \sqrt[k]{x}} d^{-s(k+1)} \cdot \left(\frac{x^{1/k}}{d}\right)^{1-ks}\right) = O(x^{1/k-s} \sum_{d \leqq \sqrt[k]{x}} d^{-s-1}).$$

With the further restriction, $0 < s < 1/k$, we have $R_{k,s}(x) = O(x^{1/k-s})$, from which (3) results.

Bibliography

[1] Eckford Cohen, Some sets of integers related to the k-free integers, *Acta Sci. Math.*, **22** (1961), 223–233.

[2] Eckford Cohen, An elementary estimate for the k-free integers, *Bull. Amer. Math. Soc.*, **69** (1963), 762–765.

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