Some torsion free rank two groups*)

By JOHN KOEHLER, S. J., in Seattle (Washington, USA)

The first theorem of this paper answers in the affirmative a conjecture of BEAUMONT and PIERCE concerning quotient divisible (q. d.) rank two groups [1, p. 40]. The rest of the paper gives a construction for rank two groups with an infinite type set. The author assumes familiarity with the notation and theorems of [1], which shall be used throughout.

Theorem. *If $T=\{\tau_0, \tau_1, \ldots, \tau_n\}$ is a finite set of distinct types such that $\tau_i \cap \tau_j = \tau_0$ if $i \neq j$ and τ_0 is non-nil, then there exists a quotient divisible group A such that $T(A)=T$.*

Proof. By Theorem 6.3 in [1], A is a q. d. torsion free rank two group if and only if $\Sigma + H(\xi) + H(\eta)$ and $H(\xi) \cap H(\eta)$ are non-nil, where $(A; x_1, x_2) \to (\Sigma, X)$ and $(\xi, \eta) \in X$. Now $\tau_0 = [H(\xi) \cap H(\eta)]$ is given as non-nil. Hence the problem reduces to that of finding a group A such that $T(A)=T$ and an independent pair x_1, x_2 such that $(A; x_1, x_2) \to (\Sigma, X)$, where Σ is non-nil, and $\Sigma(p)=\infty$ for all but a finite number of primes p such that $0<h_p(\xi(p))<\infty$ and $0<h\,(\eta(p))<\infty$ whenever $(\xi, \eta) \in X$.

Let χ_0 be the characteristic such that $\chi_0 \in \tau_0$ and $\chi_0(p)=0$ or ∞ for all p. Let χ_i' be a characteristic such that $\chi_i' \in \tau_i$, $i=1, \ldots, n$. If $i \neq j$, let $\pi_{ij}=\{p | \chi_i'(p)<\infty,\ (\chi_i' \cap \chi_j')(p) \neq \chi_0(p)\}$. Each π_{ij} is finite, since $\tau_i \cap \tau_j = \tau_0$. Furthermore, $p \in \pi_{ij}$ if and only if $0 = \chi_0(p) < \chi_i'(p) < \infty$ and $0 < \chi_j'(p)$. For $i=1, \ldots, n$ define χ_i by

$$\chi_i(p) = \begin{cases} 0 & \text{if} \quad p \in \bigcup_j \pi_{ij} \\ \chi_i'(p) & \text{otherwise.} \end{cases}$$

Then $\chi_i \in \tau_i$ since $\bigcup_j \pi_{ij}$ is finite and both χ_i and χ_i' are finite on this set. It is easy to check that $\chi_i \cap \chi_j = \chi_0$ if $i \neq j$.

The following sets of primes partition π, $1 \leqq k \leqq n$.

$$\pi_0 = \{p | \chi_k(p)=0 \text{ for all } k\};$$
$$\pi_0' = \{p | \chi_k(p)=\infty \text{ for all } k\} = \{p | \chi_0(p)=\infty\};$$
$$\pi_k = \{p | \infty > \chi_k(p) > 0 = \chi_0(p)\};$$
$$\pi_k' = \{p | \infty = \chi_k(p) > 0 = \chi_0(p)\}.$$

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Define Σ, ξ, and η by

$$\Sigma(p) = \begin{cases} 0 & \text{if } p \in \pi_0 \cup \pi_0', \\ \infty & \text{otherwise;} \end{cases}$$

$$\xi(p) = \begin{cases} 0 & \text{if } p \in \pi_0', \\ 1 & \text{otherwise;} \end{cases}$$

$$\eta(p) = \begin{cases} 0 & \text{if } p \in \pi_0', \\ 1 & \text{if } p \in \pi_0, \\ k & \text{if } p \in \pi_k' \ (k=1, \dots, n), \\ k\beta(p) & \text{if } p \in \pi_k \ (k=1, \dots, n); \end{cases}$$

where $\beta(p) = 1 + \sum_{n=0}^{\infty} p^{2^n l}$, $l = \chi_k(p)$. Thus if $p \in \pi_k$, $\eta(p)/\xi(p)$ is irrational; $h_p(\eta(p)) = h_p(\xi(p)) = 0$; and $h_p(\xi(p) - \eta(p)) = h_p(p^l + p^{2l} + p^{4l} + \dots) = l = \chi_k(p)$.

Let A be a rank two group containing independent elements x_1 and x_2 such that $(A; x_1, x_2) \to (\Sigma, X)$, $(\xi, \eta) \in X$. Note that $[H(\xi)] = [H(\eta)] = \tau_0$ is non-nil, and that $\Sigma(p) + h_p(\xi(p)) + h_p(\eta(p))$ is 0 if $p \in \pi_0$, and is ∞ otherwise. Hence A is q. d.

It remains to show that $T(A) = T$. To do this, we employ 7. 2, 7. 3, and 7. 4 of [1]. The results are illustrated in the following tables, where $0 \neq s \in R$, Δ and ϱ are as defined in 7.2 and 7.3 respectively. The blank places are not needed to calculate $T(A)$.

	$\Sigma(p)$	$\Delta(\eta - s\xi)(p)$	$(H(\xi) \cap H(\eta))(p)$	$\chi_0(p)$	$\chi_j(p)$
$p \in \pi_0$	0		0	0	0
$p \in \pi_0'$	0		∞	∞	∞
$p \in \pi_k$, $k \geqq 1$	∞	$\Delta(k\beta(p) - s) = 0$	0	0	$\chi_j(p) < \infty$ if $j = k$; 0 if $j \neq k$
$p \in \pi'$, $k \geqq 1$	∞	$\Delta(k - s) = \infty$ if $s = k$; $= 0$ if $s \neq k$	0	0	∞ if $j = k$; 0 if $j \neq k$

In the following table, let $1 \leqq k \leqq n$, $l = \chi_k(p)$, and let * indicate that the case under consideration occurs for at most a finite number of primes.

Thus $\Sigma \cap (H(\varrho - s) + \Delta(\eta - s\xi)) + H(\xi) \cap H(\eta)$ is equivalent to χ_k if $s = k$, $k = 1, \dots, n$, and is equivalent to χ_0 if $s \neq k$. $H(\xi) \sim \chi_0 \sim H(\eta)$. Hence $T(A) = \{\tau_0, \tau_1, \dots, \tau_n\} = T$ by 7. 4 of [1].

Lemma. *Let $r_1, r_2, \dots, r_k$ be a set of distinct positive integers and let $n_1, n_2, \dots, n_k$ be a set of distinct positive integers which are relatively prime in pairs. Then there exist positive integers $r_{k+1}, x_1, x_2, \dots, x_k$ such that $(x_i, n_i) = 1$ and $r_{k+1} = x_i n_i + r_i$, $i = 1, 2, \dots, k$.*

Proof. By the Chinese Remainder Theorem, there exist positive integers $y_1, y_2, \dots, y_k$ such that the numbers $y_i n_i + r_i$ are all equal to some common value

	$\varrho(p)$	$H(\varrho-s)(p)$	
$p\in\pi_k,\ p\nmid k$	$\eta(p)/\xi(p)= k(1+p^l+p^{2l}+\ldots)$	$h_p(k(1+p^l+p^{2l}+\ldots)-s)$ $=\chi_k(p)$ if $s=k$ $=\chi_0(p)$ if $s\neq k,\ h_p(k-s)=0$ $<\infty$ if $s\neq k,\ h_p(k-s)\neq 0$	*
$p\in\pi_k,\ p\mid k$	0	$h_p(-s)<\infty$	*
$p\in\pi'_k,\ p\nmid k$	k	$h_p(k-s)$ $=\infty$ if $s=k$ $=0$ if $s\neq k,\ h_p(k-s)=0$ $<\infty$ if $s\neq k,\ h_p(k-s)\neq 0$	*
$p\in\pi'_k,\ p\mid k$	0	$h_p(-s)<\infty$	*

s_{k+1}. All such values will be congruent modulo $N=n_1n_2\ldots n_k$. Let M be the product of all the primes p such that for some index i, $p|n_i$ and $p\nmid y_i$. It is easy to check that, if $x_i = y_i+MN/n_i$, $i=1,\ldots,k$, and $r_{k+1} = x_1n_1+r_1$, then the conclusion of the lemma follows.

Construction. Let $T=\{\tau_0,\tau_1,\tau_2,\ldots\}$ be an infinite type set such that $\tau_i\cap\tau_j = \tau_0$ if $i\neq j$. For $i=0,1,2,\ldots$, choose characteristics $\chi_i\in\tau_i$ such that $\chi_i\geqq\chi_0$. Somewhat as before, define

$$\pi_0=\{p|\infty>\chi_0(p)=\chi_i(p)\text{ for all } i\},$$
$$\pi'_0=\{p|\chi_0(p)=\infty\},$$
$$\pi'_k=\{p|\infty=\chi_k(p)\neq\chi_0(p)\},\qquad k=1,2,\ldots,$$
$$\pi^*_k=\{p|\infty>\chi_k(p)>\chi_0(p)\},\qquad k=1,2,\ldots;$$

having defined π_0, define inductively

$$\pi_k = \pi^*_k-\Big(\bigcup_{j\neq k}\pi'_j\cup\bigcup_{j<k}\pi_j\Big),\qquad k=1,2,\ldots.$$

$\pi_0,\pi_1,\pi_2,\ldots,\pi'_0,\pi'_1,\pi'_2,\ldots$ partition π. We wish, however, to split the primes further. Let $\pi''_0=\varnothing$; define inductively $\pi''_k=\bigcup\limits_j(\pi'_j\cap\pi^*_k)-\bigcup\limits_{j<k}\pi''_j$,

$$\pi_{ij}=\pi''_i\cap\pi'_j,\qquad i\neq j.$$

Each π_{ij} is a finite set, and $\pi_{ij}\cap\pi_{kl}\neq\varnothing$ only if $i=k$ and $j=l$.

Let $d(p) = \chi_i(p)-\chi_0(p)$ for each $p\in\pi_{ij}$; define

$$n_{ij}=\prod_{\pi_{ij}}p^{d(p)}\prod_{\pi_{ji}}p^{d(p)},$$

where $i<j$ and where we take the empty product to be 1. Note that $n_{1j},n_{2j},\ldots$ are relatively prime in pairs. Let $r_1=1,r_2 = x_{12}n_{12}+r_1,\ldots,r_k = x_{ik}n_{ik}+r_i$, where for each k and each $i<k$, r_k and x_{ik} satisfy the above lemma.

We now redefine χ_k and π_k so that no primes in π_k divide r_k or the x_{ik}, $i<k$. We do this inductively, leaving χ_0 and π_0 unchanged. Having redefined χ_i and π_i

for $i<k$, let $p\in\pi_k$, $p|r_k$ or $p|x_{ik}$. Change $\chi_k(p)$ to $\chi_0(p)$; remove p from π_k; and insert p in π_j, where j is the first index greater than k for which $\infty>\chi_j(p)>\chi_0(p)$. If there is no such j, let p be in π_0. Under these new definitions, $\pi_0, \pi_1, \ldots, \pi_0', \pi_1', \ldots$ still partition π.

Order the primes in their natural order: $p_1=2$, $p_2=3$, etc. Define ξ and η as follows:

$$\xi(p)=\begin{cases}0 & \text{if } p\in\pi_0',\\ p^{\chi_0(p)} & \text{otherwise;}\end{cases}$$

$$\eta(p)=\begin{cases}0 & \text{if } p\in\pi_0',\\ p^{\chi_0(p)}r_k & \text{if } p\in\pi_k',\\ p_m^{\chi_0(p_m)}\left(m+\sum_{i=0}^{\infty}p_m^{2^i}\right) & \text{if } p_m\in\pi_0,\\ p^{\chi_0(p)}r_k\left(1+\sum_{i=0}^{\infty}p^{2^i}\right) & \text{if } p\in\pi_k.\end{cases}$$

Define Σ arbitrarily to satisfy

$$\Sigma(p)=\begin{cases}=0 & \text{if } \chi_0(p)=\infty,\\ \geqq \sup_i\{\chi_i(p)-\chi_0(p)\} & \text{if } \chi_0(p)\neq\infty.\end{cases}$$

Let $(A;x_1,x_2)\to(\Sigma,X)$, where $(\xi,\eta)\in X$.

If, as in the preceding theorem, we perform the calculations for $\Theta(s)= =\Sigma\cap(H(\varrho-s)+\Delta(\eta-s\xi))+H(\xi)\cap H(\eta)$ for every s, $0\neq s\in R$, we obtain the results:

(1) $\Theta(r_k)\sim\chi_k'$, $k=1,2,\ldots$, where $\chi_k'(p)=\chi_0(p)+h_p(r_j-r_k)$ if $p\in\pi_j'$ and $\chi_k(p)=\chi_0(p)$ for some $j\geqq 1$. $\chi_k'(p)=\chi_k(p)$ otherwise.

(2) $\Theta(s)\sim\chi_0$ if $s\neq r_k$, unless for infinitely many k there are primes p such that

(a) $h_p(r_k-s)>0$, where $p\in\pi_k'$, $p\nmid r_k$; or

(b) $h_p(r_k\eta(p)/\xi(p)-s)>0$ where $p\in\pi_k$, $p\nmid r_k$.

(3) $H(\xi)=\chi_0$,

(4) $H(\eta)\sim\chi_0$ unless for infinitely many k, $h_p(r_k)>0$ for some $p\in\pi_k'$.

Since each $\chi_k'\geqq\chi_k>\chi_0$, it is clear that τ_0, $[\chi_1']$, $[\chi_2']$, ... are all distinct types. Since A is a group and $\tau_0=H(\xi)$, then $[\chi_j']\cap[\chi_k']=\tau_0$ if $j\neq k$. Thus $T(A)$ is infinite, and $T(A)=T$ unless the exceptions noted above occur.

The following corollaries are evident:

Corollary 1. *Let $T=\{\tau_0,\tau_1,\tau_2,\ldots\}$ be an infinite set of types such that $\tau_i\cap\tau_j=\tau_0$ if $i\neq j$. Then there is a set of types $T'=\{\tau_0,\tau_1',\tau_2',\ldots\}$ such that $\tau_i'\geqq\tau_i$ if $i\geqq 1$ and $\tau_i'\cap\tau_j'=\tau_0$ if $i\neq j$ and a rank two group A such that $T(A)\supseteqq T'$.*

Corollary 2. *Let $T=\{\tau_0,\tau_1,\tau_2,\ldots\}$ be an infinite set of types such that $\tau_i\cap\tau_j=\tau_0$ if $i\neq j$ and such that, for only finitely many i, $\chi_i(p)=\infty$ occurs, where $\chi_i\in\tau_i$. Then there is a rank two group A such that $T(A)\supseteqq T$.*

Corollary 3. *In the above construction, if τ_0 is non-nil and if we let $\Sigma(p)=\infty$ whenever $\infty \neq \chi_0 p_0$ for any $\chi_0 \in \tau_0$, then A will be a q. d. group having an infinite type set.*

Example. Define $\chi_0, \chi_1, \chi_2, \dots$ by

$$\chi_0(p)=0 \text{ for all } p,$$

$$\chi_1(p)=1 \text{ for all } p,$$

$$\chi_k(p_{k-1})=\infty,\ \chi_k(p)=0 \text{ for all other } p,$$

where $k>1$, and where the primes are given their natural order. Clearly $[\chi_i]\cap[\chi_j] = [\chi_0]$ if $i\neq j$. However $\chi_1\cap\chi_k \neq \chi_0$ if $k>1$; in fact, it is impossible to find a set of characteristics equivalent to these such that any two intersect to give χ_0.

Following the construction, we get $\pi_0=\pi_0'=\pi_1'=\pi_k=\varnothing$. If $k>1$, $\pi_k'=\{p_{k-1}\}$, $\pi_k''=\varnothing$. $\pi_1''=\pi$. Thus if $k>1$, $\pi_{1k}=\{p_{k-1}\}=\pi_k'$, $n_{1k}=p_{k-1}$, all other n_{ik} are 1. Let $r_1=1$, $x_{1k}=1$, $r_k = x_{1k}n_{1k}+1 = p_{k-1}+1$, and $x_{jk} = r_k-r_j$ for $1<j<k$. These numbers satisfy the lemma trivially for each k; also, it is not necessary to redefine any χ_k or π_k.

Define for all primes p_k: $\Sigma(p_k)=\infty$, $\xi(p_k)=1$, $\eta(p_k) = r_{k+1} = p_k+1$. Let $(A; x_1, x_2)\to(\Sigma, [(\xi,\eta)])$.

Then $H(\xi)=H(\eta)=\chi_0$. $\Theta(s)$ is easy to compute for all $0\neq s\in R$. $\Delta(\eta-s\xi)(p_k)= =\infty \Leftrightarrow s = r_{k+1} = p_k+1$. If $s=1$, $h_p(\varrho(p)-1) = h_p(p+1-1) = 1$ for all p. Hence $\Theta(1)=\chi_1$. If $s=r_k$, $k>1$, then $\Theta(r_k)(p_{k-1})=\infty$. For other primes $p\neq p_{k-1}$, $h_p(\varrho(p)-r_k) = h_p(p-p_{k-1}) = 0$. Hence $\Theta(r_k)=\chi_k$. If $s\neq r_k$, $k\geqq 1$, then $h_p(\varrho(p)-s) = h_p(p+1-s) = h_p(1-s) = 0$ for all but a finite number of p. Hence $\Theta(s)\sim\chi_0$.

Therefore $T(A)=\{[\chi_0],[\chi_1],[\chi_2],\dots\}$. Thus in this simple example, the construction gives good results.

References

[1] R. A. Beaumont and R. S. Pierce, Torsion Free Groups of Rank Two, *Memoirs Amer. Math. Soc.*, **38** (1961).

[2] L. Fuchs, *Abelian Groups* (New York, 1960).

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