Logical dependence of certain chain conditions in lattice theory

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Introduction

The purpose of this paper is to determine all logical implications that exist between certain chain conditions occurring in lattice theory.

We recall first some basic notions. A subset C of a partially ordered set P is called a *chain of* P, if C is totally ordered with respect to the ordering of P (if $x, y \in C$, then x, y are *comparable*, that is either $x<y$, or $x=y$, or $x>y$). A chain $C \subseteq P$ is said to be *maximal*, if it has the following property: $a \in L$ comparable with all $c \in C$, implies $a \in C$. A *bounded* chain is a chain C with least and greatest element: there exist $a, b \in C$, such that $a \leqq c \leqq b$ for every $c \in C$. A *finite* chain is a chain having a finite number of elements; all finite chains are bounded. Let us denote by $x \prec y$ the fact that y *covers* x, that is: $x<y$ and there is no element z such that $x<z<y$. Then a *finite maximal* chain C can be written in the form $x_0 \prec x_1 \prec \ldots \prec x_n$; the number n is called the *length* of C. By definition, two maximal chains C, C' *have the same length* if either both C, C' are infinite, or C, C' have the same finite length n. A partially ordered set P is said be *of finite length*, if there is a natural number n, such that every maximal chain C of P has a length $\leqq n$. P is said to be *of locally finite length*, if for every $a, b \in P$, with $a<b$, the segment $[a, b]=\{x | x \in L, a \leqq x \leqq b\}$ is of finite length.

As concerns the existence of maximal chains, we recall here that the following property is equivalent to the axiom of choice: *every chain of a partially ordered set is contained in a maximal chain* (see, for instance, [1]).

Now, let L be a lattice whose operations are denoted by $\cup$, $\cap$. We shall consider the following properties:

M. For every $a, b, c \in L$, $a \leqq c$ implies $a \cup (b \cap c) = (a \cup b) \cap c$.

S. For every $a, b, x \in L$ such that $a \cap b < x < a$, there is an element $t \in L$, such that $a \cap b < t \leqq b$ and $(x \cup t) \cap a = x$.

F_1. The lattice L is of finite length.
F_2. The lattice L is of locally finite length.
F_3. All bounded chains of L are finite.

J_1. For every $a, b \in L$ with $a<b$, all maximal chains of $[a, b]$ have the same length.

J_2. For every $a, b \in L$ with $a<b$, all finite maximal chains of $[a, b]$ have the same length.

C_1. For every $a, b \in L$, $a \cap b \prec a$ implies $b \prec a \cup b$.

C_2. For every $a, b \in L$, $a \cap b \prec a, b$*) implies $a, b \prec a \cup b$.

A lattice L satisfying property M, respectively S, is called *modular*, respectively *semi-modular*. According to G. SZÁSZ [11], property J_1 is the *Jordan—Dedekind chain condition*; in G. BIRKHOFF's book [1], this denomination is reserved to property J_2; in R. CROISOT's terminology [4], the Jordan—Dedekind chain condition is the logical conjunction of J_1 and F_3 (or, of J_2 and F_3). Properties C_1 and C_2 may be called *covering conditions* [1]; C_1 is also called „die Nachbarbedingung" [11].

The above properties are not independent. For instance, it is easily seen that $F_1 \Rightarrow F_2 \Rightarrow F_3$; $J_1 \Rightarrow J_2$; $C_1 \Rightarrow C_2$ a. s. o.; other less trivial logical implications are given in [1], [4], [11]. In this paper, we shall determine all logical implications between M, ..., C_2.

In general, a system Σ of properties being given, the determination of all possible logical implications between the properties belonging to Σ**), is called the *complete existential theory* of Σ. This notion is due to E. H. MOORE [9]. The complete existential theory of a system Σ, offers us a (complete) set of new results and a definitive systematization of them. Studies concerning the complete existential theory of certain axiom systems occurring in the theory of partially ordered sets, were made by A. H. DIAMOND [2], L. L. DINES [3], R. CROISOT [4], E. V. HUNTINGTON [5], [6], [7] (paper [7] contains detailed explanations about the significance of this notion), H. M. MACNEILLE [8], I. ROSENBAUM [10], J. S. TAYLOR [12], [13], and W. E. VAN DE WALLE [14].

In the sequel we shall study the complete existential theory of the system

$$\Sigma = \{M, S, F_1, F_2, F_3, J_1, J_2, C_1, C_2\}. \tag{1}$$

This problem was suggested us by D. VAIDA.

§ 1

We shall first establish some lemmas,

Lemma 1. *If* F_3 *and* C_1, *then* F_2 *and* J_1.

Proof. Let L be a lattice satisfying F_3 and C_1; and let $a, b \in L$, with $a < b$. We must prove that all maximal chains of $[a, b]$ have the same finite length. By F_3, $[a, b]$ includes a finite maximal chain; taking into account Theorem 1 from [4], p. 88, we obtain the desired conclusion.

Lemma 2. *If* C_1, *then* J_1.

Proof. Let L be a lattice satisfying C_1, and let $a, b \in L$, with $a < b$. If $[a, b]$ includes a finite maximal chain, then, reasoning as in Lemma 1, we conclude that all maximal chains of $[a, b]$ have the same (finite) length. If $[a, b]$ has no finite maximal chain, then all maximal chains of $[a, b]$ are infinite. In both cases, J_1 is verified.

*) $x \prec y, z$ means $x \prec y$ and $x \prec z$; similarly for $t, u \prec v$.

**) We consider the most general type of implications $A \Rightarrow B$, that is, A and B are logical functions of the properties of Σ (expressed by means of logical conjunction, disjunction, negation).

Lemma 3. *If* F_3 *and* J_1, *then* F_2.

Proof. Let L be a lattice satisfying F_3, but not F_2; we shall prove that L does not satisfy J_1. According to the hypothesis, L includes a segment $[a, b]$ of infinite length, although all maximal chains of $[a, b]$ are finite. This means that for every natural number n, there exists a maximal chain of $[a, b]$, whose length is $>n$; thus J_1 is not verified.

Lemma 4. *If* F_2 *and* C_2, *then* S.

Proof. Let L be a lattice satisfying F_2 and C_2, and let $a, b, x \in L$, such that

(2) $$a \cap b < x < a.$$

By F_2, the segment $[a \cap b, x]$ has a finite length n, and $[a \cap b, b]$ is also of finite length. Hence there is an element $t \in L$, such that

(3) $$a \cap b \prec t \leqq b;$$

we shall prove that

(4) $$x = (x \cup t) \cap a.$$

We remark first that (2) and (3) imply

(5) $$a \cap b = x \cap t$$

and

(6) $$x \leqq (x \cup t) \cap a < x \cup t;$$

($x \cup t = (x \cup t) \cap a$ would imply $t \leqq x \cup t \leqq a$, hence $t \leqq a \cap b$, a contradiction).

Now, relation (4) is an immediate consequence of (6) and

(7) $$x \prec x \cup t;$$

therefore, it is sufficient to prove (7). We shall do this by recurrence on the length n of $[a \cap b, x]$.

If $n=1$, then $a \cap b \prec x$; taking into account (5) and (3), we deduce $x \cap t \prec x, t$. By C_2, we obtain (7).

Now, supposing the assertion true for $n-1$, we shall prove it for $n>1$. In fact, F_2 implies the existence of an element $y \in L$ such that

(8) $$a \cap b < y \prec x,$$

the segment $[a \cap b, y]$ being of length $n-1$. Relation (8) and the inductive hypothesis imply respectively

(9) $$y \leqq x \cap (y \cup t) \leqq y \cup t$$

and

(10) $$y \prec y \cup t.$$

Since $x \cap (y \cup t) \neq y \cup t$ (otherwise $t \leqq y \cup t \leqq x < a$, hence $t \leqq a \cap b$, a contradiction), relations (9) and (10) imply

(11) $$y = x \cap (y \cup t).$$

Relations (11), (8) and (10) show that $x \cap (y \cup t) \prec x, y \cup t$. Hence, by C_2, we obtain

$$x, y \cup t \prec x \cup (y \cup t) = x \cup t,$$

completing the proof.

Lemma 5. *If* C_2, *then* J_2.

Proof. Let L be a lattice satisfying C_2, and let

$$a = x_0 \prec x_1 \prec \ldots \prec x_n = b, \tag{12}$$

$$a = y_0 \prec y_1 \prec \ldots \prec y_m = b \tag{13}$$

be two finite maximal chains, having the same end elements; we must prove that $n=m$.

If $n=1$, then $a \prec b$, hence $m=1$. We suppose the assertion true for $n-1$ and we shall prove it for n.

If $x_1 = y_1$, the maximal chains $x_1 \prec x_2 \prec \ldots \prec x_n = b$ and $x_1 = y_1 \prec y_2 \prec \ldots$ $\ldots \prec y_m = b$ have the same length, according to the inductive hypothesis, that is, $n-1 = m-1$.

If $x_1 \neq y_1$, then $x_1 \cap y_1 = a \prec x_1, y_1$. By C_2, we have

$$x_1, y_1 \prec x_1 \cup y_1 = u_1. \tag{14}$$

If there exists a finite maximal chain between a_1 and b, then the equality $n=m$ is immediately deduced, as in [4], Lemma 1, pp. 64—65. But such a chain can be constructed as follows.

We define

$$u_i = \begin{cases} x_i \cup u_{i-1}, & \text{if } x_i \neq u_{i-1} \\ x_{i+1}, & \text{if } x_i = u_{i-1} \end{cases} \qquad (i=2, 3, \ldots, n-1); \tag{15}$$

and we remark that

$$x_i, u_{i-1} \prec u_i \qquad (i=2, \ldots, n-1). \tag{16}$$

Indeed, if $x_2 \neq u_1$, then $x_2 \cap u_1 = x_1 \prec x_2, u_1$ and by C_2, $x_2, u_1 \prec x_2 \cup u_1 = u_2$. If $x_2 = u_1$, then $u_2 = x_3 \succ x_2 = u_1$. The proof of (16) is easily completed by recurrence.

Now, the maximal chain between u_1 and b will be $u_1 \prec u_2 \prec \ldots \prec u_{n-1}$; it is sufficient to prove that $u_{n-1} = b_n$.

If $x_{n-1} = u_{n-2}$, then $u_{n-1} = x_n = b$. If $x_{n-1} \neq u_{n-2}$, then $x_{n-1} \leqq x_{n-1} \cup u_{n-2} = u_{n-1} \leqq b$. But $x_{n-1} \prec b$ and $x_{n-1} \neq x_{n-1} \cup u_{n-2}$ (otherwise, $x_{n-2} \prec u_{n-2} \leqq x_{n-1}$ and $x_{n-2} \prec x_{n-1}$ would imply $u_{n-2} = x_{n-1}$, a contradiction), therefore $u_{n-1} = b$, completing the proof.

§ 2

In the sequel, for every $A, B \in \Sigma$ we shall denote by AB and $A \vee B$ the logical conjunction and disjunction, respectively, of A, B and by $\bar{A}$ the negation of A. The complete existential theory of Σ will be made by means of Venn diagrams. This is possible on account on the equivalence between the implication

$$A \Rightarrow B \tag{17}$$

and the relation

(18) $$A\bar{B}=0.$$

The last equality means that, when considering a Venn diagram, the region corresponding to property $A\bar{B}$ (A and not B) is void. On the other hand, a relation of the form

(19) $$AB=0$$

(expressed in terms of Venn diagrams) is equivalent to the implication

(20) $$A \Rightarrow \bar{B},$$

if we take into account that

(21) $$\bar{\bar{A}} = A.$$

Thus, the problem of studying the complete existential theory of Σ, is equivalent to the following:*)

Consider the Venn diagram of the 9 properties belonging to Σ; there are 2^9 possible elementary regions**). Determine which of these regions are void, and which are not.

To solve this problem, we notice first that

(22) $$\mathrm{M} \Rightarrow \mathrm{S}, \quad \text{but} \quad \mathrm{S} \not\Rightarrow \mathrm{M},$$

(23) $$\mathrm{F}_1 \Rightarrow \mathrm{F}_2 \Rightarrow \mathrm{F}_3, \quad \text{but} \quad \mathrm{F}_3 \not\Rightarrow \mathrm{F}_2 \not\Rightarrow \mathrm{F}_1;\ \mathrm{F}_3 \not\Rightarrow \mathrm{F}_1,$$

(24) $$\mathrm{J}_1 \Rightarrow \mathrm{J}_2, \quad \text{but} \quad \mathrm{J}_2 \not\Rightarrow \mathrm{J}_1,$$

(25) $$\mathrm{C}_1 \Rightarrow \mathrm{C}_2, \quad \text{but} \quad \mathrm{C}_2 \not\Rightarrow \mathrm{C}_1,$$

(see, for instance, [1], [4], or [11]). This means that the non-void elementary regions of the system $\Sigma' = \{\mathrm{M}, \mathrm{S}\}$ are

(26) $$\mathrm{MS} = \mathrm{M},\ \overline{\mathrm{M}}\mathrm{S},\ \overline{\mathrm{M}}\,\overline{\mathrm{S}} = \overline{\mathrm{S}},$$

and the non-void elementary regions of the system $\Sigma'' = \{\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3\}$ are

(27) $$\mathrm{F}_1\mathrm{F}_2\mathrm{F}_3 = \mathrm{F}_1,\ \overline{\mathrm{F}}_1\mathrm{F}_2\mathrm{F}_3 = \overline{\mathrm{F}}_1\mathrm{F}_2,\ \overline{\mathrm{F}}_1\overline{\mathrm{F}}_2\mathrm{F}_3 = \overline{\mathrm{F}}_2\mathrm{F}_3,\ \overline{\mathrm{F}}_1\overline{\mathrm{F}}_2\overline{\mathrm{F}}_3 = \overline{\mathrm{F}}_3.$$

Now, the elementary regions of the system

(28) $$\Sigma_1 = \Sigma' \cup \Sigma'' = \{\mathrm{M}, \mathrm{S}, \mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3\}$$

are obtained by forming the meet of each region (26) with each region (27).

But $\mathrm{F}_3\mathrm{C}_1 \Rightarrow \mathrm{F}_2\mathrm{J}_1 \Rightarrow \mathrm{F}_2$, by Lemma 1, and

(29) $$\mathrm{S} \Rightarrow \mathrm{C}_1,$$

(see, for instance, [11], p. 157, Theorem 50), therefore $\mathrm{F}_3\mathrm{S} \Rightarrow \mathrm{F}_2$, hence $\overline{\mathrm{F}}_2\mathrm{F}_3\mathrm{S}=0$.

*) For a detailed proof of this equivalence, see, for instance, [7].

**) An elementary region is a region corresponding to a complete elementary conjunction of $\mathrm{M}, \mathrm{S}, \ldots, \mathrm{C}_2$, for instance $\mathrm{M}\overline{\mathrm{S}}\overline{\mathrm{F}}_1\mathrm{F}_2\overline{\mathrm{F}}_3\mathrm{J}_1\overline{\mathrm{J}}_2\overline{\mathrm{C}}_1\mathrm{C}_2$.

The other meets of the regions (26) and (27), indicated in Fig. I, are non-void. More precisely, the fact that the

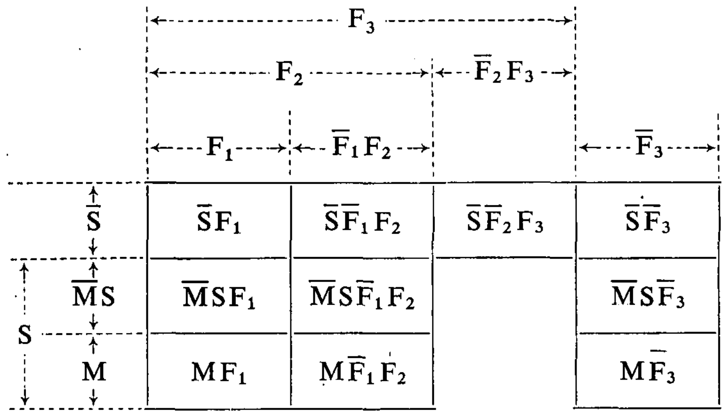

Fig. I

regions MF_1, $M\bar{F}_1F_2$, $M\bar{F}_3$, $\bar{M}S F_1$, $\bar{M}S\bar{F}_1F_2$, $\bar{M}S\bar{F}_3$, $\bar{S}\bar{F}_2F_3$ are non-void, is shown respectively by the lattices in Fig. 1, 2, ..., 6, 11 (the lattice in Fig. 2, which appears also as a sublattice in Fig. 3, 5, 6, 9, 10, 13, 14, 16, is a lattice isomorphic to the chain of natural numbers). The fact that the regions $\bar{S}F_1$, $\bar{S}\bar{F}_1F_2$ are non-void, is shown by any of the lattices in Fig. 7, 8, respectively by any of the lattices in Fig. 9, 10. Finally, the fact that the region $\bar{S}\bar{F}_3$ is non-void, is shown by any of the lattices in Fig. 12, ..., 17 (in Fig. 12, 13, 15 the dual of the lattice in Fig. 2 appears as a sublattice). Thus we have proved

Lemma 6. *The complete existential theory of system Σ_1 is shown in Fig. I* (the only non-void elementary regions are shown in Fig. I).

Now, we shall study the complete existential theory of the system

(30) $$\Sigma_2 = \Sigma_1 \cup \{J_1, J_2\} = \{M, S, F_1, F_2, F_3, J_1, J_2\}.$$

It follows from (24) that the non-void elementary regions of the system $\{J_1, J_2\}$ are

(31) $$J_1 J_2 = J_1,\ \bar{J}_1 J_2,\ \bar{J}_1 \bar{J}_2 = \bar{J}_2,$$

hence the elementary regions of Σ_2 are obtained by forming the meet of each region shown in Fig. I, with each region (31). We shall prove that some of these meets are void.

First, we have $S \Rightarrow J_1$ by Lemma 2 and (29); hence all Σ_2 elementary regions included in S satisfy J_1.

Further, it is evident that $F_3J_2 \Rightarrow J_1$, hence $F_3\bar{J}_1J_2 = 0$ and a fortiori

(32) $$\bar{S}\bar{F}_2F_3\bar{J}_1J_2 = 0;$$

by Lemma 3 we have also $\overline{F}_2F_3J_1=0$, hence a fortiori

$$\overline{S}\overline{F}_2F_3J_1=0. \tag{33}$$

Finally, $F_2J_2 \Rightarrow F_3J_2 \Rightarrow J_1$, hence all Σ_2 elementary regions included in F_2, that is in F_1 or in $\overline{F}_1F_2$, satisfy J_1.

The elementary regions of Σ_2, indicated in Fig. II, are non-void. To prove

<table>
<tr><td rowspan="2">$\overline{S}F_1\overline{J}_2$</td><td rowspan="2">$\overline{S}\overline{F}_1F_2\overline{J}_2$</td><td rowspan="4">$\overline{S}\overline{F}_2F_3\overline{J}_2$</td><td>$\overline{S}\overline{F}_3J_2$</td></tr>
<tr><td rowspan="2">$\overline{S}\overline{F}_3\overline{J}_1J_2$</td></tr>
<tr><td rowspan="2">$\overline{S}F_1J_1$</td><td rowspan="2">$\overline{S}\overline{F}_1F_2J_1$</td></tr>
<tr><td>$\overline{S}\overline{F}_3J_1$</td></tr>
<tr><td>$\overline{M}SF_1J_1$</td><td>$\overline{M}S\overline{F}_1F_2J_1$</td><td rowspan="2"></td><td>$\overline{M}S\overline{F}_3J_1$</td></tr>
<tr><td>MF_1J_1</td><td>$M\overline{F}_1F_2J_1$</td><td>$M\overline{F}_3J_1$</td></tr>
</table>

Fig. II

this assertion, we compare Fig. II to Fig. I and we remark it is sufficient to prove that the regions $\overline{S}F_1J_1$, $\overline{S}F_1\overline{J}_2$, $\overline{S}\overline{F}_1F_2J_1$, $\overline{S}\overline{F}_1F_2\overline{J}_2$, $\overline{S}\overline{F}_3J_1$, $\overline{S}\overline{F}_3\overline{J}_1J_2$, $\overline{S}\overline{F}_3\overline{J}_2$ are non-void. But the fact that the first four regions are non-void, is shown by the lattices in Fig. 7, 8, 9, 10. The fact that the regions $\overline{S}\overline{F}_3J_1$ and $\overline{S}\overline{F}_3\overline{J}_1J_2$ are non-void, is shown by any of the lattices in Fig. 12, 13, 14, respectively by any of the lattices in Fig. 15, 16. Finally the fact that the region $\overline{S}\overline{F}_3\overline{J}_2$ is non-void, is shown by the lattice in Fig. 17. We have thus examined all elementary regions of Σ_2, proving the following

Lemma 7. *The complete existential theory of system Σ_2 is shown in Fig. II.*

Now, we are in the position to study the complete existential theory of the system

$$\Sigma=\Sigma_2\cup\{C_1, C_2\}=\{M, S, F_1, F_2, F_3, J_1, J_2, C_1, C_2\}. \tag{34}$$

It follows from (25) that the non-void elementary regions of the system $\{C_1, C_2\}$ are

$$C_1C_2 = C_1,\ \overline{C}_1C_2,\ \overline{C}_1\overline{C}_2 = \overline{C}_2, \tag{35}$$

hence the elementary regions of Σ are obtained by forming the meet of each region shown in Fig. II, with each region (35). We shall prove that some of these meets are void.

First, relation (29) shows that every Σ elementary region, included in S, satisfies C_1.

Further, by Lemma 4, $\overline{S}F_2C_2=0$, hence every Σ elementary region, included in $\overline{S}F_2$, that is in $\overline{S}F_1$ or in $\overline{S}\overline{F}_1F_2$, satisfies $\overline{C}_2$.

But $\bar{J}_2C_1 \vee \bar{J}_2\bar{C}_1C_2 = \bar{J}_2(C_1 \vee \bar{C}_1C_2) = \bar{J}_2C_2 = 0$, by (35) and Lemma 5, hence a fortiori

$$\bar{S}\bar{F}_2F_3\bar{J}_2C_1 = \bar{S}\bar{F}_2F_3\bar{J}_2\bar{C}_1C_2 = \bar{S}\bar{F}_3\bar{J}_2C_1 = \bar{S}\bar{F}_3\bar{J}_2\bar{C}_1C_2 = 0. \tag{36}$$

Finally, $\bar{J}_1C_1=0$, by Lemma 2, hence

$$\bar{S}\bar{F}_3J_1J_2C_1=0. \tag{37}$$

The elementary regions of Σ, shown in Fig. III, are non-void. To prove this assertion, we compare Fig. III and Fig. II, and we remark that it is sufficient to prove

<table>
<tr><td rowspan="2">$\bar{S}F_1\bar{J}_2\bar{C}_2$</td><td rowspan="2">$\bar{S}\bar{F}_1F_2\bar{J}_2\bar{C}_2$</td><td rowspan="3">$\bar{S}\bar{F}_2F_3\bar{J}_2\bar{C}_2$</td><td colspan="6">$\bar{S}\bar{F}_3\bar{J}_2\bar{C}_2$</td></tr>
<tr><td colspan="3">$\bar{S}\bar{F}_3\bar{J}_1J_2\bar{C}_1C_2$</td><td colspan="3">$\bar{S}\bar{F}_3\bar{J}_1J_2\bar{C}_2$</td></tr>
<tr><td>$\bar{S}F_1J_1\bar{C}_2$</td><td>$\bar{S}\bar{F}_1F_2J_1\bar{C}_2$</td><td colspan="2">$\bar{S}\bar{F}_3J_1C_1$</td><td colspan="2">$\bar{S}\bar{F}_3J_1\bar{C}_1C_2$</td><td colspan="2">$S\bar{F}_3J_1\bar{C}_2$</td></tr>
<tr><td>$\bar{M}SF_1J_1C_1$</td><td>$\bar{M}S\bar{F}_1F_2J_1C_1$</td><td rowspan="2"></td><td colspan="6">$\bar{M}S\bar{F}_3J_1C_1$</td></tr>
<tr><td>$MF_1J_1C_1$</td><td>$M\bar{F}_1F_2J_1C_1$</td><td colspan="6">$M\bar{F}_3J_1C_1$</td></tr>
</table>

Fig. III

that the regions $\bar{S}\bar{F}_3J_1C_1$, $\bar{S}\bar{F}_3J_1\bar{C}_1C_2$, $\bar{S}\bar{F}_3J_1\bar{C}_2$, $\bar{S}\bar{F}_3\bar{J}_1J_2\bar{C}_1C_2$, $\bar{S}\bar{F}_3\bar{J}_1J_2\bar{C}_2$ are non-void. But this is shown by the lattices in Fig. 12, ..., 16, respectively. We have thus examined all elementary regions of Σ, proving the following

Theorem. *The complete existential theory of system Σ is shown in Fig. III.*

In other words, *the non-void elementary regions of Σ are those shown in Fig. III.* The fact that the regions indicated in Fig. III are elementary, is a consequence of the following relations: $M\bar{S}=0$, $F_1\bar{F}_2F_3=F_1\bar{F}_2\bar{F}_3=F_1F_2\bar{F}_3=\bar{F}_1F_2F_3=0$, $J_1\bar{J}_2=0$, $C_1\bar{C}_2=0$ (equivalent to (22), (23), (24), (25) respectively) and (26), (27), (31), (35).

As an application of the above theorem, let us decide whether the relation $\bar{S}\bar{F}_2\bar{J}_2C_1=0$ (that is $C_1 \Rightarrow S \vee F_2 \vee J_2$, or $\bar{S}\bar{J}_2 \Rightarrow F_2 \vee \bar{C}_1$, a. s. o.) is true or not. But $\bar{S}\bar{F}_2\bar{J}_2C_1=\bar{S}(\bar{F}_2F_3 \vee \bar{F}_3)\bar{J}_2C_1=\bar{S}\bar{F}_2F_3\bar{J}_2C_1 \vee \bar{S}\bar{F}_3\bar{J}_2C_1=0$ thus the above implication is true.

Conclusions. The present study could be continued, by adding other axioms to Σ, for instance the converses of C_1 and C_2, or some conditions involving the notion of a *dimension function* (see [11]).

Note added in proof. A simple inspection of Fig. III shows that $F_3\bar{C}_1C_2=0$ (or $F_3C_2 \Rightarrow C_1$). This assertion is equivalent to the dual of Theorem 3.3 in Kořínek's paper „Lattices in which the theorem of Jordan – Hölder is generally true", *Třídy České Acad.* **59**, No. 23 (1949).

Fig. 1

Fig. 2

Fig. 3

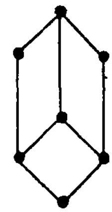

Fig. 4

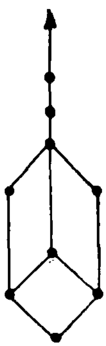

Fig. 5

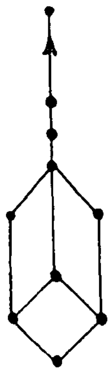

Fig. 6

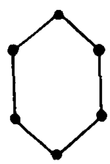

Fig. 7

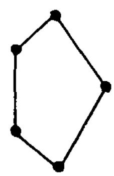

Fig. 8

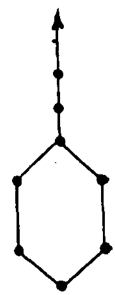

Fig. 9

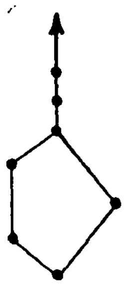

Fig. 10

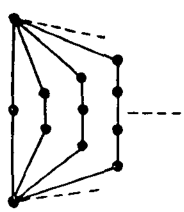

Fig. 11

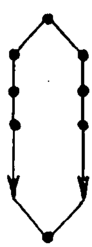

Fig. 12

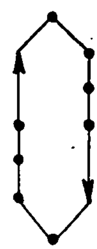

Fig. 13

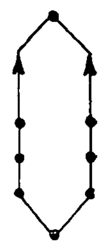

Fig. 14

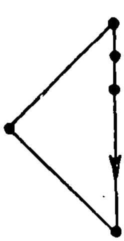

Fig. 15

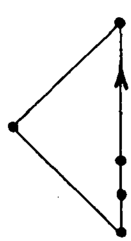

Fig. 16

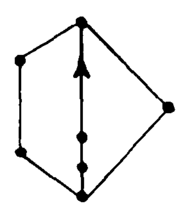

Fig. 17

References

[1] G. Birkhoff, *Lattice Theory* (New York, 1948).
[2] A. H. Diamond, The complete existential theory of the Whitehead—Huntington set of postulates for the algebra of logic, *Trans. Amer. Math. Soc.*, **35**, (1933) 940—948; correct. *ibid.*, (1934), 893.
[3] L. L. Dines, Complete existential theory of Sheffer's postulates for Boolean algebras, *Bull. Amer. Math. Soc.*, **21** (1914/15), 183—188.
[4] M. L. Dubreil—Jacotin, L. Lesieur, R. Croisot, Leçons sur la théorie des treillis, des structures algébriques ordonnées et des treillis géométriques (Paris, 1953).
[5] E. V. Huntington, Complete existential theory of the postulates for serial order, *Bull. Amer. Math. Soc.*, **23** (1917), 276—280.
[6] E. V. Huntington, Sets of completely independent postulates for cyclic order, *Proc. Nat. Acad. Sci. USA*, **10** (1924), 74—78.
[7] E. V. Huntington, A new set of postulates for betweenness, with a proof of complete independence, *Trans. Amer. Math. Soc.*, **26** (1924), 257—282.
[8] H. M. MacNeille, Partially ordered sets, *Trans. Amer. Math. Soc.*, **42** (1937), 416—460.
[9] E. H. Moore, *Introduction to a form of general analysis*. New Haven Colloquium, 1906. (New Haven, 1910).
[10] I. Rosenbaum, A new system of completely independent postulates for betweenness, *Bull. Amer. Math. Soc.*, **57** (1951), 279.
[11] G. Szász, *Einführung in die Verbandstheorie* (Budapest, 1962).
[12] J. S. Taylor, Complete existential theory of Bernstein's set of four postulates for Boolean algebras, *Ann. Math.*, **19** (1917), 64—69.
[13] J. S. Taylor, Sheffer's set of five independent postulates for Boolean algebras in terms of the operation „rejection" made completely independent, *Bull. Amer. Math. Soc.*, **26** (1920), 449—454.
[14] W. E. van de Walle, On the complete independence of the postulates for betweenness. *Trans. Amer. Math. Soc.*, **26** (1924), 249—256.

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