On an interpolation theorem of Foiaş and Lions

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Introduction

Let X be a locally compact space provided with a positive measure μ. We denote by $L_\zeta^p(E)$, where $1 \leqq p \leqq \infty$ and ζ is *a* positive μ-measurable function and E a Banach space (or, more generally, a field of Banach spaces over X; we do not consider this generalization here in order not to complicate the notation), the space of μ-measurable functions a with values in E such that $\|\zeta a\|_E$ is of μ-integrable p th power (if $p<\infty$) or μ-bounded (if $p=\infty$). We provide $L_\zeta^p(E)$ with the norm

$$\|a\|_{L_\zeta^p(E)} = \begin{cases} \left(\int\limits_X \|\zeta a\|_E^p \, d\mu\right)^{1/p} & (\text{if } p<\infty) \\ \mu\text{-}\sup\limits_X \|\zeta a\|_E & (\text{if } p=\infty). \end{cases} \tag{1}$$

A function $H(z_0, z_1)$ defined, measurable and positive for $z_0 \geqq 0$, $z_1 \geqq 0$ is said to be an *interpolation function of power p* if and only if whenever π is a linear mapping from some space, containing $L_{\zeta_0}^p(E)$ and $L_{\zeta_1}^p(E)$ as linear subspaces, into itself such that the restriction of π to $L_{\zeta_i}(E)$ maps $L_{\zeta_i}^p(E)$ continuously into itself ($i=0, 1$) then the restriction of π to $L_{H(\zeta_0,\zeta_1)}^p(E)$ maps $L_{H(\zeta_0,\zeta_1)}^p(E)$ continuously into itself. E. g. $z_0^{1-\theta} z_1^\theta$ with $0<\theta<1$ is an interpolation function of power p for any p (see example 2). In [1] FOIAŞ and LIONS found a sufficient condition for a function to be an interpolation function of power p (in the above terminology). In the present note we give two constructions of interpolation functions of power p. in a sense dual to each other The first of these constructions leads to a condition essentially the one of FOIAŞ and LIONS (see remark 2) while the second leads to a condition in a sense dual to the first one. It is also shown that under some auxiliary restrictions both constructions are equivalent. In particular this leads to a simple condition which is independent of p (see theorem 4).

The general ideas underlying these results were briefly discussed in [2] (cf. also [3]).

§ 1

Let us set

(2) $$J(t, a) = (\|a\|^p_{L^p_{\xi_0}(E)} + t^p \|a\|^p_{L^p_{\xi_1}(E)})^{1/p}, \quad a \in L^p_{\xi_0}(E) \cap L^p_{\xi_1}(E), \quad 0 < t < \infty,$$

and

(3) $$K(t, a) = \inf_{a = a_0 + a_1} (\|a_0\|^p_{L^p_{\xi_0}(E)} + t^p \|a_1\|^p_{L^p_{\xi_1}(E)})^{1/p}, \quad a \in L^p_{\xi_0}(E) + L^p_{\xi_1}(E), \quad 0 < t < \infty.$$

Let $\alpha = \alpha(t)$ and $\beta = \beta(t)$ $(0 < t < \infty)$ be two positive functions measurable with respect to dt/t.

We denote by S_α the space of elements $a \in L^p_{\xi_0}(E) + L^p_{\xi_1}(E)$ such that there exists a function $u = u(t)$ $(0 < t < \infty)$ measurable with respect to dt/t with values in $L^p_{\xi_0}(E) \cap L^p_{\xi_1}(E)$ such that

(4) $$a = \int_0^\infty u(t) \frac{dt}{t} \text{(in } L^p_{\xi_0}(E) + L^p_{\xi_1}(E)), \quad \alpha(t) J(t, u(t)) \in L^p_*,$$

and by T_β the space of elements $a \in L^p_{\xi_0}(E) + L^p_{\xi_1}(E)$ such that

(5) $$\beta(t) K(t, a) \in L^p_*.$$

(L^p_* denotes L^p with respect to the measure dt/t.) We provide S_α with the norm

(6) $$\|a\|_{S_\alpha} = \inf \|\alpha(t) J(t, u(t))\|_{L^p_*}, \quad a = \int_0^\infty u(t) \frac{dt}{t}$$

and T_β with the norm

(7) $$\|a\|_{T_\beta} = \|\beta(t) K(t, a)\|_{L^p_*}.$$

Theorem 1. *Each of the spaces S_α and T_β is an interpolation space with respect to $L^p_{\xi_0}(E)$ and $L^p_{\xi_1}(E)$; i. e. whenever π is a linear mapping from some space, containing $L^p_{\xi_0}(E)$ and $L^p_{\xi_1}(E)$ as linear subspaces, into itself such that the restriction of π to $L^p_{\xi_i}(E)$ maps $L^p_{\xi_i}(E)$ continuously into itself $(i = 0, 1)$ then the restriction of π to S_α or T_β maps S_α or T_β continuously into itself. Moreover, if*

(8) $$\|\pi a\|_{L^p_{\xi_i}(E)} \leqq M_i \|a\|_{L^p_{\xi_i}(E)}, \quad a \in L^p_{\xi_i}(E) \qquad (i = 0, 1),$$

where M_0 and M_1 are positive constants, then

(9) $$\|\pi a\| \leqq M \|a\|, \quad a \in S_\alpha \text{ or } T_\beta$$

with $\| \; \| = \| \; \|_{S_\alpha}$ or $\| \; \|_{T_\beta}$, where M is a constant that depends only upon M_0 and M_1.

Proof. i) We have

$$J(t, \pi a) \leqq (M_0^p \|a\|^p_{L^p_{\xi_0}(E)} + t^p M_1^p \|a\|^p_{L^p_{\xi_1}(E)})^{1/p} \leqq \max(M_0, M_1) J(t, a).$$

Since

$$\pi a = \int_0^\infty \pi u(t) \frac{dt}{t}$$

we therefore get

$$\|\pi a\|_{S_\alpha} \leqq \max(M_0, M_1) \|\alpha(t) J(t, u(t))\|_{L^p_*}$$

and, by making vary u, (9) follows in this case, with $M = \max(M_0, M_1)$.

ii) We have

$$K(t, \pi a) \leqq (M_0^p \|a_0\|^p_{L^p_{\zeta_0}(E)} + t^p M_1^p \|a_1\|^p_{L^p_{\zeta_1}(E)})^{1/p}.$$

Making vary a_0 and a_1 we get

$$K(t, \pi a) \leqq \max(M_0, M_1)\, K(t, a).$$

Therefore (9) follows in this case, again with $M = \max(M_0, M_1)$.

Remark 1. If α and β satisfy inequalities of the form

$$\alpha(st) \leqq \varrho(s) \alpha(t), \quad \beta(st) \leqq \sigma(s) \beta(t) \tag{10}$$

we may replace $M = \max(M_0, M_1)$ by $M = M_0 \varrho\left(\frac{M_0}{M_1}\right)$ $M = M_0 \sigma\left(\frac{M_0}{M_1}\right)$ (cf. [3]). In particular if $\varrho(s) = \sigma(s) = s^{-\theta}$ we get $M = M_0^{1-\theta} M_1^\theta$.

Theorem 2. *We have* $S_\alpha = L^p_{F(\zeta_0, \zeta_1)}(E)$ *and* $T_\beta = L^p_{G(\zeta_0, \zeta_1)}(E)$ *with equality of norms, where* $\left(\frac{1}{q} = 1 - \frac{1}{p}\right)$

$$F(z_0, z_1) = \left(\int_0^\infty (z_0^p + t^p z_1^p)^{-(q/p)} (\alpha(t))^{-q} \frac{dt}{t}\right)^{-(1/q)} \tag{11}$$

and

$$G(z_0, z_1) = \left(\int_0^\infty (z_0^{-q} + t^{-q} z_1^{-q})^{-(p/q)} (\beta(t))^p \frac{dt}{t}\right)^{1/p}. \tag{12}$$

Example 1. If $\alpha(t) = \beta(t) = t^{-\theta}$ $(0 < \theta < 1)$ we get $F(z_0, z_1) = c z_0^{1-\theta} z_1^\theta$, $G(z_0, z_1) = d z_0^{1-\theta} z_1$ where c and d are constants.

Proof. i) We have to minimalize the expression

$$\mathfrak{J}_u = \int_0^\infty (\|u(t)\|^p_{L^p_{\zeta_0}(E)} + t^p \|u(t)\|^p_{L^p_{\zeta_1}(E)}) (\alpha(t))^p \frac{dt}{t} =$$

$$= \int_0^\infty \int_X (\zeta_0^p + t^p \zeta_1^p) \|u(t)\|^p_E \, d\mu' (\alpha(t))^p \frac{dt}{t}$$

where $a = \int_0^\infty u(t)\, dt/t$. We claim that it is sufficient to consider $u(t)$ of the form $\varphi(t) a$ with $\int_0^\infty \varphi(t)\, dt/t = 1$, $\varphi(t) \geqq 0$. Indeed given any $u(t)$ let us set

$$\varphi(t) = \frac{\|u(t)\|_E}{\int_0^\infty \|u(t)\|_E\, dt/t}.$$

Then $\int_0^\infty \varphi(t)\,dt/t = 1$, $\varphi(t) \geqq 0$ and moreover $\|\varphi(t)a\|_E \leqq \|u(t)\|_E$ so that $\mathfrak{J}_{\varphi a} \leqq \mathfrak{J}_u$, which proves the assertion. Thus restricting ourselves to the case $u(t) = \varphi(t)a$ we obtain after a change of the order of integration

$$\mathfrak{J}_u = \int_X \int_0^\infty (\varphi(t))^p (\zeta_0^p + t^p \zeta_1^p)(\alpha(t))^p \frac{dt}{t} \|a\|_E^p \, d\mu.$$

The problem is now reduced to minimizing (for each $x \in X$) the expression

$$\mathscr{E}_\varphi = \int_0^\infty (\varphi(t))^p (\zeta_0^p + t^p \zeta_1^p)(\alpha(t))^p \frac{dt}{t}$$

where $\int_0^\infty \varphi(t)\,dt/t = 1$, $\varphi(t) \geqq 0$. Choose

$$\varphi(t) = (F(\zeta_0, \zeta_1))^q (\zeta_0^p + t^p \zeta_1^p)^{-(q/p)} (\alpha(t))^{-q};$$

then

$$\mathscr{E}_\varphi = \int_0^\infty (F(\zeta_0, \zeta_1))^{qp} (\zeta_0^p + k^p \zeta_1^p)^{1-q} (\alpha(t))^{-qp+p} \frac{dt}{t} = (F(\zeta_0, \zeta_1))^{qp-q} = (F(\zeta_0, \zeta_1))^p$$

so that

$$\min \mathscr{E}_\varphi \leqq (F(\zeta_0, \zeta_1))^p.$$

On the other hand, using HÖLDER's inequality

$$(F(\zeta_0, \zeta_1))^p = (F(\zeta_0, \zeta_1))^p \left(\int_0^\infty \varphi(t)(\zeta_0^p + t^p \zeta_1^p)^{1/p} \alpha(t) (\zeta_0^p + t^p \zeta_1^p)^{-(1/p)} (\alpha(t))^{-1} \frac{dt}{t} \right)^p \leqq$$

$$\leqq (F(\zeta_0, \zeta_1))^p \int_0^\infty (\varphi(t))^p (\zeta_0^p + t^p \zeta_1^p)(\alpha(t))^p \frac{dt}{t} (F(\zeta_0, \zeta_1))^{-p} = \mathscr{E}_\varphi,$$

which finishes the proof.

ii) We have to minimize the expression

$$\mathcal{K}_{v_0, v_1} = \int_0^\infty \left(\|v_0(t)\|^p_{L^p_{\zeta_0}(E)} + t^p \|v_1(t)\|^p_{L^p_{\zeta_1}(E)} \right) (\beta(t))^p \frac{dt}{t} =$$

$$= \int_0^\infty \int_X \left(\zeta_0^p \|v_0(t)\|_E^p + t^p \zeta_1^p \|v_1(t)\|_E^p \right) d\mu (\beta(t))^p \frac{dt}{t}$$

where $a = v_0(t) + v_1(t)$. We claim that it is sufficient to consider $v_0(t)$ and $v_1(t)$ of the form $\psi_0(t)a$ and $\psi_1(t)a$ with $\psi_0(t) + \psi_1(t) = 1$, $\psi_0(t) \geqq 0$, $\psi_1(t) \geqq 0$. Indeed given $v_0(t)$ and $v_1(t)$ let us set

$$\psi_0(t) = \frac{\|v_0(t)\|_E}{\|v_0(t)\|_E + \|v_1(t)\|_E}, \quad \psi_1(t) = \frac{\|v_1(t)\|_E}{\|v_0(t)\|_E + \|v_1(t)\|_E}.$$

Then $\psi_0(t)+\psi_1(t) = 1, \psi_0(t) \geqq 0, \psi_1(t) \geqq 0$ and moreover $\|\psi_0(t)a\|_E \leqq \|v_0(t)\|_E$, $\|\psi_1(t)a\|_E \leqq \|v_1(t)\|_E$ so that $\mathcal{K}_{\psi_0 a, \psi_1 a} \leqq \mathcal{K}_{v_0, v_1}$ which proves the assertion. Thus restricting ourselves to the case $v_0(t) = \psi_0(t)a$ and $v_1(t) = \psi_1(t)a$ we obtain after a change of the order of integration

$$\mathcal{K}_{v_0, v_1} = \int_X \int_0^\infty ((\psi_0(t))^p \zeta_0^p + (\psi_1(t))^p t^p \zeta_1^p)(\beta(t))^p \frac{dt}{t} \|a\|_E^p \, d\mu.$$

The problem is now reduced to minimizing (for each $x \in X$) the expression

$$\int_0^\infty ((\psi_0(t))^p \zeta_0^p + (\psi_1(t))^p t^p \zeta_1^p)(\beta(t))^p \frac{dt}{t}$$

where $\psi_0(t)+\psi_1(t) = 1, \psi_0(t) \geqq 0, \psi_1(t) \geqq 0$ from which the result in this case (see (12)) easily follows as in the preceding case. Combining theorem 1 and theorem 2 we get

Theorem 3. *Each of the functions $F(z_0, z_1)$ and $G(z_0, z_1)$ as defined by* (11) *and* (12) *is an interpolation function of power p.*

Example 2. By example 1, $z_0^{1-\theta} z_1^\theta$ with $0 < \theta < 1$ is thus an interpolation function of power p for any p. This leads to the interpolation theorem of Stein and Weiss [4].

Remark 2. It is easily seen that the condition provided by (11) is essentially the one found by Foiaş and Lions [1]. The only significant difference is that these authors allow $(\alpha(t)^{-q} dt/t$ to be replaced by an arbitrary positive measure $d\xi$ (not necessarily absolutely continuous with respect to dt/t). It should be possible to extend our approach to cover this generalization too.

§ 2.

We conclude by pointing out some relations between the functions $F(z_0, z_1)$ and $G(z_0, z_1)$. Since they are both homogeneous of degree 1 it suffices to consider the functions $f(z) = F(z, 1)$ and $g(z) = G(z, 1)$. We have then after a change of variable

$$f(z) = z \left(\int_0^\infty (1+t^p)^{-(q/p)} (\alpha(tz))^{-q} \frac{dt}{t} \right)^{-(1/q)} \tag{13}$$

and

$$g(z) = z \left(\int_0^\infty (1+t^{-q})^{-(p/q)} (\beta(tz))^p \frac{dt}{t} \right)^{1/p}. \tag{14}$$

Let us consider the special case $\alpha(t) = \beta(t)$. By Hölder's inequality we obtain

$$\int_0^\infty (1+t^p)^{-(1/p)} (1+t^{-q})^{-(1/q)} \frac{dt}{t} \leqq \frac{g(z)}{f(z)}$$

or

$$f(z) \leqq C g(z) \tag{15}$$

where C is a constant, $0<C<\infty$. Assume next that α satisfies (10) where

$$\int_0^\infty (1+t^p)^{-(q/p)}\left(\varrho\left(\frac{1}{t}\right)\right)^q \frac{dt}{t}<\infty. \tag{16}$$

Then we get

$$\int_0^\infty (1+t^p)^{-(q/p)}(\alpha(tz))^{-q}\frac{dt}{t} \leqq \int_0^\infty (1+t^p)^{-(q/p}\left(\varrho\left(\frac{1}{t}\right)\right)^q \frac{dt}{t}(\alpha(z))^{-q}$$

so that

$$Az\alpha(z)\leqq f(z) \tag{17}$$

where A is a constant, $0<A<\infty$. Assume again that β satisfies (10) where

$$\int_0^\infty (1+t^{-q})^{-(p/q)}(\sigma(t))^p \frac{dt}{t}<\infty. \tag{18}$$

Then we get

$$\int_0^\infty (1+t^{-q})^{-(p/q)}(\beta(tz))^p \frac{dt}{t} \leqq \int_0^\infty (1+t^{-q})^{-(p/q)}(\sigma(t))^p\frac{dt}{t}(\beta(z))^p$$

so that

$$g(z)\leqq Bz\beta(z) \tag{19}$$

where B is a constant, $0<B<\infty$. Therefore in the special case $\alpha(t)=\beta(t)$, $\varrho(t)=$ $=\sigma(t)$ assuming also (16) and (18) we get by (15)

$$Az\alpha(z)\leqq f(z)\leqq Cg(z)\leqq CBz\beta(z). \tag{20}$$

(Note that $A\leqq CB$!) In other words the functions $f(z)$, $g(z)$ and $z\alpha(z)$ are here equivalent.

Finally we make a few observations concerning the conditions (16) and (18). We note that, since all functions of the form $(1+t^p)^{1/p}$ are equivalent, they may be replaced by the conditions

$$\int_0^\infty \left(\min\left\{1,\frac{1}{t}\right\}\varrho\left(\frac{1}{t}\right)\right)^q \frac{dt}{t}<\infty \tag{21}$$

and, after a change of variable,

$$\int_0^\infty \left(\min\left\{1,\frac{1}{t}\right\}\sigma\left(\frac{1}{t}\right)\right)^p \frac{dt}{t}<\infty. \tag{22}$$

We note also that in view of Hölder's inequality if one of these conditions holds for two different values of the parameter, p or q, than it holds for all intermediate values.

We may sum up these results as follows.

Theorem 4. *Assume that* $\alpha(t)$ *satisfies* (10) *where*

$$\int_0^\infty \min\left\{1, \frac{1}{t}\right\} \varrho\left(\frac{1}{t}\right) \frac{dt}{t} < \infty, \quad \sup_t \min\left\{1, \frac{1}{t}\right\} \varrho\left(\frac{1}{t}\right) < \infty. \tag{23}$$

Then $z_0\alpha(z_0/z_1)$ *is equivalent to an interpolation function of power* p *for any* p.

Remark 3. Conditions of the type (23) arose in a similar context in [3].

References

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(Received August 20, 1963)