# On the numerical range of normal operators

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The *numerical range* of a (linear, bounded) operator $T$ in Hilbert space $\mathfrak{H}$ is defined as to be the set

$$W(T) = \{(T\varphi, \varphi) : \varphi \in \mathfrak{H}, \|\varphi\| = 1\}.$$

Theorem 1 below gives a characterisation of the set $W(T)$ in case $T$ is a normal operator. Using this characterisation, we shall be able to give, in Theorem 2, a negative answer, even for normal contractions[1]), to the following question of HALMOS [1]: Is the numerical range of each contraction the intersection of the numerical ranges of its unitary dilations[2])?

We shall denote by $\sigma(T)$, $\sigma_p(T)$, and $\sigma_c(T)$ the spectrum, the point spectrum, and the continuous spectrum, respectively, of a normal operator $T$. $I$ denotes the identity operator and $O$ the zero operator; $z$ denotes a complex variable; $\{z_0\}$ is the set consisting of the single complex number $z_0$.

Moreover, we shall use the following notations: For any set $\Lambda$ of points in the complex plane, $(\Lambda)^0$ is the set of the inner points of $\Lambda$, $\overline{\Lambda}$ is the closure of $\Lambda$, and $[\Lambda]^{\text{conv}}$ the (non necessarily closed) convex hull of $\Lambda$.

We start with the following

Lemma. *If $\Lambda$ is a convex set in the complex plane and $\mu$ is a non-negative Borel measure defined on $\Lambda$ such that $\mu(\Lambda) = 1$, then*

$$\int_\Lambda z\,d\mu \in \Lambda.$$

This is an obvious consequence of the theorem proved in [2].

Using this lemma we shall prove our

Theorem 1. *Let $E_T$ be the spectral measure corresponding to the normal operator $T$, and let $S$ be the family of all the convex Borelian sets $s$ in the complex plane, for which $E_T(s) = I$. Then*

$$W(T) = \bigcap_{s \in S} s.$$

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[1]) An operator $T$ is called a *contraction* if $\|T\| \leqq 1$.

[2]) The operator $T'$ in a space $\mathfrak{H}' \supset \mathfrak{H}$ is called a *dilation* of $T$, if $(T'\varphi_1, \varphi_2) = (T\varphi_1, \varphi_2)$ for every $\varphi_1, \varphi_2 \in \mathfrak{H}$. It follows that $W(T') \supset W(T)$.

Proof. Denote the intersection of all the $s \in S$ by $V(T)$. Consider a point $z_0 \in W(T)$, and a set $s \in S$. Then there exists a $\varphi \in \mathfrak{H}, \|\varphi\| = 1$, such that

$$z_0 = (T\varphi, \varphi) = \int_s z d(E_T\varphi, \varphi) = \int_s z d\|E_T\varphi\|^2, \quad \int_s d\|E_T\varphi\|^2 = \|\varphi\|^2 = 1;$$

consequently, by the lemma, $z_0 \in S$. This proves that

$$W(T) \subset V(T). \tag{1}$$

In order to prove the converse inclusion, we note first that[3])

$$[\sigma(T)]^{\mathrm{conv}} \subset \overline{W(T)}. \tag{2}$$

Indeed, let $z_0 \in \sigma(t)$ and $d$ an arbitrary disc with the centre $z_0$. If $\varphi$ is an element of the spectral subspace corresponding to $d$, $\|\varphi\| = 1$, then

$$(T\varphi, \varphi) = \int_d z d(E_T\varphi, \varphi) = \int_d z d\|E_T\varphi\|^2, \quad \int_d d\|E_T\varphi\|^2 = \|\varphi\|^2 = 1,$$

thus, by the lemma, we have $(T\varphi, \varphi) \in d$. This implies that $z_0$ is either an element, or an accumulation point of $W(T)$; consequently $\sigma(T) \subset \overline{W(T)}$. Since $W(T)$ is convex[4]), $[\sigma(T)]^{\mathrm{conv}} \subset \overline{W(T)}$ also.

Now let us consider a point $z_0 \in V(T)$. Since we have always $E_T(\sigma(T)) = I$ and *a fortiori* $E_T([\sigma(T)]^{\mathrm{conv}}) = I$, thus $[\sigma(T)]^{\mathrm{conv}}$ belongs to $S$. Consequently $z_0 \in [\sigma(T)]^{\mathrm{conv}}$; hence, by (2),

$$z_0 \in \overline{W(T)}. \tag{3}$$

We shall show that even $z_0 \in W(T)$. To do this, we have to distinguish several cases.

If $z_0 \in ([\sigma(T)]^{\mathrm{conv}})^0$ then, by (2), $z_0 \in (\overline{W(T)})^0$. Since $W(T)$ is convex, this implies that $z_0 \in W(T)$.

If $z_0$ is an extreme point[5]) of $[\sigma(T)]^{\mathrm{conv}}$ then $[\sigma(T)]^{\mathrm{conv}} - \{z_0\}$ also is convex, but does not belong to $S$, since it does not contain the point $z_0$, which, by hypothesis, belongs to every set in $S$. Thus $E_T(\{z_0\}) \neq O$, i. e. $z_0$ is a proper value of $T$, and so it necessarily belongs to $W(T)$.

It remains, by (3), to consider the case when $z_0$ is a point on the boundary of $[\sigma(T)]^{\mathrm{conv}}$ without being an extreme point. If $E_T(\{z_0\}) \neq O$, then $z_0$ is a proper value of $T$, thus $z_0 \in W(T)$. If $E_T(\{z_0\}) = O$, then consider the intersection $e$ of the set $[\sigma(T)]^{\mathrm{conv}}$ with its unique line of support through $z_0$. Deleting $z_0$ from $e$, we get two intervals, say $e_1$ and $e_2$. We have $E_T(e_i) \neq O$ $(i = 1, 2)$. Indeed: $E_T(e_1) = O$ would imply $E_T(e_1 \cup \{z_0\}) = O$, hence the set $\sigma_1 = \sigma(T) - (e_1 \cup \{z_0\})$ also would have $E_T$-measure $I$, and, since $\sigma_1$ is obviously convex, it would belong to $S$. Now, this is impossible, since $z_0$ does not belong to $\sigma_1$, although $z_0$ is, by hypothesis,

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[3]) $[\sigma(T)]^{\mathrm{conv}} = \overline{W(T)}$ holds also. See [3], p. 321.

[4]) The numerical range of every operator in convex. See [3], p. 131.

[5]) $z_0$ is called an *extreme point* of the convex set $A$, if $A$ has no line segment containing $z_0$ in its interior.

a point of $V(T)$. Thus, $E_T(e_1) \neq O$, and, by the same reason, $E_T(e_2) \neq O$. Choose vectors $\varphi_i$ $(i=1, 2)$, $\|\varphi_i\|=1$ from the spectral subspaces corresponding to the sets $e_i$. Then we have

$$(T\varphi_i, \varphi_i) = \int_{e_i} z d \|E\varphi_i\|^2,$$

hence, by the lemma, $(T\varphi_i, \varphi_i) \in e_i$. Thus $W(T)$ has points in $e_1$ and in $e_2$. By the convexity of $W(T)$, $z_0$ must then belong to $W(T)$, too.

So we have proved that every point $z_0$ of $V(T)$ belongs to $W(T)$, i. e.

$$V(T) \subset W(T). \tag{4}$$

(1) and (4) together prove the Theorem 1.

Theorem 2. *There exists a normal contraction $T$ for which the intersection of the numerical ranges of all the unitary dilations of $T$ contains $W(T)$ as a proper subset.*

Proof. Let $T$ be a normal operator with the following properties:

$$\sigma_c(T) = \{z: |z| = 1, \operatorname{Re} z \geqq 0\}, \quad \sigma_p(T) = \{0\}.$$

These properties imply that $T$ is a contraction. Consequently, $T$ has unitary dilations.

Such a $T$, is for example, the operator of multiplication by $z$ in the Hilbert space $L^2_\mu$ of functions $f(z)$ on the complex plane, $\mu$ being the measure, which is supported by the set $\alpha \cup \{0\}$, where

$$\alpha = \{z: |z| = 1, \operatorname{Re} z \geqq 0\},$$

is positive and, say, equidistributed on $\alpha$, and positive on $\{0\}$.

Consider a set $s \in S$. Since $s$ has full $E_T$-measure, the point 0, which has positive $E_T$-measure, necessarily belongs to $s$. Further, $s$ also includes the set

$$R = \{z: |z| < 1, \ \operatorname{Re} z > 0\}.$$

Indeed, in the contrary case the set $s$, being convex, would be disjoint from a whole subarc $\beta$ of $\alpha$; since $\beta$ has positive $E_T$-measure this would contradict the fact that $s$ is of full $E_T$-measure. Hence $s \supset R \cup \{0\}$. Since this is true for every set $s \in S$, we have

$$V(T) \supset R \cup \{0\}. \tag{5}$$

On the other hand, since $\overline{R}$ is convex and $E_T(\overline{R}) = I$, we have

$$\overline{R} \supset V(T). \tag{6}$$

If $z_0$ is an arbitrary point of $\alpha$, then $\overline{R} - \{z_0\} \in S$ also holds, thus $z_0 \notin V(T)$. This and (6) imply that

$$\overline{R} - \alpha \supset V(T). \tag{7}$$

The set $\{z: |z| \leqq 1, \ \operatorname{Re} z > 0\} \cup \{0\}$ is obviously convex, and its $E_T$-measure is $I$, consequently it belongs to $S$. This fact and (7) prove that

$$R \cup \{0\} \supset V(T). \tag{8}$$

(5) and (8) imply $V(T) = R \cup \{0\}$, hence, according to Theorem 1, we have

$$W(T) = R \cup \{0\}. \tag{9}$$

Now let $U$ be an arbitrary unitary dilation of $T$. Then we have[4]) $W(U) \supset W(T)$; thus, by (9),

$$W(U) \supset R. \tag{10}$$

This implies that there exists no subarc $\beta$ of $\alpha$, which would be free of the points of $\sigma(U)$. Indeed, if there were such an arc $\beta$, then the part of the closed unit disc lying on the side of the centre from the chord of $\beta$ would be of full $E_U$-measure, and convex, and so it would contain, by Theorem 1, the set $W(U)$, in contradiction to (10). Thus $\sigma(U)$ is dense on $\alpha$, and since $\sigma(U)$ is closed, we have $\sigma(U) \supset \alpha$.

We shall show that $W(U)$ contains the whole open interval $(-i, i)$.

This is obvious if $\pm i$ both belong to $\sigma_p(U)$ and hence to $W(U)$, since then by the convexity of $W(U)$ the whole interval $[-i, i]$ also belongs to $W(U)$.

Suppose now that at least one of the points $i$ and $-i$ does not belong to $\sigma_p(U)$. By symmetry it suffices to consider the case that $i$ does not belong to $\sigma_p(U)$. We shall show that in this case the open halfcircle

$$\alpha' = \{z: |z| = 1, \ \operatorname{Re} z < 0\}$$

necessarily contains at least one point of $\sigma(U)$. In the contrary case the convex set

$$\{z: |z| \leqq 1, \ \operatorname{Re} z > 0\} \cup \{-i\} \tag{11}$$

would be of full $E_U$ measure, hence, by Theorem 1, it would contain $W(U)$, and this is a contradiction, since the point 0, which belongs to $W(U)$ by (9) and (10), does not belong to the set (11). Now, since $\sigma(U)$ has a point on $\alpha'$, and since, on the other hand, it contains $\alpha$, $[\sigma(U)]^{\text{conv}}$ contains the diameter $(-i, i)$ in its interior. The relation (2), when applied to $U$ instead of $T$, yields then that $(-i, i)$ is also in the interior of $\overline{W(U)}$, and hence in $W(U)$.

Thus the diameter $(-i, i)$ belongs to the intersection of the numerical ranges of *all* the unitary dilations of $T$, and this proves, by (9), that $W(T)$ is a *proper* subset of this intersection.

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## References

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