On the structure of set mappings and the existence of free sets

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To Professor László Kalmár on his 60th birthday

Let E be an infinite set of power m and suppose that to every element x of E there corresponds a subset $S(x)$ of E such that $x \notin S(x)$. Two distinct elements of E, x and y, are called *independent* if $x \notin S(y)$ and $y \notin S(x)$. A subset F of E is called *free* if any two distinct elements of F are independent. *Throughout this paper we assume that m is regular and the power of the set $S(x)$ is $<m$ for every $x \in E$.*

Let us denote by $\mathbf{Pt}(E)$ the set of all subsets of E. Let $\mathbf{S} \subseteqq \mathbf{Pt}(E)$. We say that $\mathbf{S}$ satisfies the ascending chain condition for the ordinal number τ, if there exists no sequence $\{X_\xi\}_{\xi<\tau}$ of type τ of elements of $\mathbf{S}$ such that $X_\vartheta \subset X_\lambda$ holds for every $\vartheta<\lambda<\tau$. Let us denote by $\mathbf{Ac}(E, \tau)$ the set of all sets $\mathbf{S} \subseteqq \mathbf{Pt}(E)$ which satisfy the ascending chain condition for τ. By $S|H$ we denote the set $\{S(x) \cap H : x \in E\}$, where $H \subseteqq E$.

We consider the following two conditions for the set mapping $S(x)$:

(A) *For every $X \in \mathfrak{S} = \{S(x)\}_{x\in E}$, the set $\{Y \in \mathfrak{S} : Y \subseteqq X\}$ is well ordered by $\subseteqq$ and the ordinal type $\xi(X)$ of the set $\{Y \in \mathfrak{S} : Y \subset X\}$ is less than $\omega(m)$. The least upper bound of the ordinal numbers $\xi(X)$, where $X \in \mathfrak{S}$, is $\omega(m)$. The set of the elements $X \in \mathfrak{S}$ for which $\xi(X) = \xi$ has power less than m, and every subset of $\mathfrak{S}$ which is well ordered by $\subseteqq$ has power $<m$.*

(B) *For every decomposition $E = \bigcup_{\eta<\tau} E_\eta$ of E, where $\tau < \omega(m)$, $\overline{\overline{E}}_\eta = m$, and the sets E_η are mutually disjoint, there exists an ordinal number $\xi<\tau$ such that $S|E_\xi \in \mathbf{Ac}(E_\xi, \omega(m))$.*

In this paper we deal with the following question:

Whether or not the condition (A) (or (B)) implies the existence of free subsets of certain cardinalities of E?

In section I we shall prove that the condition (A) does not imply the existence of a free set of power m. In section II we shall prove that the condition (B) implies the existence of a free set of power $\aleph_0$.

We shall use the following notations: For any set $\mathbf{P}$ of sets let

$$\langle \mathbf{P} \rangle = \bigcup_{X \in \mathbf{P}} X.$$

For any cardinal number p we denote by $\omega(p)$ the initial number of p and by p^+ the cardinal number following p immediately. The symbols $\overline{\overline{S}}$ and $\bar{\gamma}$ denote the cardinal numbers of the set S and of the ordinal number γ, respectively. For any element x of E let $S^{-1}(x)=\{y\in E\colon x\in S(y)\}$ and for any $X\subseteqq E$ let $S(X)=\bigcup_{x\in X} S(x)$ and $S^{-1}(X)=\bigcup_{x\in X} S^{-1}(x)$.

I.

First we consider the following conditions:

(a) *There is a cardinal number* $n<m$ *such that, for every* $x\in E$, $\overline{\overline{S(x)}}<n$.

(b) *For every* $x\in E$, $\overline{\overline{S(x)}}<m$.

If the condition (a) is satisfied, then there exists a free subset of E with the power m. (See [1].)

On the other hand, it is easy to see that (b) does not imply the existence of an independent pair.

Example: $E=\{\xi\colon \xi<\omega(m)\}$ and $S(\xi)=\{\eta\colon \eta<\xi\}$ for every $\xi\in E$.

In this case the set $\mathfrak{S}$ of all sets $S(x)$ forms a well ordered set of type $\omega(m)$ with respect to inclusion $\subseteqq$.

In the following we assume that the condition (b) *holds for the sets* $S(x)$, *where* $x\in E$.

Now, we consider the following condition for the system $\mathfrak{S}=\{S(x)\}_{x\in E}$:

(c) *For every* $X\in\mathfrak{S}$, *the set* $\{Y\in\mathfrak{S}\colon Y\subseteqq X\}$ *is well ordered by* $\subseteqq$ *and the ordinal type* $\xi(X)$ *of the set* $\{Y\in\mathfrak{S}\colon Y\subset X\}$ *is less than* $\omega(m)$.

The ordinal number $\xi(X)$ is called the order of the element X. The order of the system $\mathfrak{S}$ is the least upper bound of the orders of its elements.

It is easy to prove that there exists a system $\mathfrak{S}=\{S(x)\}_{x\in E}$ of order 1 such that there exists no free subset of power greater than 2 of E. Let E_1 and E_2 two disjoint subsets of power m of E such that $E=E_1\cup E_2$. Let $\{x_\xi^1\}_{\xi<\omega(m)}$ and $\{x_\xi^2\}_{\xi<\omega(m)}$ be well orderings of E_1 and E_2 respectively. If $x=x_\gamma^1\in E_1$, then let

$$S(x)=\{x_\gamma^2\}\cup\{x_\xi^1\}_{\xi<\gamma}$$

and if $x=x_\gamma^2\in E_2$, then let similarly

$$S(x)=\{x_\gamma^1\}\cup\{x_\xi^2\}_{\xi<\gamma}.$$

It is easy to see that the system $\mathfrak{S}$ of the sets $S(x)$ has the order 1 and there exists no free subset of power greater than 2 of E.

Therefore in the sequel we assume the following condition:

(A) *The system* $\mathfrak{S}=\{S(x)\}_{x\in E}$ *satisfies the conditions* (b) *and* (c), *the order of* $\mathfrak{S}$ *is* $\omega(m)$, *the set of the elements of order* ξ *of* $\mathfrak{S}$ *has power less than* m, *and every subset of* $\mathfrak{S}$ *which is well ordered by* $\subseteqq$ *has power* $<m$.

In this section we deal with the following question:

Whether or not the condition (A) *implies the existence of a free subset of power* m *of* E?

In discussing of this question we shall need another formulation for the properties of the system $\mathfrak{S}$.

Definition 1. 1. The ordered pair $(R,\leqq)$ is said to be a ramification system if R is a set and $\leqq$ is a partial ordering relation defined on the set R satisfying the condition: for every $x\in R$ the set $\{y\in R\colon y\leqq x\}$ is well ordered by $\leqq$.

Definition 1.2. The order $\xi(x)$ of an element x of a ramification system $(R, \leqq)$ is the ordinal type of the well ordered set $\{y \in R: y < x\}$.

Definition 1.3. The order of the system $(R, \leqq)$ is the least upper bound of the orders of its elements.

Definition 1.4. By a (p, β) ramification system we mean a system of order β, where for each $\xi < \beta$ the set R_ξ of the elements of order ξ has power less than p.

Definition 1.5. We say that the property $Q(p, \beta)$ holds, if there is a (p, β) ramification system $(R, \leqq)$ of order β such that every subset of R which is well ordered by $\leqq$ has power $< \overline{\beta}$.

It is known that the property $Q(p, \omega(p))$ holds (see [2]), if

(i) $p = n^+$, where n is strongly inaccessible,

(ii) $p = n^+$, where n is regular (assuming the generalized continuum hypothesis),

(iii) p is singular.

We prove now the following

Theorem 1. *The condition* (A) *does not imply the existence of a free set of power* m.

Proof. We shall define a system $\mathfrak{S} = \{S(x)\}_{x \in E}$ which satisfies the condition (A) and for which Theorem 1 is valid, assuming that $Q(m, \omega(m))$ holds (otherwise (A) cannot be satisfied). In this case there exists a ramification system $(R, \leqq)$ satisfying the following conditions:

(1) $$R = \bigcup_{\xi < \omega(m)} R_\xi, \quad R_{\xi_1} \cap R_{\xi_2} = 0 \quad \text{for every} \quad \xi_1 < \xi_2 < \omega(m),$$

where R_ξ is the non-empty set of the elements of order ξ of R,

(2) $$0 < \overline{\overline{R}}_\xi < m$$

for every $\xi < \omega(m)$,

(3) $$\overline{\overline{R}} = m$$

(this follows from (1) and (2)),

(4) $$\overline{\overline{R'}} < m$$

for every subset R' of R which is well ordered by $\leqq$.

Consider now an arbitrary element r of R. We define the set $S(r)$ as follows. There is an ordinal number $\xi < \omega(m)$ such that $r \in R_\xi$. Let

$$S(r) = \bigcup_{\alpha \leqq \xi} R_\alpha - W(r)$$

where $W(r) = \{y \in R: y \leqq r\}$. It is obvious that for every $r \in R$, the power of the set $S(r)$ is $< m$. We prove now

(j) for the system $\mathfrak{S} = \{S(r)\}_{r \in R}$ the condition (A) holds,

(jj) there exists no free set of power m.

Since for $r, s \in R$ the relation $r < s$ holds if and only if the relation $S(r) \subset S(s)$ holds, (j) follows from (1)—(4).

Since r and s are independent if and only if $r < s$ or $s < r$, any free set is ordered, consequently it is well ordered (see 1. 1) and so because of (4) it has power $< m$. Put $E = R$. The theorem is proved.

II.

We assume in the sequel that m is regular and the condition $\overline{\overline{S(x)}} < m$ holds for $x \in E$. In this section we prove the following

Theorem 2. *The condition* (B) *implies the existence of a free set of power $\aleph_0$.*

We need, for the proof of Theorem 2, some lemmas and definitions.

Lemma 2. 1. *Let F be a set of power m, where m is regular and $m \geqq \aleph_0$. Further let $\mathbf{S} \subseteqq \mathbf{Pt}(F)$ such that*

(1) $\mathbf{S} \in \mathbf{Ac}(F, \omega(m))$,

(2) $\overline{\overline{X}} < m$ *for every* $X \in \mathbf{S}$.

Then there exists a subset K of power $< m$ of F such that

(3) $K \nsubseteq X$ *for every* $X \in \mathbf{S}$.

Proof. Consider the partial ordering of $\mathbf{S}$ with respect to the relation of inclusion. By a theorem of HAUSDORFF [3] there is a maximal ordered subset $\mathbf{P}$ of $\mathbf{S}$. By another theorem of HAUSDORFF [3] $\mathbf{P}$ has a well ordered subset $\mathbf{Q}$ which is confinal to $\mathbf{P}$. It is obvious that $\langle \mathbf{P} \rangle = \langle \mathbf{Q} \rangle$. It follows from (1) that $\overline{\overline{\mathbf{Q}}} < m$. Since m is regular and $\mathbf{Q} \subseteqq \mathbf{S}$, we obtain from (2) that $\overline{\overline{\langle \mathbf{Q} \rangle}} < m$. As $\langle \mathbf{P} \rangle = \langle \mathbf{Q} \rangle$, we have that $\overline{\overline{\langle \mathbf{P} \rangle}} < m$.

Since $\overline{\overline{F}} = m$, $F - \langle \mathbf{P} \rangle \neq 0$. Let x be an arbitrary element of $F - \langle \mathbf{P} \rangle$. Clearly the power of $\langle \mathbf{P} \rangle \cup \{x\}$ is $< m$. By the maximality of $\mathbf{P}$ there exists no set X in $\mathbf{S}$ for which $\langle \mathbf{P} \rangle \cup \{x\} \subseteqq X$ holds. Put $K = \langle \mathbf{P} \rangle \cup \{x\}$. The lemma is proved.

Lemma 2. 2. *The condition* (B) *with $\tau = 1$ implies the existence of a subset K of power $< m$ of E for which $\bigcap_{x \in K} S^{-1}(x) = 0$ holds.*

Proof. It follows from (B) with $\tau = 1$ that $S|E$ satisfies the conditions of Lemma 2. 1 with $E = F$ and $S|E = \mathbf{S}$. Consequently there is a subset K of power $< m$ of E such that $K \nsubseteq S(x)$ for every $x \in E$, i. e. $\bigcap_{x \in K} S^{-1}(x) = 0$.

Definition 2. 3. Let $\mathfrak{T} = (E, \leqq)$ be an arbitrary partial ordering of the set E. If $x \in E$, then we denote by $\mathfrak{T}(x)$ the set of the minimal elements of the set $\{y \in E: x < y\}$; moreover let $\mathfrak{T}^{(0)}(x) = x$ and $\mathfrak{T}^{(k)}(x) = \mathfrak{T}(\mathfrak{T}^{(k-1)}(x))$ for the natural number k. If $X \subseteqq E$ then we put $\mathfrak{T}(X) = \bigcup_{x \in X} \mathfrak{T}(x)$, $\mathfrak{T}^{(0)}(X) = X$, and $\mathfrak{T}^{(k)}(X) = \mathfrak{T}(\mathfrak{T}^{(k-1)}(X))$.

Definition 2. 4. Let K be the set defined in Lemma 2. 2. The partial ordering $\mathscr{F} = (E, \leqq)$ is said to be free if

(1) $\mathscr{F} = (E, \leqq)$ is a ramification system (defined in 1. 1),

(2) each element of K is minimal in $\mathscr{F}$ and if the element $x \in E - K$ is minimal in $\mathscr{F}$, then it is maximal too.

(3) if $x<y$ in $\mathscr{F}$, then x and y are independent.

Definition 2. 5. The partial ordering $\mathscr{R}=(E, \leqq)$ is said to be regular if
(1) $\mathscr{R}$ is free,
(2) for each $x \in E$, $\overline{\overline{\mathscr{R}(x)}}<m$ (see Definition 2. 3),
(3) for each $x \in E$ for which $\mathscr{R}(x) \neq 0$, $\bigcap_{y \in \mathscr{R}(x)} S^{-1}(y)=0$.

Definition 2. 6. If the partial ordering $\mathscr{R}=(E, \leqq)$ is regular, then the set $X \subseteq E$ is said to be a path if
(1) X is well ordered by $\leqq$,
(2) if $x \in X$ and $y<x$, then $y \in X$,
(3) $X \cap K \neq 0$ (for K, see Lemma 2. 2 and Definition 2. 4).
Let us denote by $\sigma(\mathscr{R})$ the set of the paths in $\mathscr{R}$, which are maximal with respect to the relation of inclusion $\subseteq$. Further, for every natural number k, let us denote by $\sigma(\mathscr{R}, k)$ the set of the paths of power k in $\mathscr{R}$.

Lemma 2. 7. *Let $\mathscr{R}$ be a regular partial ordering having only paths of finite length. Then*

$$(1)\quad \overline{\overline{\sigma(\mathscr{R})}}<m \quad \textit{and} \quad (2) \quad \bigcap_{X \in \sigma(\mathscr{R})} S^{-1}(X)=0.$$

Proof. First we prove the statement (1). It is easy to see by 2. 5/(2), that for every positive integer k

$$\overline{\overline{\sigma(\mathscr{R}, k)}}=\overline{\overline{\mathscr{R}^{k-1}(K)}}<m.$$

If $m>\aleph_0$, then by the regularity of m we obtain:

$$\overline{\overline{\sigma(\mathscr{R})}} \leqq \overline{\overline{\bigcup_{1 \leqq k<\omega} \sigma(\mathscr{R}, k)}}<m.$$

If $m=\aleph_0$, then a simple argument of D. König [4] gives the existence of a positive integer t with $\mathscr{R}^t(K)=0$, and so we have

$$\overline{\overline{\sigma(\mathscr{R})}}=\overline{\overline{\bigcup_{1 \leqq k<t} \sigma(\mathscr{R}, k)}}<\aleph_0$$

which proves the statement (1).

To prove the statement (2) let y be an arbitrary element of E. It is enough to show that there exists a path $X \in \sigma(\mathscr{R})$ for which $y \notin S^{-1}(X)$. For this purpose let x_0 be an element of K with $y \notin S^{-1}(x_0)$ (such an x_0 clearly exists — see 2. 2). Now suppose that for the positive integer k the path $X_k=\{x_i\}_{i<k}$ has been already defined such that $y \notin S^{-1}(X_k)$. If X_k is no maximal path, then let $x_k \in \mathscr{R}(x_{k-1})$ for which $y \notin S^{-1}(x_k)$ (such an x_k clearly exists — see 2.5/(3)). Since every path in $\mathscr{R}$ is finite, in a finite number of steps we obtain a maximal path $X=X_t=\{x_i\}_{i<t}$ with the positive integer t, such that $y \notin S^{-1}(X)$. The lemma is proved.

Definition 2. 8. Let $\mathfrak{X}$ denote the set of all regular partial orderings $\mathscr{R}$. We define a partial ordering $\mathfrak{M}=(\mathfrak{X}, \leqq)$ as follows. Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be two elements of $\mathfrak{X}$, then we put $\mathscr{R}_1 \leqq \mathscr{R}_2$ if
(1) the relation $x \leqq y$ in $\mathscr{R}_1$ implies the relation $x \leqq y$ in $\mathscr{R}_2$ and
(2) $\mathscr{R}_1(x) \neq 0$ implies $\mathscr{R}_1(x)=\mathscr{R}_2(x)$.

Lemma 2.9. *Let $\mathfrak{X}'$ be an ordered subset of $\mathfrak{X}$. Then*

$$(1) \quad \mathscr{R}' = \bigcup_{\mathscr{R} \in \mathfrak{X}'} \mathscr{R} \in \mathfrak{X} \quad \textit{and} \quad (2) \quad \mathscr{R} \in \mathfrak{X}' \ \textit{implies} \ \mathscr{R} \leqq \mathscr{R}'.$$

We omit the proof.

It follows from 2.9 with the aid of the Kuratowski—Zorn lemma the following

Lemma 2.10. *The set $\mathfrak{X}$ has a maximal element in $\mathfrak{M}$.*

Lemma 2.11. *If $\mathscr{R}$ is an element of $\mathfrak{X}$ having only paths of finite length, then $\mathscr{R}$ is not maximal in $\mathfrak{M}$.*

Proof. By 2.7/(1) we have $\overline{\overline{\sigma(\mathscr{R})}} < m$. Let

$$X_0, X_1, \ldots, X_\xi, \ldots \quad (\xi < \tau)$$

be a well ordering of $\overline{\overline{\sigma(\mathscr{R})}}$ for some $\tau < \omega(m)$. As $\overline{\overline{\sigma(\mathscr{R})}} < m$ and for every $\xi < \tau$ $\overline{\overline{X}}_\xi < \aleph_0$, we have $\overline{\overline{\langle \sigma(\mathscr{R}) \rangle}} < m$. Thus $\overline{\overline{S(\langle \sigma(\mathscr{R}) \rangle)}} < m$ because m is regular and for every $x \in E$, $\overline{\overline{S(x)}} < m$. Put $G = E - \langle \sigma(\mathscr{R}) \rangle - S(\langle \sigma(\mathscr{R}) \rangle)$. It follows that

$$\text{(i)} \qquad \overline{\overline{E - G}} < m.$$

Let

$$G_\xi = \{x \in G \colon X_\xi \cup \{x\} \text{ is free}\}.$$

Obviously $G_\xi = G - S^{-1}(X_\xi)$. Thus

$$\bigcup_{\xi < \tau} G_\xi = \bigcup_{\xi < \tau} (G - S^{-1}(X_\xi)) = G - \bigcap_{\xi < \tau} S^{-1}(X_\xi) = G - \bigcap_{X \in \sigma(\mathscr{R})} S^{-1}(X).$$

By means of 2.7/(2) we have:

$$\bigcup_{\xi < \tau} G_\xi = G - \bigcap_{X \in \sigma(\mathscr{R})} S^{-1}(X) = G.$$

For every $\xi < \tau$, let

$$H_\xi = G_\xi - \bigcup_{\alpha < \xi} G_\alpha.$$

It is obvious that

$$\text{(ii)} \qquad \bigcup_{\xi < \tau} H_\xi = \bigcup_{\xi < \tau} G_\xi = G.$$

Let now $F_\xi = H_\xi$ if $\overline{\overline{H}}_\xi = m$ and $F_\xi = 0$ if $\overline{\overline{H}}_\xi < m$. In accordance with (i) and (ii) we obtain that

$$\overline{\overline{E - \bigcup_{\alpha < \tau} F_\alpha}} < m.$$

It follows from the condition (B) that there is an $F_\xi \neq 0$ such that

$$S | F_\xi \in \mathbf{Ac}(F_\xi, \omega(m)).$$

Therefore we can easily conclude by Lemma 2.1 that there is a set $L \subseteqq F_\xi$ with $\overline{\overline{L}} < m$ such that $L \not\subseteqq S(x)$ for every $x \in E$, i. e. $\bigcap_{y \in L} S^{-1}(y) = 0$. Now we define the partial ordering $\mathscr{R}'$ as follows. Let $x \leqq y$ in $\mathscr{R}'$ if $x \in X_\xi$ and $y \in L$, in the other cases let $u \leqq v$ in $\mathscr{R}'$ if and only if $u \leqq v$ in $\mathscr{R}$. It is easy to see that $\mathscr{R}'$ is regular and $\mathscr{R} < \mathscr{R}'$ in $\mathfrak{M}$. Consequently $\mathscr{R}$ is not maximal. Thus Lemma 2. 11 is proved.

Lemma 2. 12. *There exists a regular partial ordering which has an infinite path X.*

Proof. This follows trivially from 2. 10 and 2. 11.

It follows from the definition of the regular partial ordering (2. 5) that the path X defined in 2. 12 is an infinite free set. Thus Theorem 2 is proved.

References

[1] A. HAJNAL, Proof of a conjecture of S. Ruziewicz, *Fundamenta Math.*, **50** (1961—1962), 123—128.
[2] P. ERDŐS and A. TARSKI, On some problems involving inaccessible cardinals, *Essays on the Foundations of Mathematics* (Jerusalem, 1961), 50—82.
[3] F. HAUSDORFF, Grundzüge einer Theorie der geordneten Mengen, *Math. Annalen*, **65** (1908), 435—505.
[4] D. KÖNIG, Über eine Schlußweise aus dem Endlichen ins Unendliche, *Acta Sci. Math.*, **3** (1927), 121—130.

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