Representation of partially ordered groups

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To Professor L. Kalmár on his 60th birthday

A well-known theorem of CAYLEY states that groups are representable as permutation groups. In his paper [3] Ch. HOLLAND has dealt with a similar problem and proved that each lattice-ordered group is isomorphic to a group of monotone permutations of a suitable fully ordered set. Here we shall consider two questions connected with the above ones. The first one is similar to CAYLEY's theorem and is concerned with partially ordered groups, while the second one is connected with the paper of HOLLAND and characterizes the partially ordered groups that can be regarded as subgroups of the lattice-ordered group of the monotone permutations of some fully ordered set.

The fundamental concepts concerning partially ordered groups can be found in the book of L. FUCHS [2].

The following result is well known.

Lemma. *A group Σ of permutations of a fully ordered set S is partially left-ordered*[1]) *by the rule*[2])

$$\alpha \leqq \beta \quad (\alpha, \beta \text{ in } \Sigma), \quad \text{if } u^{\alpha} \leqq u^{\beta} \text{ for each } u \in S.$$

Before proving the converse of this lemma we shall consider the connection of S and Σ in more detail.

We define a new partial order $\leqq'$ on the set S in the following way:

Let $a \leqq' b$ (a, b in S) if there exist a u in S and α, β in Σ such that $a = u^{\alpha}$, $b = u^{\beta}$ and $\alpha \leqq \beta$. We denote the set S partially ordered under $\leqq'$ by S'. The order of S is obviously an extension of the order of S'. Indeed, $a \leqq' b$ (a, b in S) implies by definition $a \leqq b$.

We verify that the relation $\leqq'$ is actually a partial order. We get reflexivity by the choice $\alpha = \beta = \varepsilon$ (the unity of Σ). Antisimmetry is clear because S is an extension of S'. Finally, we obtain transitivity in the following way: $a \leqq' b$ and $b \leqq' c$ imply the existence of u, v in S and $\alpha \leqq \beta$, $\gamma \leqq \delta$ in Σ with the properties $a = u^{\alpha}$, $b = u^{\beta} = v^{\gamma}$, $c = v^{\delta}$. From the partial left-order it follows $\beta^{-1}\alpha \leqq \varepsilon \leqq \gamma^{-1}\delta$, that is, $a = u^{\alpha} = b^{\beta^{-1}\alpha} \leqq' \leqq' b^{\gamma^{-1}\delta} = v^{\delta} = c$.

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[1]) A group Σ is partially left-ordered if there is a partial order $\leqq$ in the set Σ with the property: $\alpha \leqq \beta$ implies $\gamma\alpha \leqq \gamma\beta$ for all $\alpha, \beta, \gamma \in \Sigma$. We can define partial right-order similarly. A group is partially ordered if it is both partially left- and right-ordered.

[2]) We denote by u^{α} the image of the element $u \in S$ under the mapping $\alpha \in \Sigma$.

Thus we have the following

Theorem 1. *S' is a partially ordered set.*

The connection between Σ and S' is shown by

Theorem 2. *Σ is a partially ordered group if and only if its elements are monotone permutations of S'.*

Proof. Let first Σ be a partially ordered group, and let $a \leqq' b$. Then $a = u^{\alpha}$, $b = u^{\beta}$ for a suitable $u \in S$ and for some $\alpha \leqq \beta$ in Σ. By two-sided ordering, for every $\sigma \in \Sigma$, $\alpha\sigma \leqq \beta\sigma$ holds, and therefore $a^{\sigma} = u^{\alpha\sigma} \leqq' u^{\beta\sigma} = b^{\sigma}$, as asserted.

Next, let the elements of Σ be monotone permutations of S', and let $\alpha \leqq \beta$ and σ in Σ. From $\alpha \leqq \beta$ we obtain for every $u \in S'$ the relation $u^{\alpha} \leqq' u^{\beta}$, and hence because of the monotonicity $u^{\alpha\sigma} \leqq' u^{\beta\sigma}$. On using the fact that S is an extension of S', we get $u^{\alpha\sigma} \leqq u^{\beta\sigma}$ for each $u \in S$, that is, $\alpha\sigma \leqq \beta\sigma$.

Clearly, S is the union of the disjoint sets S_{ν} which are the domains of intransitivity under Σ.

Let Σ_{ν} be the partially left-ordered group of the permutations induced by the elements of Σ on S_{ν}. It is easy to see that Σ is a subdirect product of the partially left-ordered groups Σ_{ν}. (See COHN [1][3].)

Now, let $\bar{\Sigma}$ be a subgroup of Σ. We can define $\overline{S'}$ and $\overline{S_{\nu}}$ similarly as we did S' and S_{ν}. Obviously, each S_{ν} is the union of some $\overline{S_{\mu}}$, and it is easy to see that the order of S' is an extension of the order of $\overline{S'}$. It can be proved without difficulty that if all positive elements of Σ are in $\bar{\Sigma}$ then $\overline{S'} = S'$.

Theorem 3. *Let $\bar{\Sigma}$ be a normal subgroup of Σ. Then the $\overline{S_{\nu}}$'s which are subsets of the same S_{μ} are domains of imprimitivity of Σ.*

Proof. Let $\overline{S_1}$, $\overline{S_2}$ belong to S_1, $a, b \in \overline{S_1}$ and $c \in \overline{S_2}$. Then $b = a^{\varphi}$ and $c = a^{\alpha}$, where $\varphi \in \bar{\Sigma}$ and $\alpha \in \Sigma$. Hence $b^{\alpha} = a^{\varphi\alpha} = a^{\alpha\alpha^{-1}\varphi\alpha} = c^{\alpha^{-1}\varphi\alpha}$. Now, $\alpha^{-1}\varphi\alpha \in \bar{\Sigma}$ because $\bar{\Sigma}$ is a normal subgroup of Σ, thus $b^{\alpha} \in \overline{S_2}$.

The following result gives information about the representability of partially left-ordered groups.

Theorem 4. *If Σ is a partially left-ordered group then there exists a fully ordered set S such that the partially left-ordered group of the permutations of S contains a subgroup o-isomorphic to Σ.*

Proof. Let Σ be a partially left-ordered group and let Σ_0 denote the underlying partially ordered set (i. e. in Σ_0 we disregard from the group operation). It is known (see SZPILRAJN [4]) that the order of Σ_0 is the intersection of orders of some fully ordered sets Σ_{ν}. We can suppose that the set of indices ν is ordered. Finally, let S be the union of the disjoint sets Σ'_{ν} such that Σ'_{ν} is order-isomorphic to Σ_{ν}. We denote by u_{ν} the element of Σ'_{ν} corresponding to the element u of Σ_0. We define a full order in S by putting $u_{\nu} \leqq v_{\mu}$ if either ν precedes μ in the ordering of the indices or if $\nu = \mu$ and $u_{\nu} \leqq v_{\nu}$ in Σ'_{ν}.

Now we define, for each $a \in \Sigma$, a mapping σ_a of S such that $u_{\nu}^{\sigma_a} = (ua)_{\nu}$ for each ν. Obviously, σ_a is a permutation of S. The mapping $a \to \sigma_a$ is a one-to-one corres-

[3]) COHN considers only the case of complete direct product.

pondence, for if $\sigma_a=\sigma_b$, then $a_\nu=e_\nu^{\sigma_a}=e_\nu^{\sigma_b}=b_\nu$ for every ν, that is, $a=b$. From $u_\nu^{\sigma_a\sigma_b}=(ua)_\nu^{\sigma_b}=(uab)_\nu=u_\nu^{\sigma_{ab}}$ it follows that the mapping $a\to\sigma_a$ is an isomorphism. Let $a\leqq b$. Then $ua\leqq ub$, and therefore $u_\nu^{\sigma_a}=(ua)_\nu\leqq(ub)_\nu=u_\nu^{\sigma_b}$. Thus $\sigma_a\leqq\sigma_b$, and the mapping $a\to\sigma_a$ is an o-preserving isomorphism. Let finally σ_a be a positive element, that is, $e_\nu\leqq e_\nu^{\sigma_a}=(ea)_\nu=a_\nu$ for all ν. Since Σ_0 is the intersection of the Σ_ν, we get $e\leqq a$, showing that $a\to\sigma_a$ is an o-isomorphism.

Next we shall consider partially ordered groups which are representable by monotone permutations of a fully ordered set. The monotone permutations of a fully ordered set S, as is well known, form a lattice-ordered group under the usual ordering (see: Lemma). HOLLAND proved in [3] that to each lattice-ordered group Σ there exists a fully ordered set S such that the group of all monotone permutations of S, under the usual ordering, has a subgroup o-isomorphic to Σ. We get the obvious

Theorem 5. *A partially ordered group is representable by monotone permutations of a fully ordered set if and only if it is isomorphic to a subgroup of a lattice-ordered group.*

Let the elements of a group Σ operate on a set S. We call an ordering $\leqq$ of S a Σ-ordering if, for each $\alpha\in\Sigma$, $a\leqq b$ implies $a^\alpha\leqq b^\alpha$ $(a, b\in S)$. Let, for example, Σ be a partially ordered group and P a convex subgroup of Σ. The set of the right cosets of P is in the induced partial order ($P\alpha\leqq P\beta$ if there are elements $\gamma\in P\alpha$, $\delta\in P\beta$ such that $\gamma\leqq\delta$) a Σ-ordered set by the definition $(P\alpha)^\sigma=P\alpha\sigma$. If the ordering of the Σ-ordered set of the right cosets of a convex subgroup P can be extended to a full Σ-ordering, we call P an *admissible* subgroup.

Remark. A convex subgroup P of a commutative group Σ is admissible if and only if the induced order of the group Σ/P is extendible to a full order, that is, Σ/P is torsionfree. Let namely $\leqq$ be the induced order in Σ/P and $\leqq'$ a full order of Σ/P which is an extension of $\leqq$. Let $P\alpha\leqq' P\beta$ and τ an element of Σ. Because of $P\varrho P\sigma=P\varrho\sigma$, $P\alpha P\tau\leqq' P\beta P\tau$ and $P\alpha\tau\leqq' P\beta\tau$ are equivalent. The commutativity of Σ completes the proof.

Example. We construct a group containing a subgroup which is not admissible. Let the elements of Σ be the numbers of the form $\pm 2^\alpha$ and the elements of P the numbers of the form 2^α, where α are integers, and the group operation is multiplication of numbers. Let the positive elements be the elements $\geqq 1$; then P is convex and there are two cosets of P, namely P and $-P$. P can not be an admissible subgroup because $P(-1)=-P$ and $-P(-1)=P$.

Theorem 6. *A partially ordered group Σ is o-isomorphic to a subgroup of the lattice-ordered group of all monotone permutations of a fully ordered set S if and only if there exists a set Λ of admissible subgroups P of Σ such that to each non-positive α in Σ there is some P in Λ satisfying $P\alpha<P$ in the induced ordering*[4]).

Proof. Let first Σ be o-isomorphic to a subgroup of the partially ordered group of all monotone permutations of a fully ordered set S. We can assume, on the basis of this o-isomorphism, that the elements of Σ are monotone permutations of S. Let $a\in S$, and define P_a to consist of all $\alpha\in\Sigma$ such that $a^\alpha=a$. Obviously, for each $a\in S$, P_a is a convex subgroup. There will not be any ambiguity in denoting

[4]) The last assumption implies that Σ is directed.

the orders of S, Σ and the set of cosets of P_a by the same sign $\leqq$. Let us fix an a in S. $a^\alpha = a^\beta$ is equivalent to $a^{\alpha\beta^{-1}} = a$, further to $\alpha\beta^{-1} \in P_a$, that is, $a^\alpha = a^\beta$ if and only if α and β are in the same right coset of P_a. Now, we order the cosets of P_a by setting $P_a\alpha \leqq P_a\beta$ if and only if $a^\alpha \leqq a^\beta$. This relation is, obviously, independent of the representation of the cosets. It is also clear that this relation is an order, moreover a full order, because the elements a^α are in the fully ordered set S. Now, $P_a\alpha \leqq P_a\beta$, i. e., $a^\alpha \leqq a^\beta$ implies $a^{\alpha\sigma} \leqq a^{\beta\sigma}$, i. e., $P_a\alpha\sigma \leqq P_a\beta\sigma$. Therefore P_a is admissible.

Finally, let α be a non-positive element in Σ. From the non-positivity we infer the existence of an $a \in S$ such that $a^\alpha < a$, because from $a^\alpha \geqq a$, for each $a \in S$, the positivity of α would follow. From $a^\alpha < a$ it follows $P_a\alpha < P_a$; this means that the subgroup P_a has the property required in theorem 6.

Conversely, let Λ be a set of admissible subgroups P_ν, such that to each non-positive $\alpha \in \Sigma$ there is some P_ν in Λ with the property $P_\nu\alpha < P_\nu$. We can suppose Λ to be fully ordered. Let S_ν denote the set of the right cosets of P_ν; S_ν is a fully Σ-ordered set. We denote the orders in S_ν and in Λ by $\leqq$. Let finally S be the union of S_ν with the following full order: $P_\nu\alpha < P_\mu\beta$ if either $\nu < \mu$ in Λ or if $\nu = \mu$ and $P_\nu\alpha < P_\nu\beta$ in S_ν. We define, for each $\sigma \in \Sigma$, the mapping $P_\nu\alpha \to (P_\nu\alpha)^\sigma = P_\nu\alpha\sigma$ of S into itself. Obviously, these are mappings of S onto S, moreover they are one-to-one. These permutations are monotone. For, let $P_\nu\alpha < P_\mu\beta$. If $\nu < \mu$ in Λ, then $P_\nu\alpha\sigma < P_\mu\beta\sigma$ for each $\sigma \in \Sigma$. If $P_\nu\alpha < P_\nu\beta$ in S_ν, then $P_\nu\alpha\sigma < P_\nu\beta\sigma$, because S_ν is Σ-ordered.

Now, to $\sigma \in \Sigma$ we make correspond the monotone permutation $P_\nu\alpha \to P_\nu\alpha\sigma$ of S. Then Σ will be isomorphic to a subgroup of the monotone permutations of S. We have to show that this isomorphism is an o-isomorphism. It is enough to prove that a monotone permutation $P_\nu\alpha \to P_\nu\alpha\sigma$ is positive if and only if σ is a positive element of Σ. Let σ be positive. Then $\alpha < \alpha\sigma$ for each α in Σ and $P_\nu\alpha < P_\nu\alpha\sigma = (P_\nu\alpha)^\sigma$ for all P_ν in Λ and α in Σ. Now, let σ be non-positive. By hypothesis there exists a subgroup P_ν in Λ such that $P_\nu\sigma < P_\nu$, and so the mapping $P_\nu\alpha \to P_\nu\alpha\sigma$ is again non-positive.

Remark. The intersection of all P_ν in Λ is the unity of Σ. In order to prove this proposition, it is enough to show that for each $\alpha \neq \varepsilon$ in Σ there is a P_ν in Λ such that $\alpha \notin P_\nu$. If α is non-positive then there is a P_ν in Λ such that $P_\nu\alpha < P_\nu$ and so $\alpha \notin P_\nu$. If $\alpha > \varepsilon$, then α^{-1} is non-positive, and if $\alpha^{-1} \notin P_\nu$ then $\alpha \notin P_\nu$ too. If we replace the condition "for each non-positive $\alpha \in \Sigma$ there exists a $P \in \Lambda$ such that $P\alpha < P$ holds" by "the intersection of all $P_\nu \in \Lambda$ is the unity of Σ", we are able to prove only the existence of an o-preserving isomorphism.

References

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