On completely continuous and uniformly bounded operators in l^p spaces

By L. GEHÉR in Szeged

To Professor L. Kalmár on his 60th birthday

Introduction and terminology. In the sequel, a *transformation* T means a linear, bounded (and everywhere defined) mapping of a Banach space $\mathfrak{B}_1$ into another Banach space $\mathfrak{B}_2$; T is called *invertible* if it has a bounded (everywhere defined) inverse T^{-1}. In case $\mathfrak{B}_1 = \mathfrak{B}_2$, T is said to be an *operator*. An operator T defined on a Banach space is called uniformly bounded if there exists a constant K with $\|T^n\| \leqq K$ $(n=0, 1, \dots)$. *Contraction* means an operator with norm not greater than 1. Two operators A and B defined on $\mathfrak{A}$ and $\mathfrak{B}$, respectively, are said to be *similar* if there exists an invertible transformation C mapping $\mathfrak{A}$ onto $\mathfrak{B}$ such that $A = C^{-1}BC$.

A few years ago B. SZ.-NAGY [1] proved the following theorem:

Every completely continuous and uniformly bounded operator defined on a Hilbert space $\mathfrak{H}$ is similar to a contraction of $\mathfrak{H}$.

The theorem also has been proved for l_n^p spaces [2].

In this paper we are going to give a generalization of the above mentioned theorem of B. SZ.-NAGY in l^p spaces. The following theorem will be proved:

Theorem. *Every completely continuous and uniformly bounded operator on l^p $(1 \leqq p < \infty)$ is similar to a contraction defined on a proper subspace of l^p isomorphic to l^p.*

Every subspace of a Hilbert space $\mathfrak{H}$ isomorphic to $\mathfrak{H}$ being also isometric to $\mathfrak{H}$, our result contains that of B. SZ.-NAGY (at least for separable Hilbert spaces).

Proof of the theorem. 1°. Let T be a completely continuous and uniformly bounded operator defined on l^p. Then the spectrum Σ of T consists of countable many points, the only possible accumulation point of which is the point 0. Further, T being uniformly bounded, its spectrum Σ lies on the closed unit disc, and on the unit circle there are only finitely many eigenvalues of T. Denote by $\mu_1, \mu_2, \dots, \mu_k$ the eigenvalues lying on the unit circle, each of them being written as many times as its geometric multiplicity (for a uniformly bounded operator, each eigenvalue lying on the unit circle has the same algebraic multiplicity as its geometric multiplicity, cf. [1]). Let $a < 1$ be a positive number such that the rest Σ' of the spectrum lies in the disc $|z| < a$. Then, as well-known,

$$P = -\frac{1}{2\pi i} \oint_{|z|=a} R_z \, dz, \quad \text{where} \quad R_z = (T - zI)^{-1},$$

is a projection commuting with T, such that the spectrum of the restriction $T_{\mathfrak{B}}$ of T to $\mathfrak{B} = Pl^p$ is Σ'. $\mathfrak{B}$ has finite codimension. (A subspace of a Banach space has finite codimension k, if it has a k-dimensional complementary subspace.) $\mathfrak{B}$ being isomorphic to l^p, there exists an operator T' on l^p similar to $T_{\mathfrak{B}}$.*) For a moment, suppose that the theorem is true for T', i. e. that there exists a subspace $\mathfrak{B}_1'$ of l^p, such that T' is similar to a contraction on $\mathfrak{B}_1'$.

The space of the elements $(\alpha_1, \alpha_2, \ldots, \alpha_k, x)$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are scalars and $x \in l^p$, supplied with the norm $\left(\sum_{i=1}^{k} |\alpha_i|^p + \|x\|^p \right)^{1/p}$ is a Banach space isometric to l^p, thus it can be identified with l^p. The elements $(\alpha_1, \alpha_2, \ldots, \alpha_k, x)$ $(x \in \mathfrak{B}_1')$ form a subspace $\mathfrak{B}_1$ of this space. The operator

$$T_1(\alpha_1, \alpha_2, \ldots, \alpha_k, x) = (\mu_1 \alpha, \mu_2 \alpha_2, \ldots, \mu_k \alpha_k, T_1' x) \qquad (x \in \mathfrak{B}_1')$$

defined on this subspace is evidently a contraction and similar to T.

---

*) We have made use of the fact that every finite codimensional subspace of l^p is isomorphic to l^p. Although this is a consequence of a general theorem of Pełczyński [3], in this special case we sketch a proof for it. Let $\mathfrak{B}$ be a k-codimensional subspace of l^p (k is finite, $k > 0$) and P a (bounded) projection of l^p onto $\mathfrak{B}$. Introducing the basis vectors $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, ... in l^p, every element $x \in \mathfrak{B}$ may be written in the form

$$x = \sum_{n=1}^{\infty} \alpha_n P e_n \quad \text{with} \quad \sum_{n=1}^{\infty} |\alpha_n|^p < \infty,$$

but not uniquely. It can be proved, however, that we can choose a set of k subscripts $N = \{n_1, n_2, \ldots, n_k\}$ such that a representation

$$x = \sum_{n \notin N} \alpha_n' P e_n \quad \text{with} \quad \sum_{n \notin N} |\alpha_n'|^p < \infty$$

is possible and is uniquely determined, for each $x \in \mathfrak{B}$. I. e. $\{Pe_n\}_{n \notin N}$ is a basis for $\mathfrak{B}$ and P furnishes, by the closed graph theorem, an isomorphic mapping of the space spanned by the vectors e_n $(n \notin N)$ onto $\mathfrak{B}$. To construct such a set N consider a basis $q_i = \alpha_{i1} e_1 + \alpha_{i2} e_2 + \ldots$ $(i = 1, 2, \ldots, k)$ of the complementary subspace $(I - P) l^p$ of $\mathfrak{B}$. Then the matrix

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \cdots & \cdots \\ \alpha_{21} & \alpha_{22} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{k1} & \alpha_{k2} & \cdots & \cdots & \cdots \end{pmatrix}$$

is of rank k, i. e. it has a subdeterminant of order k different from 0, e. g.

$$D = \begin{vmatrix} \alpha_{1s_1} & \alpha_{1s_2} & \cdots & \alpha_{1s_k} \\ \alpha_{2s_1} & \alpha_{2s_2} & \cdots & \alpha_{2s_k} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{ks_1} & \alpha_{ks_2} & \cdots & \alpha_{ks_k} \end{vmatrix} \neq 0.$$

An easy calculation with determinants shows that the set $N = \{s_1, s_2, \ldots, s_k\}$ satisfies the above requirement.

So it is enough to deal with the case when the spectrum is contained in the interior of the unit disc.

2°. Suppose that T has no eigenvalue lying on the unit circle. In virtue of the well-known theorem of Gelfand, $\lim_{n\to\infty} \|T^n\|^{1/n} = \varrho < 1$, where ϱ denotes the spectral radius of T. Thus for any number σ with $\varrho < \sigma < 1$ there exists a natural number N such that $\|T^n\| \leqq \sigma^n$ for $n > N$. We define a new norm in l^p with

$$|x| = \left(\sum_{n=0}^{\infty} \|T^n x\|^p\right)^{1/p}.$$

It follows from

$$\|x\| \leqq |x| = \left(\sum_{n=0}^{\infty} \|T^n x\|^p\right)^{1/P} \leqq \|x\| \left(\sum_{n=0}^{N} \|T^n\|^p + \sum_{n=N+1}^{\infty} \sigma^{np}\right)^{1/p} = C\|x\| \qquad (C<\infty)$$

that this new norm is equivalent to the old one. Denote by $\boldsymbol{l}^p$ this new space and by $\boldsymbol{T}$ the operator on $\boldsymbol{l}^p$ defined by $\boldsymbol{T}x = Tx$. Evidently, $\boldsymbol{T}$ is a contraction. The two norms being equivalent, $\boldsymbol{T}$ is similar to T. Now, we have only to prove that $\boldsymbol{l}^p$ may be isometrically imbedded into l^p. Let Θ be a rearrangement of the double sequence into a simple sequence. Rearranging the double sequence $(x, Tx, T^2x, \ldots)$, where $x, Tx, T^2x, \ldots$ stand instead of the sequences corresponding to $x, Tx, T^2x, \ldots$, respectively, by means of Θ, we easily get an isometric transformation of $\boldsymbol{l}^p$ into l^p.

Remark. Suppose that there exists a decreasing sequence $\mathfrak{B}_1 \supset \mathfrak{B}_2 \supset \ldots$ of finite codimensional subspaces of l^p reducing T, with $\bigcap_{n=1}^{\infty} \mathfrak{B}_n = 0$. (For instance, this is the case if the point 0 is not an eigenvalue of T. Indeed, let $\{a_n\}_1^\infty$ be a decreasing sequence of positive numbers, converging to 0, such that the circles with radius a_n $(n=1, 2, \ldots)$ and centre 0 contain no eigenvalue of T. Then with the aid of the projections

$$P_n = -\frac{1}{2\pi i} \oint_{|z|=a_n} R_z\, dz$$

we get a sequence of invariant subspaces having the required property.) Then T is similar to a contraction defined on a *finite codimensional* subspace of l^p. The proof is based on the fact that if $T_1, T_2, \ldots$ denote the restrictions of T onto $\mathfrak{B}_1, \mathfrak{B}_2, \ldots$, respectively, then $\|T_n\| \to 0$ for $n \to \infty$. To prove this suppose the contrary, i.e. that there exist a positive number $\varepsilon > 0$ and a sequence $\{x_n\}_1^\infty$, with $\|x_n\| = 1$ and $x_n \in \mathfrak{B}_n$ $(n=1, 2, \ldots)$ such that

$$\|T_n x_n\| = \|Tx_n\| \geqq \varepsilon.$$

T being completely continuous, there exists a convergent subsequence $\{Tx_{i_n}\}_1^\infty$ of $\{Tx_n\}_1^\infty$. Let $\lim_{n\to\infty} Tx_{i_n} = x$. Then by $\|Tx_{i_n}\| \geqq \varepsilon$ it follows that $\|x\| \geqq \varepsilon$. On the other hand, in virtue of $x_i \in \mathfrak{B}_n$ and $\bigcap_{n=1}^{\infty} \mathfrak{B}_n = 0$, we get $x=0$, which is a contradiction.

Using this, the proof can be finished so as in the former case.

References

B. Sz.-Nagy, Completely continuous operators with uniformly bounded iterates, *Publ. Math. Inst. Hung. Acad. Sci.*, **4** (1959), 89—93.

L. Gehér, Über ähnliche lineare Transformationen in endlichdimensionalen Räumen, *Publ. Math. Inst. Hung. Acad. Sci.*, **4** (1959), 95—99.

A. Pełczyński, Projections in certain Banach spaces, *Studia Math.*, **19** (1960), 209—228.

(Received January 31, 1965)