On certain representations of real numbers and on sequences of equivalent events

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Dedicated to Professor L. Kalmár at the occasion of his 60th birthday

Introduction

In § 1 of this paper we shall deal with certain representations of real numbers. Let a_n $(n=0, 1, \dots)$ be an absolutely monotonic sequence of numbers, i. e. such that

(1) $$\Delta^k a_n > 0 \qquad (k=0, 1, \dots;\; n \geqq k),$$

where $\Delta^0 a_n = a_n$, $\Delta^1 a_n = \Delta a_n = a_{n-1} - a_n$ $(n \geqq 1)$, and

$$\Delta^k a_n = \Delta(\Delta^{k-1} a_n) \qquad (k \geqq 1, n \geqq k).$$

Let us also suppose that the sequence is *normed*, i. e.

(2) $$a_0 = 1,$$

and *regular*, i. e.

(3) $$a_n \to 0 \quad \text{and} \quad \Delta^n a_n \to 0 \qquad (n \to \infty).$$

Then every real $x \in (0, 1]$ admits a uniquely determined representation of the form

(4) $$x = \sum_{k=0}^{\infty} \Delta^k a_{n_k}$$

where the sequence of integers $1 \leqq n_0 < n_1 < n_2 < \dots$ depends on x.

This representation can also be written in the form

(5) $$x = \sum_{n=1}^{\infty} \varepsilon_n \Delta^{\varepsilon_1 + \dots + \varepsilon_{n-1}} a_n$$

where $\varepsilon_n = \varepsilon_n(x)$ equals 0 or 1; clearly $\varepsilon_n = 1$ if the number n occurs in the sequence n_k, and $\varepsilon_n = 0$ if not.

In § 2 we deal with the probability distribution of $n_k = n_k(x)$ provided that x is chosen at random with uniform distribution in (0, 1]. We shall show that the sequence of random variables n_k is then a Markov chain. In § 3 we deal with the joint probability distribution of the random variables ε_n $(n=1, 2, \dots)$ defined above. We prove that if A_n denotes the random event that $\varepsilon_n(x)=1$ then the events A_n

($n=1, 2, \ldots$) form a sequence of *equivalent* (symmetrically dependent) events, such that

(6) $$P(A_{m_1} A_{m_2} \ldots A_{m_k}) = \Delta^k a_k$$

for $1 \leqq m_1 < m_2 < \ldots < m_k$. (Here and in what follows $P(A)$ stands for the probability of the event A.)

In § 4 we show that the strong law of large numbers for equivalent events implies that for almost all x the limit

(7) $$\lim_{k \to \infty} \frac{k}{n_k(x)}$$

exists. (Previously in § 2 we obtain the weaker result that the distribution of k/n_k tends to a limit distribution.) On the other hand the above mentioned connection between equivalent events and the representation (4) or (5) leads to an effective construction of any sequence of equivalent events. A consequence of this is discussed in § 5. In § 6 we construct the corresponding measure preserving transformation to each sequence a_n, while in § 7 we discuss an example.

§ 1. Representation of real numbers by series of successive differences

We start with the following

Theorem 1. *Let a_n ($n=0, 1, \ldots$) be a normed, regular, absolutely monotonic sequence of real numbers. Then any real number $x \in (0, 1]$ can be represented in the form*

(1. 1) $$x = \sum_{k=0}^{\infty} \Delta^k a_{n_k}$$

where the increasing sequence of natural numbers n_k is uniquely determined by x.

Proof. Let n_0 be the first natural number such that

(1. 2) $$a_{n_0} < x;$$

such a number exists because $a_0 = 1$ and $a_n \to 0$. Let n_1 be the first natural number such that

(1. 3) $$a_{n_0} + \Delta a_{n_1} < x;$$

such a number n_1 exists because $\Delta a_n \to 0$. Moreover, by the definition of n_0 we have $a_{n_0} < x \leqq a_{n_0 - 1}$, i. e. $x - a_{n_0} \leqq \Delta a_{n_0}$, hence it follows $n_1 > n_0$. Similarly if $n_0, n_1, \ldots, n_r$ are already determined so that

(1. 4) $$\sum_{k=0}^{r} \Delta^k a_{n_k} < x \leqq \sum_{k=0}^{r-1} \Delta^k a_{n_k} + \Delta^r a_{n_r - 1},$$

let n_{r+1} be the least natural number such that

(1. 5) $$\sum_{k=0}^{r+1} \Delta^k a_{n_k} < x.$$

It follows from (1. 4) that

$$0 < x - \sum_{k=0}^{r} \Delta^k a_{n_k} \leqq \Delta^{r+1} a_{n_r} \tag{1.6}$$

which implies — as by supposition $\Delta^{r+1} a_n$ is decreasing in n — that $n_{r+1} > n_r$. Thus $n_{r+1} > r+1$. Therefore — using again the monotonicity of $\Delta^{r+1} a_n$ — it follows from (1. 6) that

$$0 < x - \sum_{k=0}^{r} \Delta^k a_{n_k} < \Delta^{r+1} a_{r+1}. \tag{1.7}$$

In view of the condition $\Delta^n a_n \to 0$ it follows that if the numbers n_k are determined by the algorithm described above, then (1. 1) holds. This proves Theorem 1.

Let us note that according to a well-known theorem of F. HAUSDORFF [1] every normed absolutely monotonic sequence can be represented in the form

$$a_n = \int_0^1 t^n \, dF(t) \tag{1.8}$$

where $F(t)$ is non-decreasing on the closed interval [0, 1], is continuous from the left in the interior, and such that $F(0)=0$ and $F(1)=1$. Evidently,

$$\lim_{n\to\infty} a_n = F(1) - F(1-0),$$

thus condition $\lim a_n = 0$ implies that $F(t)$ is continuous at $x=1$. We have further

$$\Delta^k a_n = \int_0^1 (1-t)^k t^{n-k} \, dF(t) \tag{1.9}$$

for $k=0, 1, \ldots$ and $n \geqq k$; thus in particular

$$\Delta^k a_k = \int_0^1 (1-t)^k \, dF(t). \tag{1.10}$$

Hence

$$\lim_{k\to\infty} \Delta^k a_k = F(+0).$$

Thus the condition of regularity $\lim \Delta^n a_n = 0$ implies that $F(t)$ is continuous at $t=0$. Thus every normed, regular, absolutely monotonic sequence a_n can be represented in the form (1. 8) where $F(t)$ is the distribution function of a probability distribution in the *open* interval (0, 1).

In view of formula (1. 9) the representation (1. 1) can be written in the form

$$x = \int_0^1 \left(\sum_{k=0}^{\infty} (1-t)^k t^{n_k - k} \right) dF(t). \tag{1.11}$$

Thus it follows that for every $x \in (0, 1]$ there is exactly one function $g(t)$ of the form

$$g(t) = \sum_{k=0}^{\infty} (1-t)^k t^{n_k - k}$$

with $1 \leqq n_0 < n_1 < n_2 < \ldots$, such that

$$x = \int_0^1 g(t)\, dF(t).$$

§ 2. Statistical theory of the difference-series representation of real numbers

Let x be a random variable, uniformly distributed in the interval (0, 1). Let us consider the representation of x in the form

$$x = \sum_{k=0}^{\infty} \Delta^k a_{n_k} \tag{2. 1}$$

where a_n is a given normed, regular, absolutely monotonic sequence. According to Theorem 1 the natural numbers $n_k = n_k(x)$ are uniquely determined by x; thus they are well defined random variables. We shall study now the probability laws governing the behaviour of these random variables. It is easy to see that if $n_1, \ldots, n_k$ are fixed, then x belongs to an interval of length $\Delta^{k+1} a_{n_k}$. It follows that denoting by $P(A|B)$ the conditional probability of the event A under condition B, we have

$$P(n_k = n \mid n_0 = m_0, n_1 = m_1, \ldots, n_{k-1} = m_{k-1}) = \frac{\Delta^{k+1} a_n}{\Delta^k a_{m_{k-1}}} \tag{2. 2a}$$

provided that $1 \leqq m_0 < m_1 < \ldots < m_{k-1} < n$. Thus the conditional distribution of n_k by given $n_0, \ldots, n_{k-1}$ depends on n_{k-1} only, that is the sequence of random variables n_k $(k = 0, 1, \ldots)$ is a Markov chain with the transition probabilities

$$P(n_k = n \mid n_{k-1} = m) = \frac{\Delta^{k+1} a_n}{\Delta^k a_m}. \tag{2. 2b}$$

As the probability on the right-hand side of (2. 2b) depends in general on k too, the Markov chain n_k is in general inhomogeneous. It is easy to see that the Markov chain is homogeneous if and only if $a_n = (1-p)^n$ $(n = 0, 1, \ldots)$ where $0 < p < 1$. In this particular case

$$P(n_k = n \mid n_{k-1} = m) = p(1-p)^{n-m-1}. \tag{2. 3}$$

This particular case corresponds to the representation of the real number x in the form

$$x = \sum_{k=0}^{\infty} p^k (1-p)^{n_k - k}. \tag{2. 4}$$

In this case if A_n denotes the event that n is contained in the sequence n_k then the events A_n $(n=1, 2, \ldots)$ are independent and each has the probability $P(A_n)=p$. Especially if $p=\frac{1}{2}$ the representation (2. 4) reduces to

(2. 5) $$x=\sum_{k=0}^{\infty}\frac{1}{2^{n_k}} \quad \text{where} \quad 1\leqq n_0<n_1<\ldots,$$

by other words to the representation of x in the binary number system.

Let us return to the random variables n_k in the general case. The unconditional distribution of n_k can be determined as follows: As mentioned above if $n_0, n_1, \ldots, n_k$ are fixed, then x belongs to an interval of length $\Delta^{k+1}a_{n_k}$. Now if only n_k is fixed, $n_k=n$, then the values of $n_0, n_1, \ldots, n_{k-1}$ can be chosen in $\binom{n-1}{k}$ different ways; thus we have

(2. 6) $$P(n_k=n)=\binom{n-1}{k}\Delta^{k+1}a_n.$$

Especially in the case when $a_n=(1-p)^n$, we have

(2. 7) $$P(n_k=n)=\binom{n-1}{k}p^{k+1}(1-p)^{n-k-1}$$

i. e. n_k-k-1 has a negative binomial distribution of order $k+1$.

In the general case it follows from (2. 6) and (1. 9) that

(2. 8) $$P(n_k=n)=\binom{n-1}{k}\int_0^1(1-t)^{k+1}t^{n-k-1}\,dF(t)$$

for $n\geqq k+1$.

The distribution (2. 8) may be called a *mixed negative binomial distribution of order* $k+1$. The characteristic function of $\frac{n_k}{k+1}$ is

(2. 9) $$M\left(e^{\frac{iun_k}{k+1}}\right)=e^{iu}\int_0^1\left(\frac{1-t}{1-te^{\frac{iu}{k+1}}}\right)^{k+1}dF(t).$$

(Here and in what follows M stands for "expectation".

We obtain by passing to the limit

(2. 10) $$\lim_{k\to\infty}M\left(e^{\frac{iun_k}{k+1}}\right)=\int_0^1 e^{\frac{iu}{1-t}}\,dF(t).$$

It follows that the probability distribution of $\frac{k+1}{n_k}$ tends to the distribution having the distribution function $1-F(1-z)$

(2. 11) $$\lim_{k\to\infty}P\left(\frac{k}{n_k}\leqq z\right)=1-F(1-z).$$

In the special case $a_n = (1-p)^n$ we have

$$F(t) = \begin{cases} 0 & \text{if} \quad t \leqq 1-p \\ 1 & \text{if} \quad t > 1-p, \end{cases}$$

thus (2. 11) implies that in this case k/n_k tends in probability to p. This is of course well known, because

$$\frac{k}{n_k} = \frac{\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{n_k}}{n_k} \tag{2. 12}$$

and the (weak) law of large numbers applies to the independent random variables ε_n, each having the expectation p.

We shall show in the next paragraph that much more is true than (2. 11): not only does the distribution of k/n_k tend for $k \to \infty$ to the limit distribution $1-F(1-z)$, but the random variables k/n_k themselves tend for $k \to \infty$ with probability 1 to a random variable $\varkappa$ having the distribution function $1-F(1-z)$.

Using the formula (2. 8) we can of course compute all the moments of n_k. Especially we have

$$M(n_k) = (k+1) \int_0^1 \frac{dF(t)}{1-t}. \tag{2. 13}$$

Thus the expectation of $\frac{n_k}{k+1}$ does not depend on k; it is finite if and only if the integral on the right of (2. 13) is convergent, otherwise it is equal to $+\infty$.

§ 3. Connection with the theory of equivalent events

Let A_n denote the event that the natural number n is contained in the sequence $n_k(x)$ where x is a random variable, uniforly distributed in the interval (0, 1). We have evidently

$$P(A_n) = \sum_{k=0}^{n-1} P(n_k = n). \tag{3. 1}$$

It follows from (2. 8) that

$$P(A_n) = \int_0^1 \left(\sum_{k=0}^{n-1} \binom{n-1}{k} (1-t)^{k+1} t^{n-k-1} dF(t) = \int_0^1 (1-t)\, dF(t) \tag{3. 2}$$

for $n = 1, 2, \ldots$. Before proceeding further we have to compute the r-step transition probabilities of the Markov-chain n_k. Clearly we have for $r \geqq 2$ and $n \geqq m+r$

$$P(n_{k+r} = n \mid n_k = m) = \frac{\Delta^{k+r+1} a_n}{\Delta^{k+1} a_m} \cdot \sum_{m < m_1 < \ldots < m_{r-1} < n} 1, \tag{3. 3}$$

thus

$$P(n_{k+r} = n \mid n_k = m) = \binom{n-m-1}{r-1} \frac{\Delta^{k+r+1} a_n}{\Delta^{k+1} a_m}. \tag{3. 4}$$

It follows that for $m<n$

$$P(n_{k+r}=n,\, n_k=m)=\binom{n-m-1}{r-1}\binom{m-1}{k}\Delta^{k+r+1}a_n. \tag{3. 5}$$

Thus we have

$$P(A_m A_n)=\sum_{k=0}^{m-1}\sum_{l=k+1}^{k+n-m}\binom{n-m-1}{l-k-1}\binom{m-1}{k}\Delta^{l+1}a_n. \tag{3. 6}$$

Taking (1. 9) into account we obtain for $1\leqq m<n$

$$P(A_m A_n)=\int_0^1 (1-t)^2\, dF(t). \tag{3. 7}$$

We shall show now that for any $r\geqq 1$ and for $1\leqq m_1<m_2<\ldots<m_r$ we have

$$P(A_{m_1}A_{m_2}\ldots A_{m_r})=\int_0^1 (1-t)^r\, dF(t). \tag{3. 8}$$

The proof is essentially the same as for $r=2$. We obtain in the same way as (3. 5) was shown — using that n_k is a Markov chain — that for $k_1<k_2<\ldots<k_r$, $m_1<m_2<\ldots<m_r$

$$P(n_{k_1}=m_1,\ldots,n_{k_r}=m_r)=\binom{m_1-1}{k_1}\prod_{j=1}^{r-1}\binom{m_{j+1}-m_j-1}{k_{j+1}-k_j-1}\cdot\Delta^{k_r+1}a_{m_r}. \tag{3. 9}$$

Of course the probability (3. 9) is positive only if $m_1\geqq k_1+1$ and $m_{j+1}-m_j\geqq k_{j+1}-k_j$ $(j=1,2,\ldots,r-1)$. From (3. 9) one obtains (3. 8) by means of the identity

$$P(A_{m_1}A_{m_2}\ldots A_{m_r})=\sum_{k_1<k_2<\ldots<k_r}P(n_{k_1}=m_1,\ldots,n_{k_r}=m_r). \tag{3. 10}$$

As clearly

$$\int_0^1 (1-t)^r\, dF(t)=\Delta^r a_r \tag{3. 11}$$

we have proved the following

Theorem 2. *Let A_n denote the event that the natural number n is contained in the sequence $\{n_k(x)\}$ defined by Theorem* 1, *where x is a random variable uniformly distributed in the interval* (0, 1). *Then the events A_n $(n=1,2,\ldots)$ are equivalent, and one has for $1\leqq m_1<m_2<\ldots<m_r$ $(r=1,2,\ldots)$*

$$P(A_{m_1}A_{m_2}\ldots A_{m_r})=\Delta^r a_r. \tag{3. 12}$$

Remark. Note that the sequence $W_r=\Delta^r a_r$ is absolutely monotonic too, because setting

$$G(t)=1-F(1-t+0) \tag{3. 13}$$

we have

$$w_r=\int_0^1 t^r\, dG(t). \tag{3. 14}$$

It is easy to see also that

$$\Delta^h w_r = \Delta^{r-h} a_r. \tag{3.15}$$

Conversely let us be given a sequence of equivalent events B_n $(n=1, 2, \dots)$ in a probability space $[\Omega, \mathcal{A}, P]$ where Ω is a non empty set, the generic element of which will be denoted by ω, $\mathcal{A}$ is a σ-algebra of subsets of Ω and P a probability measure on $\mathcal{A}$. It is known (see [2], [3]) that there exists an $\mathcal{A}$-measurable function $\beta=\beta(\omega)$ on Ω — called the density of the sequence of events B_n — such that $0 \leqq \beta \leqq 1$ and for $r=1, 2, \dots$ and $m_1 < m_2 < \dots < m_r$ one has

$$P(B_{m_1} B_{m_2} \dots B_{m_r}) = \int_\Omega \beta^r \, dP. \tag{3.16}$$

Let us consider first the case when $\beta=1$ on a set B of positive probability. Let β_n denote the indicator of the set B_n. It was shown in [3] that if $n_1 < n_2 < \dots < n_k$ $m_1 < m_2 < \dots < m$ and $n_i \neq m_j$, then

$$P(B_{n_1} B_{n_2} \dots B_{n_k} B_{m_1} B_{m_2} \dots B_{m_l}) = \int_\Omega \beta^k \beta_{m_1} \beta_{m_2} \dots \beta_{m_l} \, dP. \tag{3.17}$$

It follows that

$$P\left(\prod_{n=r}^{\infty} B_n\right) = \int_B \prod_{r \leqq j < s} \beta_j \cdot dP. \tag{3.18}$$

As (3.18) holds for $s=r$ too (the empty product is equal to 1), we have

$$P\left(\prod_{n=r}^{\infty} B_n\right) = P(B).$$

Thus we obtain, putting $\prod_{n=1}^{\infty} B_n = B^*$,

$$P(B) = P(B^*) = P(BB^*).$$

This implies that the sets B and B^* are identical up to a set of P-measure 0. Let us denote now by $\bar{B}$ the complementary event of B, i. e. $\bar{B} = \Omega - B$. It follows that the events $A_n = \bar{B} B_n$ also are equivalent, and have the density α defined as follows:

$$\alpha(\omega) = \begin{cases} \beta(\omega) & \text{if } \omega \in \bar{B}, \\ 0 & \text{if } \omega \in B. \end{cases}$$

As a matter of fact we have

$$P(A_{m_1} A_{m_2} \dots A_{m_r}) = P(B_{m_1} B_{m_2} \dots B_{m_r}) - P(B) = \int_\Omega \alpha^k \, dP.$$

As $P(\alpha=1)=0$, we have shown that without restriction of generality one can suppose that $P(\beta=1)=0$.

Similarly one can suppose without restricting the generality that $P(\beta=0)=0$. As a matter of fact if C denotes the set on which $\beta=0$ and $0 < P(C) < 1$ then the set C is disjoint to all the sets B_n (up to a set of probability 0) and thus instead of the probability space $[\Omega, \mathcal{A}, P]$ we may consider the space $[\Omega, \mathcal{A}, P^*]$ where

$P^*(A) = \frac{P(A\bar{C})}{P(\bar{C})}$ and the events B_n will be equivalent with respect to this probability space too, with the same density β. Thus the case of an arbitrary sequence of equivalent events can be reduced to a sequence of equivalent events the density β of which is such that $P(\beta=0)=P(\beta=1)=0$. Let us call such a sequence a *regular* sequence of equivalent events. If A_n is a regular sequence of equivalent events with density β and if we put

$$w_k = P(A_{n_1}A_{n_2}\dots A_{n_k}) = \int_\Omega \beta^k \, dP$$

then clearly we have

$$\lim_{k\to\infty} w_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \Delta^k w_k = 0.$$

Putting $a_k = \Delta^k w_k$, clearly $w_k = \Delta^k a_k$ and the sequence a_k is a normed regular absolutely monotonic sequence. Thus the events $\{A_n\}$ can be realized as the events connected with the representation of the random real number x uniformly distributed in (0, 1) in the form (1. 1), so that the event A_n is identified with the event that n is contained in the sequence n_k.

§ 4. The strong law of large numbers for the Markov chain n_k

We first give — to make this paper self-contained — a short proof of the following known result:[2])

Theorem 3. *Let A_n be an arbitrary sequence of equivalent events; let α_n denote the indicator of A_n and α the density of the sequence A_n. Then we have*

$$P\left(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \alpha_k = \alpha\right) = 1. \tag{4. 1}$$

Proof. Let us consider the random variables

$$\delta_k = \alpha_k - \alpha \tag{4. 2}$$

and let us put for $k_1 < k_2 < \dots < k_r$, $r = 1, 2, \dots$

$$P(A_{k_1}A_{k_2}\dots A_{k_r}) = w_r. \tag{4. 3}$$

It follows from (3. 17) that

$$M(\delta_{k_1}\delta_{k_2}\delta_{k_3}\delta_{k_4}) = \begin{cases} A & \text{if } k_1 = k_2 = k_3 = k_4, \\ B & \text{if } k_1 = k_2 \text{ and } k_3 = k_4 \neq k_1, \\ & \text{or if } k_1 = k_3 \text{ and } k_2 = k_4 \neq k_1, \\ & \text{or if } k_1 = k_4 \text{ and } k_2 = k_3 \neq k_1, \\ 0 & \text{otherwise,} \end{cases} \tag{4. 4}$$

[2]) Theorem 3 can also be deduced from BIRKHOFF's ergodic theorem.

where

(4. 5) $$A = w_1 - 4w_2 + 6w_3 - 3w_4$$

and

(4. 6) $$B = w_2 - 2w_3 + w_4.$$

This implies

(4. 7) $$M\left(\left(\frac{\delta_1 + \dots + \delta_n}{n}\right)^4\right) = 0\left(\frac{1}{n^2}\right).$$

Thus the series

(4. 8) $$\sum_{n=1}^{\infty} \left(\frac{\delta_1 + \dots + \delta_n}{n}\right)^4$$

is convergent with probability 1 and therefore (4. 1) holds.

In view of (2. 11) and the results of § 3 this implies that the following theorem holds:

Theorem 4. *If the sequence $n_k(x)$ is defined according to Theorem 1 then the limit*

(4. 9) $$\lim_{k \to \infty} \frac{k}{n_k(x)} = \varkappa(x)$$

exists for almost all x in $(0, 1)$; denoting by $\mu(A)$ the Lebesgue measure of the set A one has

(4. 10) $$\mu(\varkappa(x) \leqq y) = 1 - F(1-y) \quad \textit{for} \quad 0 \leqq y \leqq 1.$$

§ 5. Consequences for equivalent events

In the preceding § we applied the theory of equivalent sequences of events to prove the existence almost everywhere of the limit $\lim\limits_{k \to \infty} \dfrac{k}{n_k(x)}$. Conversely, our results lead to the proof of a property of equivalent events which seems not to be noticed up to now. This is expressed by

Theorem 5. *Let A_n $(n = 1, 2, \dots)$ be a regular sequence of equivalent events. Let us set*

$$P(A_{n_1} A_{n_2} \dots A_{n_k}) = w_k \qquad (n_1 < n_2 < \dots < n_k;\ k = 1, 2, \dots).$$

Denote by α_n the indicator of the event A_n and define the random variables ν_k as follows: ν_k is the least value of n such that $\alpha_1 + \alpha_2 + \dots + \alpha_n = k$. By other words, ν_k denotes the index of the k-th event in the sequence of events A_n $(n = 1, 2, \dots)$ which takes place. Then the random variables ν_k form a Markov chain with the transition probabilities

(5. 1) $$P(\nu_{k+1} = n \mid \nu_k = m) = \frac{\Delta^{n-k-1} w_n}{\Delta^{m-k} w_m}.$$

§ 6. The measure preserving transformation corresponding to a series of successive differences

To every representation (7) — i. e. to every normed, regular, absolutely monotonic sequence $\{a_n\}$ — there corresponds a measure preserving transformation T of the interval (0, 1) defined as follows: If

$$x = \sum_{k=0}^{\infty} \Delta^k a_{n_k(x)} \tag{6. 1}$$

then

$$n_k(Tx) = n_{k+\varepsilon_1(x)}(x) = 1 \qquad (k=0, 1, \ldots). \tag{6. 2}$$

Clearly $1 \leqq n_0(Tx)$, because if $n_0(x)=1$ then $\varepsilon_1(x)=1$ and thus $n_0(Tx) = n_1(x)-1 \geqq 1$ and if $n_0(x) \geqq 2$ then $n_0(Tx) = n_0(x)-1 \geqq 1$; the inequality $n_{k+1}(Tx) > n_k(Tx)$ is evident. The inverse transformation $T^{-1}y$ can be defined as follows: $T^{-1}y$ is two-valued, namely if

$$y = \sum_{k=0}^{\infty} \Delta^k a_{n_k} \tag{6. 3}$$

then $T^{-1}y$ has the two values x_1 and x_2 where

$$x_1 = \sum_{k=0}^{\infty} \Delta^k a_{n_k+1}, \quad x_2 = a_1 + \sum_{k=1}^{\infty} \Delta^k a_{n_{k-1}+1}. \tag{6. 4}$$

Clearly if y belongs to the interval I_r defined by fixing the values of $n_0, n_1, \ldots, n_r$ in (6. 3) ($1 \leqq n_0 < n_1 < \ldots < n_r$) and having the length $\Delta^{r+1}a_{n_r}$ then x_1 belongs to an interval I_r' of length $\Delta^{r+1}a_{n_r+1}$ and x_2 to an interval I_r'' of length $\Delta^{r+2}a_{n_r+1}$. As

$$\Delta^{r+1} a_{n_r+1} + \Delta^{r+2} a_{n_r+1} = \Delta^{r+1} a_{n_r} \tag{6. 5}$$

it follows that denoting by $\mu(A)$ the Lebesgue measure of the set A one has

$$\mu(T^{-1}I_r) = \mu(I_r') + \mu(I_r'') = \mu(I_r). \tag{6. 6}$$

It follows from (6. 6) that Tx is measure preserving.

The transformation Tx can of course also be defined by

$$\varepsilon_k(Tx) = \varepsilon_{k+1}(x) \qquad (k=0, 1, \ldots). \tag{6. 7}$$

Thus T is equivalent to the shift transformation in the sequence-space $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots)$.

It is easy to see that the transformation T is ergodic if and only if $a_n = q^n$ with $0<q<1$, because it follows from (6. 7) and Theorem 4 that

$$\varkappa(Tx) = \varkappa(x) \tag{6. 8}$$

and thus each level set of $\varkappa$ is an invariant set of T; thus T is ergodic if and only if $\varkappa$ is constant almost everywhere, i. e. if $a_n = q^n$. Especially in the case $a_n = 2^{-n}$, T is the well known transformation $Tx = (2x)$ where (Z) denotes the fractional part of Z.

§ 7. An example

As an example let us consider the sequence $a_n = \frac{1}{n+1}$ $(n = 0, 1, 2, \ldots)$. Evidently,

$$\Delta^k a_n = \frac{1}{(n+1)\binom{n}{k}}, \tag{7.1}$$

hence

$$\Delta^n a_n = a_n = \frac{1}{n+1}. \tag{7.2}$$

Thus a_n is a normed, regular, absolutely monotonic sequence. Theorem 1 asserts for this case that every real number x with $0 < x \leqq 1$ has a unique representation of the form

$$x = \sum_{k=0}^{\infty} \frac{1}{(n_k+1)\binom{n_k}{k}} \tag{7.3}$$

where the n_k are integers, $1 \leqq n_1 < n_2 < \ldots$. The function $F(t)$ figuring in (1. 8) is in this example equal to t $(0 \leqq t \leqq 1)$. The transition probabilities (2. 2b) are in this example

$$P(n_k = n \mid n_{k-1} = m) = \frac{(m+1)\binom{m}{k}}{(n+1)\binom{n}{k+1}} \tag{7.4}$$

and the distribution of n_k is given by

$$P(n_k = n) = \frac{k+1}{n(n+1)} \quad \text{for} \quad n \geqq k+1. \tag{7.5}$$

Thus the random variables n_k have an infinite expectation. The equivalent events A_n can in this case be interpreted as the events of the following Pólya urn model: Let us consider an urn containing one white and one red ball. Let us draw one of the balls at random (each having the probability $\frac{1}{2}$ to be drawn) and put it back into the urn together with another ball of the same colour, then draw another ball from the urn which now contains 3 balls, each ball having the same probability to be drawn, put it back together with another ball of the same colour and continue this procedure indefinitely. Let A_n denote the event that at the n-th occasion a red ball has been drawn from the urn. Clearly in this interpretation a red ball is drawn the $k+1$-st time at the n_k-th drawing; the limit $\varkappa$ of k/n_k is in this case of course uniformly distributed in the interval (0, 1).

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Über ein Problem von S. B. Stetschkin

Von K. TANDORI in Szeged

Herrn Professor Ladislaus Kalmár zum 60. Geburtstag gewidmet

In dieser Note werden wir den folgenden Satz beweisen, der ein Problem von S. B. STETSCHKIN[1]) im positiven Sinne beantwortet.

Satz. *Es gibt eine trigonometrische Reihe*

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$$

mit $a_k\to 0$, $b_k\to 0$ $(k\to\infty)$ *derart, daß für ihre Partialsummen* $s_n(x)$ *überall gilt:* $\underline{\lim}\, s_n(x)<\overline{\lim}\, s_n(x)$ $(n\to\infty)$.

Hilfssätze

Im folgenden bezeichnen wir mit $c_1, c_2, \ldots$ positive, absolute Konstanten.

Hilfssatz I. *Es seien* a *und* N *gegebene natürliche Zahlen. Dann gibt es ein trigonometrisches Polynom*

$$P(x)=P(a,N;x)=\sum_{k=\nu(a,N)}^{\mu(a,N)}(a_k\cos kx+b_k\sin kx)\qquad (N<\nu(a,N)<\mu(a,N))$$

mit den folgenden Eigenschaften: $|a_k|\leqq c_1$, $|b_k|\leqq c_1$,

$$|P(x)|\leqq c_2\max\left\{\frac{1}{a}\left(\frac{1}{x^2}+\frac{1}{(2\pi-x)^2}\right),a\right\}\qquad(-\infty<x<\infty),$$

und es gibt für jedes $x\in[-\pi/128a,\pi/128a]$ *Indizes* $p=p(x)$, $q=q(x)$ *derart, daß*

$$s_p(x)\geqq c_3a,\quad \textit{und}\quad s_q(x)\leqq -c_3a\qquad (c_3\leqq 1),$$

wobei $s_n(x)$ *die* n*-te Partialsumme von* $P(x)$ *bezeichnet.*

[1]) Siehe П. Л. Ульянов, Решенные и нерешенные проблемы теории тригонометрических и ортогональных рядов, *Успехи матем. наук*, **19 : 1 (115)** (1964), 3—69.

Wir setzen endlich

$$Q(x) = \sum_{i=0}^{68\varrho-1} \left(\frac{4c_8}{c_3}\right)^i Q_1\left(x-2i\frac{\pi}{128a}\right).$$

Auf Grund von (4) und (5) ist offensichtlich, daß $Q(x)$ allen Bedingungen des Hilfssatzes II genügt.

Beweis des Satzes

Es sei $(c_4 \leqq) a_1 < \ldots < a_i < \ldots$ eine Folge von natürlichen Zahlen, für die die Ungleichung

$$c_6(a_1 + \ldots + a_i) \leqq \frac{c_7}{2} a_{i+1} \qquad (i=1, 2, \ldots) \tag{6}$$

besteht und es sei $M_i = m(a_{i-1}^2, M_{i-1})$ $(i=1, 2, \ldots; M_0 = 0)$. Wir setzen

$$\sum_{i=1}^{\infty} \frac{1}{a_i} Q(a_i^2, m(a_{i-1}^2, M_{i-1}); x).$$

Auf Grund des Hilfssatzes II und (6) ist es klar, daß die Koeffizienten dieser trigonometrischen Reihe nach 0 streben und überall gilt:

$$\underline{\lim}\, s_n(x) = -\infty, \quad \overline{\lim}\, s_n(x) = \infty.$$

Damit haben wir unseren Satz bewiesen.

(Eingegangen am 11. März 1964)