General theory of summability. I

By RICHARD JAJTE in Łódź (Poland)

In this paper we give the general definition of a summability method of functions defined on a σ-bicompact space. It seems that this definition includes all the cases so far considered, in particular the broad classes of summability methods of functions considered by K. KNOPP and his pupils. Owing to the generality of our methods we may give, on the one hand, a uniform theory and on the other we can find more applications of this theory. The present paper includes the fundamental theorems concerning the most important properties of the methods considered. These theorems constitute a generalization of the well-known theorems for matrix-transformations (the theorems of TOEPLITZ, KNOPP, MAZUR—ORLICZ, HENSTOCK, STEINHAUS; see references at the end of the paper). In a subsequent paper we shall consider some special classes of methods and some applications of the general theory in other branches.

§ 1

Definition 1. A locally bicompact space is called a σ-bicompact space if it is the sum of a sequence of bicompact sets.[1]) Let X be a σ-bicompact Hausdorff space[2]). By $\Omega(X)$ we denote the set of real functions defined and continuous on X with bicompact supports and by $\Omega^+(X)$ the set of functions continuous and non-negative on X with bicompact supports. Let $J(f)$ be a distributive functional defined on $\Omega(X)$ and non-negative on $\Omega^+(X)$. Let μ and $\int f(x)\,d\mu$ denote the Lebesgue measure and integral generated by the functional $J(f)$.[3])

Definition 2. A function $f(x)$ defined on X shall be called convergent in ∞ to the number ξ if for any $\varepsilon>0$ the set

$$\{x: |f(x)-\xi| \geqq \varepsilon\}$$

is contained in a bicompact set.

[1]) A topological space is called a locally bicompact space if every point $x \in X$ possesses a neighbourhood with a bicompact closure and if the whole space X is not bicompact.

[2]) The space X is considered as fixed in all the further considerations.

[3]) See [7], § 6.

Definition 3. Let $S=\{S_\tau\}$ $(0\leqq\tau<\infty)$ be a family of bicompact subsets of the space X such that $S_{\tau'}\subseteqq S_{\tau''}$ if $\tau'<\tau''$ and $\bigcup\limits_{0\leqq\tau<\infty} S_\tau=X$. The improper integral

$$(S)\int f(x)\,d\mu$$

exists and is equal to α if $\int\limits_{S_\tau} f(x)\,d\mu$ exists for $0\leqq\tau<\infty$ and $\lim\limits_{\tau\to\infty}\int\limits_{S_\tau} f(x)\,d\mu=\alpha$.

Definition 4. Let $\Phi=\{\Phi(t,x)\}$ $(t_0\leqq t<T\leqq\infty)$ denote a family of continuous functions defined on X. A function $f(x)$ defined on X is said to be summable by the method $M=M(\mu,S,\Phi)$ to the number ξ if

(a) the integrals $(S)\int\Phi(t,x)f(x)\,d\mu$ exist for every $t\in[t_0,T)$, and

(b) the limit $\lim\limits_{t\to T-}(S)\int\Phi(t,x)f(x)\,d\mu=\xi$ exists.

Evidently, by specifying the space X and the classes S and Φ we can obtain some well-known classes of summability methods for number sequences or for functions defined on the half line (see e. g. [3]). The case considered here is, however, much more general for it includes even the summability of functions defined in non-metric spaces. Let us observe that the term "summability of functions in ∞" has a formal character, since X can stand e.g. for a closed circle its center excluded. Then the center z_0 of this circle shall play the part of ∞ and we obtain a summability method of a function at the point z_0.

Definition 5. A function defined on X is said to be locally bounded if for every point $x_0\in X$ there exists a neighbourhood $U(x_0)$ in which this function is bounded.

For what follows the following conditions are important.

$$(c_1)\qquad \overline{\lim_{t\to T-}}\,(S)\int|\Phi(t,x)|\,d\mu<\infty\,;$$

$$(c_2)\qquad \lim_{t\to T-}(S)\int\Phi(t,x)\,d\mu=\alpha\,;$$

$$(c_2')\qquad \lim_{t\to T-}(S)\int\Phi(t,x)\,d\mu=1\,;$$

(c_3) $\lim\limits_{t\to T-}\int\limits_A\Phi(t,x)\,d\mu=\alpha(A)$ exists for every measurable set A contained in some bicompact set;

(c_3') $\lim\limits_{t\to T-}\int\limits_A\Phi(t,x)\,d\mu=0$ for every measurable set A contained in some bicompact set.

Definition 6. We call a method M a convergence preserving method (a convergence preserving method for null functions) if it sums all μ-measurable locally bounded and in ∞ convergent (convergent to zero) functions. We call a method M permanent (permanent for null functions) if it sums all the measurable, locally bounded and in ∞ convergent (convergent to zero) functions to their ordinary limits.

Theorem 1. *A method M is convergence preserving for null functions if and only if the conditions* (c_1) *and* (c_3) *are satisfied.*

Proof. The necessity of condition (c_3) is evident. To prove the necessity of condition (c_1) we complete the space X with the point x_∞ into the space X_∞. The set of functions continuous on X and having a limit in ∞ equal to zero (we denote it by $C_0(X_\infty)$) is a Banach space with the norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

The functional

$$A_{t,\tau}(f) = \int_{S_\tau} \Phi(t, x) f(x)\, d\mu$$

is linear (i. e. additive and continuous) in $C_0(X_\infty)$. We shall show that its norm is equal to

$$\int_{S_\tau} |\Phi(t, x)|\, d\mu$$

Since the set S_τ is bicompact, we have $\mu(S_\tau) = K < \infty$ and there exists for $\eta > 0$ an open set $U \supset S_\tau$ with a bicompact closure such that $\mu(U - S_\tau) < \eta$. We denote by A_1 and A_2 the subsets of the set S_τ in which $\Phi(t, x) \geqq \varepsilon/K$, $\Phi(t, x) \leqq -\varepsilon/K$, respectively. The sets A_i, in view of the continuity of the function $\Phi(t, x)$, are bicompact. In virtue of the well-known theorem of Urysohn (see e. g. [7], p. 34) there exists a continuous function $u(x)$ defined on the whole space X and satisfying the conditions

(i) $0 \leqq u(x) \leqq 1$,

(ii) $u(x) = 1$ for $x \in A_1$ and $u(x) = 0$ for $x \in A_2$.

By the same theorem of Urysohn there exists a continuous function $v(x)$ defined on the whole space X and satisfying the conditions

(j) $0 \leqq v(x) \leqq 1$,

(jj) $v(x) = 1$ for $x \in S_\tau$ and $v(x) = 0$ for $x \in X_\infty - U$.

Let us put $\psi(x) = [2u(x) - 1] \cdot v(x)$. We have $|\psi(x)| \leqq 1$ and

$$\psi(x) = \begin{cases} 1 & \text{for } x \in A_1, \\ -1 & \text{for } x \in A_2, \\ 0 & \text{for } x \notin U. \end{cases}$$

Obviously $\psi \in C_0(X_\infty)$. Moreover we have

$$A_{t,\tau}(\psi) \geqq \int_{S_\tau} |\Phi(t, x)|\, d\mu - 2\varepsilon.$$

Since $\|\psi\|_\infty = 1$ this implies

$$\|A_{t,\tau}\| = \int_{S_\tau} |\Phi(t, x)|\, d\mu.$$

The functional

$$A_t(f) = \lim_{\tau\to\infty} A_{t,\tau}(f)$$

is defined in $C_0(X_\infty)$, therefore according to the Banach—Steinhaus theorem ([1], Theorem 5, p. 80)

$$\sup_{0\leq\tau<\infty} \|A_{t,\tau}\| = (S)\int |\Phi(t,x)|\,d\mu < \infty.$$

Choose a set S_τ and an open set $U\supset S_\tau$ such that

$$\int_{U-S_\tau} |\Phi(t,x)|\,d\mu < \varepsilon.$$

We have

$$A_t(\psi) = A_{t,\tau}(\psi) + \int_{U-S_\tau} \Phi(t,x)\psi(x)\,d\mu \geqq \|A_{t,\tau}\| - 2\varepsilon + \int_{U-S_\tau} \Phi(t,x)\psi(x)\,d\mu.$$

But

$$\int_{U-S_\tau} \Phi(t,x)\psi(x)\,d\mu > -\int_{U-S_\tau} |\Phi(t,x)|\,d\mu > -\varepsilon.$$

Therefore

$$A_t(\psi) \geqq \|A_{t,\tau}\| - 3\varepsilon.$$

Passing to the limit as $\varepsilon\to 0$ and $\tau\to\infty$ we obtain immediately

$$\|A_t\| = (S)\int |\Phi(t,x)|\,d\mu.$$

Since by assumption $\lim\limits_{t\to T-} A_t(f)$ exists for every $f\in C_0(X_\infty)$ so by the Banach—Steinhaus theorem we have

$$\overline{\lim_{t\to T-}}\, \|A_t\| < \infty,$$

which ends the proof of the necessity of condition (c_1).

Now we shall prove that the conditions (c_1) and (c_3) are sufficient.

Let $f(x)$ be a measurable and locally bounded function and $\lim\limits_{x\to\infty} f(x)=0$. We may assume (considering, if necessary, the function $Cf(x)$) that $|f(x)|\leqq 1$. We put

$$\delta(Z) = \lim_{t\to T-} \int_Z \Phi(t,x)\,d\mu.$$

By condition (c_3) this limit exists for any measurable set contained in a bicompact set. Let ε_n be a sequence decreasing to zero. Then there exists an increasing sequence of bicompact sets $\{V_n\}$ such that for $x\notin V_n$ we have $|f(x)|<\varepsilon_n$. Let

$$Z_k^{(n)} = \{x\in V_n\colon k/2^n \leqq f(x) < k+1/2^n\} \quad \text{for} \quad k=-2^n, -2^n+1, \ldots, 2^n-2,$$

$$Z_{2^n-1}^{(n)} = \{x\in V_n\colon 1-1/2^n \leqq f(x) \leqq 1\}.$$

Put

$$\delta_n(f) = 2^{-n} \sum_{k=-2^n}^{2^n-1} k\delta(Z_k^{(n)}).$$

Now we shall show that the limit $\delta=\lim_{n\to\infty}\delta_n(f)$ exists. Let $m>n$. We have

$$|\delta_m(f)-\delta_n(f)|=\left|2^{-m}\sum_{k=-2^m}^{2^m-1}k\delta(Z_k^{(m)})-2^{-n}\sum_{k=-2^n}^{2^n-1}k\delta(Z_k^{(n)})\right|\leqq$$

$$\leqq\left|\sum_{k=-2^m}^{2^m-1}k/2^m\delta(V_n\cap Z_k^{(m)})-\sum_{k=-2^n}^{2^n-1}k/2^n\delta(Z_k^{(n)})\right|+N\cdot\varepsilon_n,$$

where

$$N=\sup_{t_0\leqq t<T}(S)\int|\Phi(t,x)|\,d\mu.\text{[4]}$$

We observe that the sets $V_n\cap Z_k^{(m)}$ are formed by dividing the sets $Z_k^{(n)}$, thus the sum under the modulus-sign on the right hand side of the inequality can be written as follows

$$\sum_{k=-2^m}^{2^m-1}\delta(V_n\cap Z_k^{(m)})[k/2^m-\alpha_k]\quad\text{with}\quad|\alpha_k-k/2^m|<2^{-n}.$$

Hence

$$|\delta_m(f)-\delta_n(f)|\leqq N\cdot\varepsilon_n+2^{-n}\cdot\sum_{k=-2^m}^{2^m-1}|\delta(V_n\cap Z_k^{(m)}|\leqq N(\varepsilon_n+2^{-n})\to 0$$

for $n\to\infty$, which implies convergence of the sequence $\delta_n(f)$. We shall prove that $M\text{-}\lim_{x\to\infty}f(x)=\delta$. Let $\eta>0$ be given. Putting

$$g_n(x)=\begin{cases}k/2^n & \text{for } x\in Z_k^{(n)},\\ 0 & \text{for } x\notin V_n,\end{cases}$$

we have

$$\left|(S)\int\Phi(t,x)f(x)\,d\mu-\delta\right|\leqq\left|\int_{V_n}\Phi(t,x)[f(x)-g_n(x)]\,d\mu\right|+$$

$$+\left|\int_{V_n}\Phi(t,x)g_n(x)\,d\mu-\delta\right|+\left|(S)\int_{X-V_n}\Phi(t,x)f(x)\,d\mu\right|.$$

We fix n such that $\max(\varepsilon_nN,N2^{-n},|\delta_n(f)-\delta|)<\eta/4$. Then we have

$$\left|(S)\int\Phi(t,x)f(x)\,d\mu-\delta\right|\leqq N2^{-n}+\left|\int_{V_n}\Phi(t,x)g_n(x)\,d\mu-\delta_n(f)\right|+$$

$$+|\delta_n(f)-\delta|+N\varepsilon_n<3/4\eta+\left|\sum_{k=-2^n}^{2^n-1}k/2^n\int_{Z_k^{(n)}}\Phi(t,x)\,d\mu-\delta_n(f)\right|<\eta$$

for t sufficiently large, which ends the proof of our theorem.

Theorem 2. *A method M is convergence preserving if and only if the conditions* (c_1), (c_2) *and* (c_3) *are satisfied.*

[4]) Without diminishing the generality of our considerations we may assume that $\sup_{0\leqq t<T}(S)\int|\Phi(t,x)|\,d\mu<\infty$.

Proof. The necessity of conditions (c_2) and (c_3) is evident, while necessity of the condition (c_1) follows from the foregoing theorem. The sufficiency of the conditions (c_1), (c_2) and (c_3) follows from theorem 1 by considering the function $g(x) = f(x) - \lim_{x\to\infty} f(x)$.

Theorem 3. *A method M is permanent for null functions if and only if the conditions* (c_1) *and* (c_3') *are satisfied.*

Proof. The necessity of condition (c_3') is evident. Necessity of condition (c_1) follows from theorem 1. The sufficiency follows from the proof of theorem 1 and from the remark that $\delta_n(f) \equiv 0$.

Theorem 4. *A method M is permanent if and only if the conditions* (c_1), (c_2') *and* (c_3') *are satisfied.*

Proof. The sufficiency of conditions (c_1), (c_2') and (c_3') follows from the theorem 3 by considering the function $g(x) = f(x) - \lim_{x\to\infty} f(x)$.

Definition 7. A method M is called a row-finite method if $\Phi(t, x) = 0$ beyond a bicompact set Z_t for $t_0 \leqq t < T$.

Theorem 5. *For each convergence preserving method M there exists a row-finite method M_1 equivalent to M for bounded functions (i. e. summing the same measurable and bounded functions to the same limit).*

We omit the easy proof of this theorem.

§ 2

A number ξ is called the limit value of the function $f(x)$ in ∞ if for any bicompact set $Z \subset X$ and any $\varepsilon > 0$ there exists a point $x \in X - Z$ such that $|f(x) - \xi| < \varepsilon$. We say that $+\infty$ $(-\infty)$ is the limit value of the function $f(x)$ if for any bicompact set Z and for any number K there exists a point $x \in X - Z$ such that $f(x) > K$ $(f(x) < K)$. The upper (lower) bound of the limit values shall be denoted by $\overline{\lim} f(x)$ $(\underline{\lim} f(x))$. The interval

$$K(f) = [\underline{\lim} f(x), \overline{\lim} f(x)]$$

(finite or infinite) shall be called the core of the function $f(x)$ in ∞.

Let $f(x)$ be a measurable, locally bounded function and $M = M(\mu, S, \Phi)$ a permanent method. We set $\varphi(t) = (S)\int \Phi(t, x) f(x) d\mu$ and

$$K_M(f) = [\underline{\lim}_{t\to T-} \varphi(t), \overline{\lim}_{t\to T-} \varphi(t)].$$

It can easily be shown that if $\Phi(t, x) \geqq 0$ then

$$K_M(f) \subseteqq K(f).$$

We call a number ξ the essential limit value in ∞ of the function $f(x)$ if for any $\varepsilon > 0$ and for any bicompact set Z there exists a point $x \in X - Z$ at which the function $f(x)$ is continuous and such that $|f(x) - \xi| < \varepsilon$. We say that $+\infty$ $(-\infty)$

is the essential limit value of the function $f(x)$ if for any bicompact set Z and for any number L there exsits a point $x \in X - Z$ at which the function $f(x)$ is continuous and such that $f(x) > L(f(x) < L)$. The upper (lower) bound of the essential limit values shall be denoted by $\overline{\lim}\, \text{ess}\, f(x)$ ($\underline{\lim}\, \text{ess}\, f(x)$). The interval

$$K_{\text{ess}}(f) = [\underline{\lim}\, \text{ess}\, f(x), \overline{\lim}\, \text{ess}\, f(x)]$$

shall be called the essential core of the function $f(x)$ in ∞. We have obviously

$$K_{\text{ess}}(f) \subseteqq K(f).$$

Theorem 6. *Given a fixed measure μ and a family S determining the improper integral $(S)\int \psi(x)\,d\mu$, for each number ξ from the essential core of the function $f(x)$ there exists a method $M(\mu, S, \Phi)$ permanent and positive (i. e. $\Phi(t, x) \geqq 0$) such that $M\text{-}\lim_{x\to\infty} f(x) = \xi$.*

We omit the proof. (See e. g. [2], p. 77).

§ 3

Definition 8. A method $M = M(\mu, S, \Phi)$ is said to satisfy the condition (w) if there exists a family of bicompact sets $\{Z_t\}$ $(t_0 \leqq t < T)$ such that $Z_{t'} \subseteqq Z_{t''}$ for $t' < t''$, $\bigcup_{t_0 \leqq t < T} Z_t = X$ and $\lim_{t\to T-} (S) \int_{X - Z_t} |\Phi(t, x)|\,d\mu = 0$.

Theorem 7. *Let two permanent methods M_1 and M_2 satisfying the condition (w) be given. If M_2 is more general than M_1 for bounded functions (i. e. any measurable bounded function summable by M_1 is also summable by M_2), then this methods are consistent for bounded functions (i. e. any measurable bounded function summable by M_1 to the number ξ is summable by M_2 to the same number).*[5]

Proof. Let $M_1 = M_1(\mu_1, S_1, \Phi_1)$, $M_2 = M_2(\mu_2, S_2, \Phi_2)$. Suppose, there exists a measurable bounded function $f(x)$ defined on the space X and summable in ∞ by the methods M_i to different numbers. We may assume that $M_1\text{-}\lim f(x) = 0$ and $M_2\text{-}\lim f(x) = 1$. We shall prove that in this case there exists a measurable and bounded function summable by M_1 but not summable by M_2.

Let $\{Z_t^{(1)}\}$ and $\{Z_t^{(2)}\}$ be two families of bicompact subsets of the space X such that

$$\lim_{t\to T-} (S_i) \int_{X - Z_t^{(i)}} |\Phi_i(t, x)|\,d\mu_i = 0 \qquad (i = 1, 2).$$

Put

$$V_t = Z_t^{(1)} \cup Z_t^{(2)}.$$

By W_0 we denote the empty set. Let $W_1 = V_{t_0}$. In virtue of the permanency of the methods M_i there exists a number $t_1 \in [t_0, T)$ such that

$$\left| \int_{W_1} \Phi_i(t, x) f(x)\,d\mu_i \right| < 1/2 \quad \text{for} \quad t \geqq t_1 \qquad (i = 1, 2).$$

[5]) Compare [5] and [8].

Put $W_2 = V_{t_1}$. Let us choose $t_2 > \max(t_1, T-\frac{1}{2})$ such that for $t \geqq t_2$

$$\Big|\int_{W_1} \Phi_i(t,x) f(x)\, d\mu_i\Big| < 1/4, \quad \Big|\int_{W_2 - W_1} \Phi_i(t,x) f(x)\, d\mu_i\Big| < 1/4 \qquad (i=1,2).$$

Let $W_3 = V_{t_2}$ etc. We define inductively a number sequence $\{t_k\}$ increasing to T and a sequence of bicompact sets $\{W_k\}$ such that for $t \geqq t_k$ we have

$$\Big|\int_{W_j - W_{j-1}} \Phi_i(t,x) f(x)\, d\mu_i\Big| < 2^{-k} \quad \text{for} \quad j=1,2,\ldots,k; \; i=1,2,$$

and $W_k = V_{t_{k-1}}$. Let $\{\xi_k\}$ be a numerical sequence such that $\overline{\lim\limits_{k\to\infty}}\, \xi_k = 1$, $\underline{\lim\limits_{k\to\infty}}\, \xi_k = 0$, $|\xi_k| \leqq 1$, $\lim\limits_{k\to\infty} |\xi_k - \xi_{k-1}| = 0$. Put $g(x) = \xi_k f(x)$ for $x \in W_k - W_{k-1}$ $(k=1,2,\ldots)$. Then we have for $i=1,2$

$$(S_i)\int \Phi_i(t,x) g(x)\, d\mu_i = \sum_{j=1}^{k} \xi_j \int_{W_j - W_{j-1}} \Phi_i(t,x) f(x)\, d\mu_i +$$

$$+ \xi_{k+1} \int_{W_{k+2} - W_k} \Phi_i(t,x) f(x)\, d\mu_i + (\xi_{k+2} - \xi_{k+1}) \int_{W_{k+2} - W_{k+1}} \Phi_i(t,x) f(x)\, d\mu_i +$$

$$+ (S_i) \int_{X - V_{t_{k+1}}} \Phi_i(t,x) g(x)\, d\mu_i.$$

The following estimations hold for the terms on the right hand side of our equality:

$$\Big|(\xi_{k+2} - \xi_{k+1}) \int_{W_{k+2} - W_{k+1}} \Phi_i(t,x) f(x)\, d\mu_i\Big| \leqq$$

$$\leqq |\xi_{k+2} - \xi_{k+1}| \sup_{t_0 \leqq t < T} (S)\int |\Phi_i(t,x)|\, d\mu_i \cdot \sup_{x\in X} |f(x)|,$$

$$\Big|\sum_{j=1}^{k} \xi_j \int_{W_j - W_{j-1}} \Phi_i(t,x) f(x)\, d\mu_i\Big| \leqq k\cdot 2^{-k} \quad \text{for} \quad t \geqq t_k,$$

$$\Big|(S_i) \int_{X - V_{t_{k+1}}} \Phi_i(t,x) g(x)\, d\mu_i\Big| \leqq (S_i) \int_{X - Z_t^{(i)}} |\Phi_i(t,x)|\, d\mu_i \cdot \sup_{x\in X} |f(x)| \quad \text{for} \quad t_k \leqq t < t_{k+1},$$

$$\Big|\int_{W_{k+2} - W_k} \Phi_i(t,x) f(x)\, d\mu_i - (S_i)\int \Phi_i(t,x) f(x)\, d\mu_i\Big| \leqq$$

$$\leqq \Big|\int_{W_k} \Phi_i(t,x) f(x)\, d\mu_i\Big| + \Big|(S_i) \int_{X - W_{k+2}} f(x) \Phi_i(t,x)\, d\mu_i\Big| \leqq$$

$$\leqq k\cdot 2^{-k} + (S_i) \int_{X - Z_t^{(i)}} |\Phi_i(t,x)|\, d\mu_i \cdot \sup_{x\in X} |f(x)| \quad \text{for} \quad t_k \leqq t < t_{k+1}.$$

Considering in the above inequality k as a function of t given by the inequality

$t_k \leqq t < t_{k+1}$ we see that

$$\int_{W_{k+2}-W_k} \Phi_i(t,x) f(x)\, d\mu_i \to M_i\text{-}\lim_{x\to\infty} f(x) \qquad (i=1,2)$$

as $t \to T-$, while the other terms on the right hand side of the inequality tend to zero as $t \to T-$. From the divergence of the sequence $\{\xi_k\}$ we see that the generalized limit M_2-lim $g(x)$ does not exist while M_1-lim $g(x) = M_1$-lim $f(x) = 0$, which ends the proof of our theorem.

Theorem 8. *Let a convergence preserving method $M(\mu, S, \Phi)$ be given. Let $f(x)$ be a measurable, bounded function defined on X and let $|f(x)| \leqq \lambda$. Put*

$$\omega_\alpha(x) = \begin{cases} 1 & \text{if } f(x) \leqq \alpha, \\ 0 & \text{if } f(x) > \alpha. \end{cases}$$

If the limit $M\text{-}\lim\limits_{x\to\infty} \omega_\alpha(x) = \omega(\alpha)$ exists for any $|\alpha| < \lambda$ then the function $\omega(\alpha)$ has a finite variation in the interval $[-\lambda, \lambda]$, the limit $M\text{-}\lim\limits_{x\to\infty} f(x)$ exists, and

$$M\text{-}\lim_{x\to\infty} f(x) = \int_{-\lambda}^{\lambda} \alpha\, d\omega(\alpha).$$

Proof. Put

$$h_t(\alpha) = (S)\int \Phi(t,x)\,\omega_\alpha(x)\, d\mu.$$

Let us observe that for any t the functions $h_t(\alpha)$ have bounded variations with the common bound $\sup\limits_{t_0 \leqq t < T} (S)\int |\Phi(t,x)|\, d\mu$. This follows from the estimation

$$\sum_{i=1}^{n-1} |h_t(\alpha_{i+1}) - h_t(\alpha_i)| \leqq (S)\int |\Phi(t,x)| \sum_{i=1}^{n-1} \left(\omega_{\alpha_{i+1}}(x) - \omega_{\alpha_i}(x)\right) d\mu \leqq$$

$$\leqq (S)\int |\Phi(t,x)|\, d\mu, \quad \text{where} \quad -\lambda \leqq \alpha_1 < \alpha_2 < \ldots < \alpha_n \leqq \lambda.$$

If for any α the limit

$$\lim_{t\to T-} h_t(\alpha) = \omega(\alpha)$$

exists, then obviously $w(\alpha)$ has a finite variation and

$$\operatorname{Var} \omega(\alpha) \leqq \sup_{t_0 \leqq t < T} (S)\int |\Phi(t,x)|\, d\mu.$$

Put $Z_i = \{x : \omega_{\alpha_{i+1}}(x) > \omega_{\alpha_i}(x)\}$ and denote by $\chi_Z(x)$ the characteristic function of the set Z. Let $|\alpha_{i+1} - \alpha_i| < \varepsilon$ $(i = 1, 2, \ldots, n-1)$. We have

$$\left|(S)\int \Phi(t,x) f(x)\, d\mu - \sum_{i=1}^{n-1} \alpha_i [h_t(\alpha_{i+1}) - h_t(\alpha_i)]\right| =$$

$$= \left|(S)\int \Phi(t,x) \left[f(x) - \sum_{i=1}^{n-1} \alpha_i \chi_{Z_i}(x)\right] d\mu\right| \leqq$$

$$\leqq \sup_{x\in X} |f(x) - \sum \alpha_i \chi_{Z_i}(x)| \cdot (S)\int |\Phi(t,x)|\, d\mu \leqq \varepsilon \cdot (S)\int |\Phi(t,x)|\, d\mu.$$

Hence it follows that

$$(S)\int \Phi(t,x)f(x)\,d\mu=\int_{-\lambda}^{\lambda}\alpha\,dh_t(\alpha).$$

Since

$$\left|\int_{-\lambda}^{\lambda}\alpha\,d[\omega(\alpha)-h_t(\alpha)]\right|\leqq[\alpha|\omega(\alpha)-h_t(\alpha)|]_{-\lambda}^{\lambda}+\int_{-\lambda}^{\lambda}|\omega(\alpha)-h_t(\alpha)|\,d\alpha\to 0 \text{ as } t\to T-,$$

then

$$\lim_{t\to T-}(S)\int \Phi(t,x)f(x)\,d\mu=\int_{-\lambda}^{\lambda}\alpha\,d\omega(\alpha),$$

which ends the proof.

Theorem 9. *For each permanent method M there exists a measurable function taking only the values* 0 *and* 1, *which is not summable by the method M.*

Proof. Suppose that there exists a permanent method summing all measurable functions with the values 0 and 1. By the foregoing theorem this method would sum all measurable and bounded functions. Let $\{t_n\}$ be a sequence increasing to T. The method $\tilde{M}$ described by the functional

$$\tilde{M}(f)=\lim_{n\to\infty}(S)\int \Phi(t_n,x)f(x)\,d\mu$$

would then sum all the measurable and bounded functions and it would satisfy the condition (w) (definition 8). Thus there would exist a method satisfying the condition (w) and not weaker for bounded functions than all the permanent methods. Thus by the consistence theorem (theorem 7) all permanent methods satisfying the condition (w) would be consistent for bounded functions, which is impossible. This contradiction proves our theorem.

References

[1] S. Banach, *Théorie des opérations linéaires* (Warszawa, 1932).
[2] R. G. Cooke, *Infinite matrices and sequence spaces* (London, 1950).
[3] G. H. Hardy, *Divergent series.* (Oxford, 1956).
[4] R. Jajte, On a theorem of Toeplitz, *Colloquium Math.* (To appear)
[5] S. Mazur and W. Orlicz, Sur les méthodes de sommation, *Comptes Rendus Acad. Sci. Paris,* **196** (1933), 32—34.
[6] S. Mazur and W. Orlicz, On linear methods of summability, *Studia Math.*, **14** (1954), 129—160.
[7] M. A. Наймарк, *Нормированные кольца* (Moscow, 1956).
[8] G. Petersen, Summability methods and bounded sequences, *J. London Math. Soc.*, **31** (1956), 324—326.
[9] H. Steinhaus, Remarks on the generalization of the idea of limit (in Polish), *Prace Mat.-Fiz.* **22** (1911), 121—134.
[10] O. Toeplitz, Über allgemeine lineare Mittelbildungen, *Prace Mat.-Fiz.*, **22** (1911), 113—120.

UNIVERSITY ŁÓDŹ

(Received November 13, 1963)