Cesàro operators*)

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Introduction

If f is a sequence of complex numbers, $f=\langle f(0), f(1), f(2), \dots\rangle$, the sequence $C_0 f$ of averages plays a role in the theory of Cesàro limits; by definition

$$(C_0 f)(n) = \frac{1}{n+1} \sum_{i=0}^{n} f(i)$$

for $n=0, 1, 2, \dots$. Our study of Cesàro operators began with the following questions. Is it true that if $f \in l^2$, then $C_0 f \in l^2$? If it is true, is the linear transformation C_0 bounded? If C_0 is bounded, what is its spectrum? Along with these discrete questions, it is natural to ask the corresponding continuous ones; they concern the operator C_1 defined on $L^2(0, 1)$ by

$$(C_1 f) = \frac{1}{x} \int_0^x f(y)\, dy$$

for $0<x<1$, and the operator C_∞ defined on $L^2(0, \infty)$ by

$$(C_\infty f)(x) = \frac{1}{x} \int_0^x f(y)\, dy$$

for $0<x<\infty$.

It turns out that all three Cesàro operators (that is, C_0, C_1, and C_∞) are everywhere defined bounded linear transformations on their respective Hilbert spaces (that is, on l^2, $L^2(0, 1)$, and $L^2(0, \infty)$). For C_0 and C_∞ this fact is proved by HARDY, LITTLEWOOD, and PÓLYA [5, Chapter IX]; the proof below (Theorem 1) is somewhat more conceptual and less computational than theirs.

For C_0 we completely determine the norm, the spectrum, and the various parts of the spectrum (Theorem 2). There is, however, much about C_0 that remains unknown. Thus, for instance, very little is known about the structure of the lattice of invariant subspaces of C_0 — a problem that belongs to a subject of great current

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interest. Another instance: while we prove that C_0 is hyponormal (Theorem 3), the problem of whether or not it is subnormal remains open.

In view of our incomplete information about C_0, it may be surprising to learn that the structures of C_1 and C_∞ are completely known. We prove that $1-C_1^*$ is a unilateral shift of multiplicity 1 (Theorem 4), and $1-C_\infty^*$ is a bilateral shift of multiplicity 1 (Theorem 5). (The operator C_1 has been studied by DE BRANGES also [3]; our methods are completely different from his.) From these facts, via the Beurling theory [1], it is easy to determine the spectra of C_1 and C_∞, and to derive a satisfactory description of their invariant subspace lattices.

Boundedness

The proof that the Cesaro operators are bounded can be made to depend on a criterion due essentially to I. SCHUR [7]. (In the notation of the statement below, SCHUR discusses the case $p(x)\equiv 1$ only; his proof is different from ours. Cf. also [6, Chapter X].) Since this criterion does not seem to be explicit in the literature, we proceed to state and to prove it with sufficent generality to make it appropriate for most applications.

Schur test. *If X is a measure space, if $k(\geqq 0)$ is a measurable function on $X\times X$, if $p(>0)$ is a measurable function on X, and if α and β are constants such that*

$$\int k(x,y)p(y)\,dy \leqq \alpha p(x)$$

and

$$\int k(x,y)p(x)\,dx \leqq \beta p(y),$$

then the equation

$$(Af)(x)=\int k(x,y)f(y)\,dy$$

defines an operator (a bounded linear transformation) on L^2, and $\|A\|^2\leqq\alpha\beta$.

Proof. If f is a bounded measurable function that vanishes outside some measurable set of finite measure, then

$$\int\left|\int k(x,y)f(y)\,dy\right|^2 dx=\int\left|\left(\int \sqrt{k(x,y)}\sqrt{p(y)}\right)\cdot\left(\frac{\sqrt{k(x,y)}}{\sqrt{p(y)}}f(y)\right)dy\right|^2 dx\leqq$$

$$\leqq\int\left(\int k(x,y)p(y)\,dy\right)\cdot\left(\int\frac{k(x,y)}{p(y)}|f(y)|^2\,dy\right)dx\leqq$$

$$\leqq\int\alpha p(x)\left(\int\frac{k(x,y)}{p(y)}|f(y)|^2\,dy\right)dx=$$

$$=\alpha\int\frac{|f(y)|^2}{p(y)}\left(\int k(x,y)p(x)\,dx\right)dy\leqq\alpha\int\frac{|f(y)|^2}{p(y)}\beta p(y)\,dy=\alpha\beta\|f\|^2.$$

Since the functions such as f are dense in L^2, the proof is complete.

Theorem 1. *Each of the Cesàro operators C_0, C_1, and C_∞ is bounded.*

Proof. For C_0 consider the measure space $\{0, 1, 2, \dots\}$ with the counting measure, and let the kernel k_0 be defined by

$$k_0(i,j) = \begin{cases} 0 & \text{if } 0 \leqq i < j, \\ \dfrac{1}{i+1} & \text{if } 0 \leqq j \leqq i. \end{cases}$$

If $p_0(n) = \dfrac{1}{\sqrt{n+1}}$, then

$$\sum_j k_0(i,j)p_0(j) = \sum_{j=0}^{i} \frac{1}{i+1} \frac{1}{\sqrt{j+1}} <$$

$$< \frac{1}{i+1} \int_0^i \frac{dx}{\sqrt{x}} = \frac{1}{i+1} 2\sqrt{i} < \frac{1}{i+1} 2\sqrt{i+1} = 2p_0(i).$$

If $j \neq 0$, then

$$\sum_i k_0(i,j)p_0(i) = \sum_{i=j}^{\infty} \frac{1}{i+1} \frac{1}{\sqrt{i+1}} <$$

$$< \int_{j-1}^{\infty} \frac{dx}{(x+1)^{3/2}} = \frac{2}{\sqrt{j}} = \frac{2}{\sqrt{j+1}} \frac{\sqrt{j+1}}{\sqrt{j}} \leqq 2\sqrt{2}p_0(j).$$

Since also

$$\sum_i k_0(i,0)p_0(i) = 1 + \sum_{i=1}^{\infty} k_0(i,0)p_0(i) < 1 + 2 = 3p_0(0),$$

it follows that

$$\sum_i k_0(i,j)p_0(i) < 3p_0(j)$$

for all j, and the Schur test implies the boundedness of C_0.

For C_1 the measure space is $(0, 1)$ with Lebesgue measure, and the kernel is defined by

$$k_1(x,y) = \begin{cases} 0 & \text{if } 0 < x \leqq y, \\ \dfrac{1}{x} & \text{if } 0 < y < x. \end{cases}$$

If $p_1(x) = \dfrac{1}{\sqrt{x}}$, then

$$\int_0^1 k_1(x,y)p_1(y)\,dy = \frac{1}{x} \int_0^x \frac{dy}{\sqrt{y}} = \frac{1}{x} 2\sqrt{x} = 2p_1(x),$$

and

$$\int_0^1 k_1(x,y)p_1(x)\,dx = \int_y^1 \frac{dx}{x^{3/2}} = \frac{2}{\sqrt{y}} - 2 < 2p_1(y),$$

and the Schur test applies again.

For C_∞ the measure space is $(0, \infty)$ with Lebesgue measure, and the kernel k_∞ is defined formally the same way as k_1; the difference is that x and y now vary in $(0, \infty)$ instead of $(0, 1)$. If, as before, $p_\infty(x) = \frac{1}{\sqrt{x}}$, then

$$\int_0^\infty k_\infty(x, y) p_\infty(y)\, dy = \frac{1}{x} \int_0^x \frac{dy}{\sqrt{y}} = \frac{2}{\sqrt{x}} = 2p_\infty(x),$$

and

$$\int_0^\infty k_\infty(x, y) p_\infty(x)\, dx = \int_y^\infty \frac{dx}{x^{3/2}} = \frac{2}{\sqrt{y}} = 2p_\infty(y),$$

and, once more, the Schur test yields the desired result.

An examination of the proof of Theorem 1 yields (via the last assertion of the Schur test) estimates for the norms of C_0, C_1, and C_∞. For C_0 this estimate turns out to be quite crude, and even for C_1 and C_∞, where it is sharp, the method is not sharp enough to tell what the norms of the operators actually are. To settle this question, and others, we turn now to detailed separate examinations of the three Cesàro operators.

The discrete Cesàro operator

Since C_0 is defined on a sequence space, it is naturally associated with a matrix, which is in fact just the kernel k_0. Since

$$k_0 = \begin{pmatrix} 1 & 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & 0 & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ & & & \ddots \end{pmatrix}, \quad k_0^* = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \\ 0 & \frac{1}{2} & \frac{1}{3} & \\ 0 & 0 & \frac{1}{3} & \\ & & & \ddots \end{pmatrix},$$

it follows that

$$k_0 k_0^* = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ & & & \ddots \end{pmatrix}.$$

It turns out therefore that the product $C_0 C_0^*$ is almost the same as the sum $C_0 + C_0^*$; the difference $C_0 + C_0^* - C_0 C_0^*$ is the diagonal operator D_0 with matrix

$$\begin{pmatrix} 1 & 0 & 0 & \\ 0 & \frac{1}{2} & 0 & \\ 0 & 0 & \frac{1}{3} & \\ & & & \ddots \end{pmatrix}.$$

Since $(1-C_0)(1-C_0^*) = 1-D_0$, it follows that

$$\|1-C_0\| = 1,$$

and hence that $\|C_0\| \leqq 2$.

It is perhaps worth while to remark that there are other ways of proving the last inequality. One way is to compute $C_0C_0^*$ immediately, and then apply the Schur test to it (with the same p_0 as in the proof of Theorem 1). Since $C_0C_0^*$ is Hermitian, only half the computation is necessary, and, moreover, the inequalities do yield the sharp result $\|C_0C_0^*\| \leqq 4$. To infer, via this approach, that C_0 itself is bounded, one more step is necessary; we need to know that if k is an infinite matrix with rows in l^2 such that kk^* is bounded, then k itself is bounded (cf. [7] and [5, Chapter VIII]). The proof of this can be carried out by looking at the n-th section $k^{(n)}$ of k and showing that the n-th section of kk^* domaintes $k^{(n)}k^{(n)*}$. (Recall that an infinite matrix is bounded if and only if its sections are uniformly bounded.)

It is easy to prove that the inequality $\|C_0\| \leqq 2$ cannot be improved:

$$\|C_0\| = 2.$$

Indeed if $f_\alpha(n) = \dfrac{1}{(n+1)^\alpha}\left(\alpha > \dfrac{1}{2},\ n=0, 1, 2, \ldots\right)$, then $f_\alpha \in l^2$ and $\|C_0^* f_\alpha\| \to 2\|f_\alpha\|$ as $\alpha \to \frac{1}{2}+$. The proof of the latter assertion is a straightforward computation. Since $(C_0^* f_\alpha)(m) = \sum\limits_{n=m}^{\infty} \dfrac{1}{(n+1)^{\alpha+1}}$, $m=0, 1, 2, \ldots$, it follows that

$$\|C_0^* f_\alpha\|^2 = \sum_{m=0}^{\infty}\left(\sum_{n=m}^{\infty}\frac{1}{(n+1)^{\alpha+1}}\right)^2 > \sum_{m=0}^{\infty}\left(\int_{m+1}^{\infty}\frac{dx}{x^{\alpha+1}}\right)^2 = \sum_{m=0}^{\infty}\left(\frac{1}{\alpha}\,\frac{1}{(m+1)^\alpha}\right)^2 =$$

$$= \frac{1}{\alpha^2}\sum_{m=0}^{\infty}\frac{1}{(m+1)^{2\alpha}} = \frac{1}{\alpha^2}\|f_\alpha\|^2,$$

and this implies the limit assertion.

For our next purpose we need the following lemma: if A is an operator such that $\|A\| \leqq 1$ and if $\|Af\| = \|f\|$ for some nonzero vector f, then $\|A^*g\| = \|g\|$ for some non-zero vector g. For the proof, write $g = Af$, so that $\|g\| = \|f\|$, and observe that

$$\|f\|^2 = (A^*Af, f) \leqq \|A^*Af\| \cdot \|f\| \leqq \|f\|^2.$$

It follows that $\|A^*Af\| = \|f\|$, so that $\|A^*g\| = \|g\|$.

We know that the supremum of $\|C_0 f\|$ (and hence of $\|C_0^* f\|$) for vectors f on the unit sphere is 2; we shall show that the supremum is not attained. Since $\|(1-D_0)f\| < \|f\|$ unless $f=0$, it follows that

$$\|(1-C_0^*)f\|^2 = ((1-C_0)(1-C_0^*)f, f) \leqq \|(1-C_0)(1-C_0^*)f\| \cdot \|f\| < \|f\|^2$$

unless $f=0$. The preceding paragraph is applicable, and we may infer that both $\|(1-C_0)f\|$ and $\|(1-C_0^*)f\|$ are strictly less than $\|f\|$, except when $f=0$. It follows of course that $\|C_0 f\|$ and $\|C_0^* f\|$ are strictly less than $2\|f\|$, except when $f=0$. (Proof: $\|C_0 f\| = \|f-(1-C_0)f\| \leqq \|f\| + \|(1-C_0)f\|$.)

The following statement sums up what we have just proved about norms and what we shall go on to prove about spectra.

Theorem 2. (1) $\|1-C_0\| = 1$ *and* $\|C_0\|=2$. (2) *If* $\|f\|=1$, *then* $\|(1-C_0)f\|<1$ *and* $\|(1-C_0^*)f\|<1$. (3) *The point spectrum of* C_0 *is empty.* (4) *If* $|1-\lambda|<1$, *then* λ *is a simple proper value of* C_0^*. (5) *The point spectrum of* C_0^* *is the open disc* $\{\lambda : |1-\lambda|<1\}$: (6) *The spectrum of* C_0 *is the closed disc* $\{\lambda : |1-\lambda|\leqq 1\}$.

Proof. (1) and (2) were proved above. To prove (3), observe first that if $C_0 f=g$, then $f(0)=g(0)$, and if $n\geqq 1$, then $f(n) = (n+1)g(n)-ng(n-1)$. Consequently, if $C_0 f=\lambda f$, then $f(n) = \lambda\big((n+1)f(n)-nf(n-1)\big)$ or $\big(\lambda(n+1)-1\big)f(n) = \lambda\, nf(n-1)$ whenever $n\geqq 1$. If m is the smallest integer for which $f(m)\neq 0$, then $\lambda = \frac{1}{m+1}$, so that $0<\lambda\leqq 1$. It follows that if $n\geqq 1$, then

$$|f(n)| = \left|\frac{\lambda n}{\lambda n-(1-\lambda)} f(n-1)\right| \geqq |f(n-1)|,$$

which, for a non-zero f in l^2, is impossible.

To prove (4), observe first that $(C_0^* f)(n) = \sum_{i=n}^{\infty} \frac{1}{i+1} f(i)$ (cf. the matrix k_0^*). If $C_0^* f=g$, then $f(n) = (n+1)\big(g(n)-g(n+1)\big)$ for $n=0, 1, 2, \ldots$. Consequently if $C_0^* f=\lambda f$, then $f(n) = \lambda(n+1)\big(f(n)-f(n+1)\big)$ or $\lambda(n+1)f(n+1) = \big(\lambda(n+1)-1\big)f(n)$. It follows that 0 is not a proper value of C_0^* (if $\lambda=0$, then $f(n)=0$ for all n), and it follows also that $f(n+1) = \left(1-\frac{1}{\lambda(n+1)}\right)f(n)$. This implies that if $n\geqq 1$, then

$$f(n) = \prod_{j=1}^{n}\left(1-\frac{1}{j\lambda}\right)f(0),$$

and we can conclude, even before we know which values of λ can be proper values of C_0^*, that all the proper values are simple.

Suppose now that $|1-\lambda|<1$, or, equivalently, that $\operatorname{Re}\frac{1}{\lambda} > \frac{1}{2}$. It is convenient to rewrite the condition once more; if $\mu=\frac{1}{\lambda}$, then the condition is that $2\operatorname{Re}\mu = 1+\varepsilon$ for some positive number ε. Our task is to prove that if this condition is satisfied, and if

$$f(n) = \prod_{j=1}^{n}\left(1-\frac{\mu}{j}\right),$$

for $n\geqq 1$, then $f\in l^2$. Since

$$\left|1-\frac{\mu}{j}\right|^2 = 1-\frac{2\operatorname{Re}\mu}{j}+\frac{|\mu|^2}{j^2} = 1-\frac{1+\varepsilon}{j}+\frac{|\mu|^2}{j^2} \leqq \exp\left(\frac{|\mu|^2}{j^2}-\frac{1+\varepsilon}{j}\right),$$

it follows that

$$|f(n)|^2 \leqq \frac{\exp\left(|\mu|^2 \sum_{j=1}^{n} \frac{1}{j^2}\right)}{\exp\left((1+\varepsilon) \sum_{j=1}^{n} \frac{1}{j}\right)} < \frac{c}{\exp((1+\varepsilon)\log n)} = \frac{c}{n^{1+\varepsilon}},$$

where $c = \exp\left(|\mu|^2 \sum_{j=1}^{\infty} \frac{1}{j^2}\right)$. This completes the proof of (4). (We note in passing that if f is a proper vector of C_0^* with proper value λ, then $\sum_{n=0}^{\infty} f(n) z^n = (1-z)^{\frac{1}{\lambda}-1}$ whenever $|z| < 1$.)

Since $\|1-C_0\| = 1$, the spectrum of $1-C_0$ is included in the closed disc $\{\lambda: |\lambda| \leqq 1\}$, and, consequently, the spectrum of C_0 is included in the closed disc $\{\lambda: |1-\lambda| \leqq 1\}$. The preceding paragraph implies that the spectrum of $1-C_0^*$ includes the open disc $\{\lambda: |\lambda| < 1\}$, and hence that the same is true of the spectrum of $1-C_0$. This, in turn, implies that the spectrum of C_0 includes the open disc $\{\lambda: |1-\lambda| < 1\}$, and the proof of (6) is complete.

In view of what was just proved, the proof of (5), and hence of the theorem, can be completed by showing that if $|1-\lambda| = 1$, then λ is not a proper value of C^*, or, equivalently, $1-\lambda$ is not a proper value of $1-C_0^*$. This, however, is an immediate consequence of (2): if $\|f\| = 1$ and $(1-C_0^*)f = (1-\lambda)f$, then $\|(1-C_0^*)f\| = |1-\lambda|$, and therefore $|1-\lambda|$ cannot be equal to 1.

We conclude our discussion of the discrete Cesàro operator by reporting a fact that may not be important but that is at least an interesting curiosity.

Theorem 3. *The operator C_0 is hyponormal, that is, $C_0^* C_0 - C_0 C_0^*$ is positive.*

Proof. The matrix $k_0^* k_0$ is "L-shaped", meaning that it is of the form

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \\ \alpha_1 & \alpha_1 & \alpha_2 & \\ \alpha_2 & \alpha_2 & \alpha_2 & \\ & & & \ddots \end{pmatrix},$$

with $\alpha_n = \sum_{j=n}^{\infty} \frac{1}{(j+1)^2}$. Since $k_0 k_0^*$ is also L-shaped $\left(\text{with } \alpha_n = \frac{1}{n+1}\right)$, and since the difference of two L-sharped matrices is another one, the problem of proving the hyponormality of C_0 reduces to the problem of deciding when an L-shaped matrix is positive. An infinite matrix is positive if and only if all its finite sections have positive determinants; the problem has reduced to the evaluation of the determinant of

$$\begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1 & \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_2 & \alpha_2 & \alpha_2 & \dots & \alpha_n \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_n & \alpha_n & \alpha_n & \dots & \alpha_n \end{vmatrix}.$$

This is easy. Subtract the second column from the first, then subtract the third column from the second, and continue this way through the columns. The resulting matrix has the same determinant as the original one and is triangular; its determinant therefore is the product of its diagonal elements. The diagonal elements are $\alpha_0-\alpha_1, \alpha_1-\alpha_2, \ldots, \alpha_{n-1}-\alpha_n$, and α_n. Conclusion: an finite L-shaped matrix is positive if and only if its determining sequence is positive and decreasing. The proof of the theorem is completed by verifying that the sequence $\left\{\sum_{j=n}^{\infty}\frac{1}{(j+1)^2}-\frac{1}{n+1}\right\}$ has these properties.

The finite continuous Cesàro operator

For C_1 the facts are simpler and the proofs are easier than for C_0; to get at those facts, it is convenient to recall a few simple results about unilateral shifts. An operator U on a Hilbert space H is a unilateral shift of multiplicity 1 if H has an orthonormal basis $\{e_0, e_1, e_2, \ldots\}$ such that $Ue_n=e_{n+1}$, $n=0, 1, 2, \ldots$. A unilateral shift of multiplicity m (here m can be any cardinal number, finite or infinite) is the direct sum of m unilateral shifts of multiplicity 1. Each unilateral shift is an isometry, and so therefore is the direct sum of a unilateral shift and a unitary operator. Conversely, every isometry is a direct sum of a unilateral shift and a unitary operator, it being understood that either summand may be absent. If U is an isometry, then U^*U-UU^* is the projection on the co-range of U (the orthogonal complement of the range of U), and consequently the rank of U^*U-UU^* (the co-rank of U) is the multiplicity of the shift component of U.

If U is a unilateral shift, then the spectrum of U is the closed unit disc, the point spectrum of U is empty, and the point spectrum of U^* is the open unit disc. Each number in the open unit disc is a proper value of U^* of multiplicity equal to the multiplicity of U. The proper vectors of U^* form a total set (that is, they span the entire underlying Hilbert space). All these facts are known; see [1, 2, 4].

There are several ways of characterizing simple unilateral shifts (that is, unilateral shifts of multiplicity 1). For our purposes the most convenient one is this: an operator U is a simple unilateral shift if and only if (1) U is an isometry, (2) the co-rank of U is 1, and (3) U^* has a total set of proper vectors with proper values of modulus strictly less than 1. Indeed, a unilateral shift has these three properties. If, conversely, U is an operator satisfying (1), (2), and (3), then, by (1), it is he direct sum of a unilateral shift and a unitary operator, and, by (2), its shift component is simple. It remains only to use (3) to prove that its unitary component is absent. Suppose therefore that W is a unitary direct summand of U. If $U^*f=\lambda f$ with $|\lambda|<1$, and if g is the component of f in the domain of W, then $W^*g=\lambda g$; since W^* is unitary, it follows that $g=0$. Thus each proper vector of U^* corresponding to a proper value of modulus strictly less than 1 belongs to the domain of the shift component of U; if such vectors span the whole space, then the unitary component of U cannot be present.

Theorem 4. *The operator* $1-C_1^*$ *is a simple unilateral shift.*

Proof. Since C_1 is given by the kernel k_1, where $k_1(x,y)=1/x$ if $0<y\leqq x$ and $k_1(x,y)=0$ otherwise, it follows that C_1^* is given by the kernel k_1^*, where

$$k_1^*(x,y)=\begin{cases}0 & \text{if } 0<y\leqq x,\\ \dfrac{1}{y} & \text{if } 0<x<y.\end{cases}$$

In other words if $f\in L^2(0,1)$, then

$$(C_1^*f)(x)=\int_x^1 \frac{1}{y}f(y)\,dy.$$

The operator $C_1C_1^*$ is given by the kernel

$$\int_0^1 k_1(x,u)k_1^*(u,y)\,du=\int_0^{\min(x,y)}\frac{1}{x}\frac{1}{y}\,du=\frac{\min(x,y)}{xy}.$$

Since

$$k_1(x,y)+k_1^*(x,y)=\begin{cases}\dfrac{1}{x} & \text{if } 0<y\leqq x,\\ \dfrac{1}{y} & \text{if } 0<x<y,\end{cases}$$

it follows that $C_1C_1^*=C_1+C_1^*$, and hence that

$$(1-C_1)(1-C_1^*)=1.$$

Conclusion: $1-C_1^*$ is an isometry.

If we write $1-C_1^*=U$, then $U^*U-UU^*=C_1C_1^*-C_1^*C_1$. Since $C_1^*C_1$ is given by the kernel

$$\int_0^1 k_1^*(x,u)k_1(u,y)\,du=\int_{\max(x,y)}^1\frac{du}{u^2}=\frac{1}{\max(x,y)}-1,$$

it follows that the kernel of $C_1C_1^*-C_1^*C_1$ is the constant function 1. Conclusion: the co-rank of $1-C$ is equal to 1.

Before completing the proof of the theorem, we remark on the kernel techniques used in the proof so far. Since the kernels in question are neither in L^2 (that is, the operators are not in the Hilbert—Schmidt class), nor symmetric (the two textbook cases), it is not quite automatic that if an operator is given by a kernel, then its adjoint is given by the conjugate transpose kernel, and that the product of two operators given by kernels is given by the product kernel. Since, however, the kernels k in question (that is, k_1 and k_1^*) have positive values, and have the property that if f and g are in L^2, then the function given on the unit square by $k(x,y)f(x)g(y)$ is in L^1, no unboundedness or infinity pathology can occur; the necessary changes in the order of integration are immediate consequences of FUBINI's theorem.

To complete the proof of the theorem it is sufficient to show that $1 - C_1$ has a total set of proper vectors corresponding to proper values of modulus strictly less than 1. This is trivial modulo the Weierstrass approximation theorem. If $f_n(x) = x^n$, $n = 0, 1, 2, \ldots$, then the set $\{f_0, f_1, f_2, \ldots\}$ is total in $L^2(0, 1)$. Since $(C_1 f_n)(x) = = \frac{1}{x}\int_0^x y^n\,dy = \frac{x^n}{n+1} = \frac{1}{n+1} f_n(x)$, it follows that $(1 - C_1) f_n = \left(1 - \frac{1}{n+1}\right) f_n$, and the proof is complete.

It may be worth while to remark that Theorem 4 implies that all the spectral assertions of Theorem 2 ((3), (4), (5), and (6)) remain true, word for word, if in their statement C_0 is replaced by C_1^*. The norm assertion (1) is also invariant under this change; the only part of the theorem that changes is (2). Since $1 - C_1^*$ is an isometry, $\|(1 - C_1^*) f\| = \|f\|$ always and $\|(1 - C_1) f\| = \|f\|$ often. What can be said, however, is that if $\|f\| = 1$, then $\|C_1 f\| < 2$ and $\|C_1^* f\| < 2$. This follows either by an examination of the cases of equality in the Schur test, or by a direct argument valid for isometries with no proper values.

Here is another useful comment about unilateral shifts, and hence about $1 - C_1^*$. The basis that a simple unilateral shift shifts is uniquely determined to within a multiplicative constant. The reason is that the co-range is one-dimensional and e_0 is in the co-range. Since the projection on the co-range of $1 - C_1^*$ is $C_1 C_1^* - C_1^* C_1$, and since, as we have seen, this projection is given by the kernel that is identically 1, it follows that the co-range of $1 - C_1^*$ is the set of all constant functions. The most natural choice for e_0 is the constant function 1. Once e_0 is chosen, the other terms of the shifted basis are determined; they are the successive images of e_0 under iterations of $1 - C_1^*$.

There is another approach to Theorem 4, more analytic than the one given above; we proceed to sketch it. If $U = 1 - C_1^*$ and $f_\alpha(x) = x^\alpha$ whenever $\operatorname{Re}\alpha > -\frac{1}{2}$, then $U^* f_\alpha = \frac{\alpha}{\alpha+1} f_\alpha$. A change of parameters is convenient: if $\beta = \bar{\alpha} + \frac{1}{2}$ and $g_\beta = f_{\bar{\beta} - \frac{1}{2}}$ whenever $\operatorname{Re}\beta > 0$, then $U^* g_\beta = \overline{\varphi(\beta)}\, g_\beta$, where $\varphi(\beta) = \frac{\beta - \frac{1}{2}}{\beta + \frac{1}{2}}$.

By means of these proper vectors, the operator U can be represented as a multiplication on a Hilbert space of analytic functions on the right half plane, as follows. For f in $L^2(0, 1)$ define $\hat{f}$ by

$$\hat{f}(\beta) = (f, g_\beta) = \int_0^1 f(t)\, t^{\beta - \frac{1}{2}}\,dt;$$

the transform of U by the mapping $f \to \hat{f}$ is multiplication by φ. Indeed,

$$(Uf)^\wedge(\beta) = (Uf, g_\beta) = (f, U^* g_\beta) = \Phi(\beta)\hat{f}(\beta).$$

Making the change of variables $t = e^{-u}$ $(0 < u < \infty)$, we obtain

$$\hat{f}(\beta) = \int_0^\infty f(e^{-u})\, e^{-u/2} e^{-u\beta}\,du = \int_0^\infty g(u)\, e^{-u\beta}\,du,$$

where g is the element of $L^2(0, \infty)$ defined by $g(u)=f(e^{-u})e^{-u/2}$. Thus the space of functions $\hat{f}$ is the space of Laplace transforms of functions in $L^2(0, \infty)$. By the Paley—Wiener theorem [6, Chapter VIII] this is precisely the space H^2 of the right half plane, and therefore the preceding paragraph exhibits U as multiplication by φ on that H^2 space. Switching to the unit disc via the conformal mapping $w=\varphi(z)$, we obtain a representation of U as multiplication by the independent variable on H^2 of the disc, and Theorem 4 follows.

We conclude our discussion of the finite continuous Cesàro operator by mentioning a curious by-product of Theorem 4. One of our earlier proofs of that theorem made use of the completeness of the set of Laguerre functions in $L^2(0, \infty)$. The proof actually offered above is independent of such considerations; since it turns out that our earlier argument is reversible, Theorem 4 can be used to prove that the Laguerre functions span $L^2(0, \infty)$. Here is how it goes. If $f\in L^2(0, 1)$, write

$$(Tf)(x) = f(e^{-x})e^{-x/2}$$

for $0<x<\infty$, and verify that T is an isometry from $L^2(0, 1)$ onto $L^2(0, \infty)$. Transform the shift $1-C_1^*$ by T; that is, consider on $L^2(0, \infty)$ the operator $V = T(1-C_1^*)T^{-1}$. If $f\in L^2(0, \infty)$, then Vf can be calculated explicitly:

$$(Vf)(x) = f(x)-e^{-x/2}\int_0^x f(y)e^{y/2}\,dy.$$

If, as usual, the Laguerre polynomials are defined by

$$L_n(x) = \frac{1}{n!}e^x\frac{d^n}{dx^n}(x^n e^{-x}),$$

and the Laguerre functions by

$$f_n(x)=e^{-x/2}L_n(x), \qquad n=0, 1, 2, \dots,$$

then the f_n's form an orthonormal set in $L^2(0, \infty)$. A straightforward argument, based on the standard identity

$$L_n(x) = \frac{d}{dx}(L_n(x)-L_{n+1}(x))$$

(see [8, Chapter VI]) implies that $Vf_n=f_{n+1}$. Since $Te_0=f_0$, it follows that $Te_n=f_n$ for $n=0, 1, 2, \dots$, and the completeness of the f_n's follows from that of the e_n's.

The infinite continuous Cesàro operator

We shall get at the facts about C_∞ by reducing its study to that of C_1. It is convenient to begin by establishing a simple result about the relation between unilateral shifts and bilateral shifts. An operator W on a Hilbert space K is a simple bilateral shift if K has an orthonormal basis $\{\dots, e_{-2}, e_{-1}, e_0, e_1, e_2, \dots\}$ such that $We=e_{n+1}$ for all n. It follows from this definition that a simple bilateral shift is a unitary operator. If H is the span of $\{e_0, e_1, e_2, \dots\}$, then H is invariant under

W and the restriction of W to H is a unilateral shift. If R is the operator on K such that $Re_n = e_{-n-1}$ for all n, then R is a symmetry (a unitary involution). The symmetry R is related to the shift W in the following three ways:

$$(1)\quad Re_0 = W^{-1}e_0, \qquad (2)\quad RH = H^{\perp}, \qquad (3)\quad RW = W^{-1}R.$$

What makes these assertions important is that they serve to characterize simple bilateral shifts, in the following sense. Suppose that K is a Hilbert space, W is a unitary operator on K, R is a symmetry on K, H is a subspace of K invariant under W, and e_0 is a vector in H. If the vectors $W^n e_0$, $n=0, 1, 2, \ldots$, form an orthonormal basis for H, and if the conditions (1), (2), and (3) are satisfied, then W is a simple bilateral shift.

The proof is straightforward. We begin by writing $e_n = W^n e_0$ for all n $(=0, \pm 1, \pm 2, \ldots)$. If n and m are arbitrary integers, find a positive integer j such that both $n+j$ and $m+j$ are positive; it follows that

$$(e_n, e_m) = (W^n e_0, W^m e_0) = (W^{n+j} e_0, W^{m+j} e_0) = (e_{n+j}, e_{m+j}) = \delta_{n+j, m+j} = \delta_{nm},$$

and hence that the e_n's form an orthonormal set in K. By assumption $\{e_0, e_1, e_2, \ldots\}$ spans H; it follows that $\{Re_0, Re_1, Re_2, \ldots\}$ spans $H^{\perp}$. Since $Re_n = RW^n e_0 = W^{-n}Re_0 = W^{-n}W^{-1}e_0 = e_{-n-1}$, it follows that $\{e_{-1}, e_{-2}, e_{-3}, \ldots\}$ spans $H^{\perp}$, and hence that the e_n's form an orthonormal basis for K. Since the definition of the e_n's makes it obvious that W shifts them, the proof of the characterization of simple bilateral shifts is complete.

Theorem 5. *The operator* $1 - C_\infty^*$ *is a simple bilateral shift.*

Proof. We apply the preceding characterization of simple bilateral shifts with $K = L^2(0, \infty)$, $W = 1 - C_\infty^*$, and

$$(Rf)(x) = -\frac{1}{x} f\left(\frac{1}{x}\right)$$

whenever $f \in K$. The role of H is played by those elements of K that vanish on $(1, \infty)$, and the role of e_0 is played by the characteristic function of $(0, 1)$. We observe that H differs from $L^2(0, 1)$ in notation only.

If $f \in K$, then

$$(Wf)(x) = f(x) - \int_x^\infty \frac{1}{y} f(y)\, dy$$

for $0 < x < \infty$. With this explicit representation of W, the verifications needed to justify the application of the characterization theorem for bilateral shifts become a matter of routine integrations. They are not only routine, but they are almost identical with the integrations indicated in our study of C_1. (Note that if H is identified with $L^2(0, 1)$, then the restriction of W to H must be identified with $1 - C_1^*$.) With these remarks we consider the proof of Theorem 5 complete.

It follows from Theorem 5 (just as the corresponding facts for C_1 followed from Theorem 4) that $\|1 - C_\infty\| = 1$ and $\|C_\infty\| = 2$; if $\|f\| = 1$, then $\|C_\infty f\| < 2$ and

$\|C_\infty^* f\| < 2$. Using in addition well known (and easily recaptured) facts about the spectrum of a bilateral shift, we obtain the following description of the spectrum of C_∞: the point spectra of both C_∞ and C_∞^* are empty, and the spectrum of C_∞ is the circle $\{\lambda: |1-\lambda| = 1\}$.

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