On products of normal subgroups

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The purpose of this note is to demonstrate that the product of two "good" normal subgroups of a group need not be "good": we do this for the cases when "good" is interpreted, in turn, as locally soluble, locally SI^*, $\overline{SI}$, locally SN^*, locally residually nilpotent, and residually nilpotent. The first four interpretations answer questions recorded in section 2. 4 of PLOTKIN's survey [3][1]); the fifth answers question 13. 1. 3 of [3]; and the last contradicts an assertion of SESEKIN ([4], part of Corollary 1 to Lemma 8) quoted in [3] (in the paragraph immediately preceding 13. 1. 3). In fact, we construct two examples which show even more:

Theorem 1. *There exists a finitely generated group which is the product of two locally soluble normal subgroups but is neither an SI-group nor a radical group* (in the sense of PLOTKIN [3]; note that, according to § 15 of [3], it follows that the group is not an SN^*-group).

Theorem 2. *There exists a finitely generated* (*torsion-free, metabelian*) *group which is the product of two residually nilpotent normal subgroups but is not residually nilpotent.*

The proofs depend on the following:

Lemma. *To each group H which is a split extension of a group G by an abelian group A, it is possible to construct a group H^* which contains a subgroup isomorphic to H and is the product of two normal subgroups isomorphic to the restricted* (*standard*) *wreath product G wr A. Moreover, if H is finitely generated, then so is the corresponding H^*.*

A similar construction has been used independently by HALL (unpublished) for dealing with some of PLOTKIN's questions which are answered by Theorem 1. A result similar to Theorem 2 can also be deduced from a theorem of HALL and HARTLEY (to appear) according to which every group is embeddable in a suitable product of two normal free subgroups.

Proof of the Lemma. Let H be a split extension of a group G by an abelian group A; it can be assumed that G is a normal subgroup of H, $A \cap H = 1$, and there

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[1]) We note that the remaining question of this type in 2. 4 of [3], namely the question relating to the class $\overline{SN}$, has a positive answer: it is straightforward to see that even all extensions of $\overline{SN}$-groups by $\overline{SN}$-groups are $\overline{SN}$-groups.

is a monomorphism $\beta: A \to H$ for which $G \cap A\beta = 1$ and $G(A\beta) = H$. Consider the unrestricted (standard) wreath product P of H and A, and write its base group H^A as the group of functions from A to H with valuewise multiplication. Call K that subgroup of H^A which consists of those functions whose values are all in G and are in fact equal to 1 at all but finitely many elements of A. This K is normal in P, and $KA \cong G \text{ wr } A$. For each element a in A, let $a\delta$ be the constant function on A with value $a\beta$; then $\delta: A \to H^A$ is a monomorphism; moreover, $A\delta$ and A generate an abelian subgroup in P. We need next the element f of H^A defined by

$$f(a) = a\beta \qquad \text{for every } a \text{ in } A.$$

Straightforward calculation shows that

$$(*) \qquad f^{-1}af = a\delta \cdot a \qquad \text{for every } a \text{ in } A,$$

so that $A^f \leqq (A\delta)A$, and therefore A and A^f commute elementwise. Consequently, KA and KA^f normalize each other, and so their product H^* is a subgroup of P. As $KA^f = (KA)^f$, we have that H^* is the product of two normal subgroups isomorphic to $G \text{ wr } A$. To each element g of G, let the element $g\gamma$ of H^A be defined by

$$(g\gamma)(1) = g, \quad (g\gamma)(a) = 1 \qquad \text{whenever } 1 \neq a \in A.$$

Then $\gamma: G \to H^A$ is a monomorphism; moreover, $G\gamma \leqq K$. Each element of H is uniquely a product $g(a\beta)$ with $g \in G$, $a \in A$; the mapping $\alpha: g(a\beta) \to (g\gamma)(a\delta)$ is therefore well defined; in fact, it is a monomorphism of H into H^A. As $G\gamma \leqq K$, and as (*) shows that $A\delta \leqq AA^f$, it follows that $H\alpha \leqq H^*$: so H has a subgroup isomorphic to H. Finally, suppose that H is finitely generated, and note that in this case so is A. It is easily seen that $G\gamma$ and A generate KA; hence H^* is generated by G_γ, A, and A^f. By (*), A and A^f generate the same subgroup as A and $A\delta$; hence H^* is generated by $G\gamma$, $A\delta$, and A. The subgroup generated by $G\gamma$ and $A\delta$ is precisely $H\alpha$. Thus we conclude that H^* is generated by the finitely generated subgroups $H\alpha$ and A, and so H^* itself is finitely generated.

Proof of Theorem 1. We use the terminology and results of HALL [1]. Let G be the wreath power $\text{Wr } C^Z$ of an infinite cyclic group C corresponding to the naturally ordered set Z of rational integers, and let A be the group of all order-preserving permutations of Z. Then A is again an infinite cyclic group, and there is a natural split extension H of G by A. Clearly, this H is finitely generated. According to Theorem D of HALL [1], G' is a minimal normal subgroup of H. It is easy to see that G' is not locally nilpotent; consequently, no group containing H can be an SI-group or a radical group. On the other hand, G is locally soluble and therefore so is $G \text{ wr } A$. The corresponding group H^* of the Lemma provides the example required for the theorem.

Proof of Theorem 2. Let G be the group defined on the generators $g_1, g_2, \dots$ by the relations $g_i = g_{i+1}^2$, $i = 1, 2, \dots$; then G is an abelian group of rank 1. Let A be the group of automorphisms of G generated by the automorphism $\alpha: g \to g^2$, and let H be the natural split extension of G by A, with a an element from that coset of G in H which corresponds to α. Then g_1 and a generate H; moreover, the relation

$$[g, a] = g \qquad \text{for every } g \text{ in } G$$

shows that G lies in every term of the lower central series of H: consequently, H is not residually nilpotent, and so no group containing H can be residually nilpotent. On the other hand, according to Lemma 14 of HALL [2], $G \text{ wr } A$ is residually nilpotent. Thus the corresponding group H^* of the Lemma provides the required example.

References

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