Positive definite kernels generated by operator-valued analytic functions

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Let $\Theta(z)$ be a function of the complex variable z in the unit disc $D_0 = \{z: |z| < 1\}$, whose values are linear transformations of a Hilbert space $\mathfrak{H}$ into a Hilbert space $\mathfrak{H}'$, of norm $\|\Theta(z)\| \leqq 1$, and which is analytic in D_0:

$$\Theta(z) = A_0 + zA_1 + z^2A_2 + \dots \quad (z \in D_0).$$

This function generates the "kernel"

$$K(u, v) = \frac{I - \Theta(v)^* \Theta(u)}{1 - \bar{v}u} \qquad (u, v \in D_0). \tag{1}$$

In his Yale dissertation 1963 ROVNYAK has proved, among other things, the following proposition (*cf.* [1] theorem 4. 3):

The kernel $K(u, v)$ *is positive definite, i. e.*

$$\sum_m \sum_n (K(u_m, u_n)h_m, h_n) \geqq 0 \tag{2}$$

for every finite system $\{u_n\}_1^N$ *of points of* D_0, *and every corresponding system* $\{h_n\}_1^N$ *of vectors in* $\mathfrak{H}$.

The proof given by ROVNYAK is rather indirect, so it might be of some interest to present here a simple proof.

This proof is based on the following *formula:*

$$(K(u, v)h, k) = \tag{3}$$
$$= \frac{r^2 - \bar{v}u}{1 - \bar{v}u} \frac{1}{2\pi} \int_0^{2\pi} \left[\left(\frac{\Theta(z) - \Theta(u)}{z - u} h, \frac{\Theta(z) - \Theta(v)}{z - v} k \right) + \left(\frac{\Omega(z)}{z - u} h, \frac{1}{z - v} k \right) \right] dt$$

where $z = re^{it}$; $\max\{|u|, |v|\} < r < 1$; $h, k \in \mathfrak{H}$; and

$$\Omega(z) = I - \Theta(z)^* \Theta(z) \quad (\Omega(z) \geqq O).$$

Indeed, if we choose r such that $\max\{|u_n|\}_1^N < r < 1$, we shall have, in virtue of (3),

$$\sum_m \sum_n (K(u_m, u_n)h_m, h_n) =$$
$$= \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \left[\left\| \sum_n \frac{\Theta(z) - \Theta(u_n)}{z - u_n} h_n \right\|^2 + \left(\Omega(z) \sum_n \frac{1}{z - u_n} h_n, \sum_n \frac{1}{z - u_n} h_n \right) \right] dt$$

$(z = re^{it})$; since $\Omega(z) \geqq O$, this yields (2).

Now, to prove (3), we first observe that

$$\left(\frac{\Theta(z)}{z-u}h, \frac{\Theta(z)}{z-v}k\right) + \left(\frac{\Omega(z)}{z-u}h, \frac{1}{z-v}k\right) = \frac{1}{z-u}\,\frac{1}{\bar z-\bar v}(h, k),$$

and

$$\left(\frac{\Theta(u)}{z-u}h, \frac{\Theta(v)}{z-v}k\right) = \frac{1}{z-u}\,\frac{1}{\bar z-\bar v}(\Theta(u)h, \Theta(v)k),$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{(z-u)(\bar z-\bar v)} = \frac{1}{r^2}\cdot\frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{d\zeta}{\left(\zeta-\frac{u}{r}\right)\left(1-\frac{\bar v}{r}\zeta\right)} = \frac{1}{r^2}\,\frac{1}{1-\frac{\bar v}{r}\frac{u}{r}} = \frac{1}{r^2-\bar v u}.$$

Next we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\Theta(z)}{z-u}h, \frac{\Theta(v)}{z-v}k\right)dt = \frac{1}{r^2}\,\frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{(\Theta(r\zeta)h, \Theta(v)k)}{\left(\zeta-\frac{u}{r}\right)\left(1-\frac{\bar v}{r}\zeta\right)}d\zeta =$$

$$= \frac{1}{r^2}\,\frac{\left(\Theta\left(r\frac{u}{r}\right)h, \Theta(v)k\right)}{1-\frac{\bar v}{r}\frac{u}{r}} = \frac{(\Theta(u)h, \Theta(v)k)}{r^2-\bar v u}$$

and analogously, passing through complex conjugates,

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\Theta(u)}{z-u}h, \frac{\Theta(z)}{z-v}k\right)dt = \frac{(\Theta(u)h, \Theta(v)k)}{r^2-\bar v u}.$$

Putting these results together we obtain the desired result (3).

Remark. In the case the space $\mathfrak{H}$ is separable, the radial limit $\Theta(e^{it}) = \lim_{r\to 1}\Theta(re^{it})$ exists almost everywhere, in the sense of strong operator convergence, *cf.* [2], n° 1. Hence $\Omega(e^{it}) = \lim_{r\to 1}\Omega(re^{it})$ also exists a. e., at least in the sense of weak operator convergence. In virtue of LEBESGUE's theorem on the integration of bounded sequences of functions we obtain from (3) as $r\to 1$:

$$(K(u, v)h, k) =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\left(\frac{\Theta(e^{it})-\Theta(u)}{e^{it}-u}h, \frac{\Theta(e^{it})-\Theta(v)}{e^{it}-v}k\right) + \left(\frac{\Omega(e^{it})}{e^{it}-u}h, \frac{1}{e^{it}-v}k\right)\right]dt.$$

(Added by proof reading.) An alternative simple proof of (2) was kindly indicated to me in a letter of February 10, 1965, by DR. ROVNYAK.

References

[1] J. ROVNYAK, Some Hilbert spaces of analytic functions, *Yale dissertation* (1963).
[2] B. SZ.-NAGY—C. FOIAŞ, Sur les contractions de l'espace de Hilbert. VIII. Fonctions caractéristiques. Modèles fonctionnels, *Acta Sci. Math.*, **25** (1964), 38—71.

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