Asymptotic values of entire functions of finite order with density conditions

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1. Introduction

Let

$$f(z) = \sum_{1}^{\infty} c_n z^{\lambda_n}$$

be an entire function of finite order ϱ, and let the sequence $\{\lambda_n\}$ satisfy the density condition:

$$\lambda_{n+1} - \lambda_n \geqq p = \frac{1}{\Delta}. \tag{1. 1}$$

PÓLYA (1) has proved (under a somewhat less restrictive density-condition) [1]) the following result:

Theorem A. [1, Satz VII, p. 625] *If $f(z)$ is of mean-type, and $|f(z)|$ is bounded on the positive real axis, then*

$$\Delta \cdot \varrho \geqq \tfrac{1}{2}. \tag{1. 2}$$

Actually the assumption that $f(z)$ is of mean-type can be omitted [3, Theorem 1].

It seems very likely that if $|f(z)|$ is bounded on any curve joining 0 and ∞, conclusion (1. 2) still holds. However, we can only prove the following weaker result:

Theorem 1. *If Γ is a continuous curve without self-intersections joining* 0 *and* ∞, *and $|f(z)|$ is bounded on Γ, then*

$$\Delta \cdot \varrho \geqq \frac{1}{\pi^2}. \tag{1. 3}$$

Corollary. *If $\Delta \cdot \varrho < \frac{1}{\pi^2}$, $f(z)$ has no finite asymptotic value.*

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1) In a recent paper [2], A. EDREI has replaced the Pólya density condition by a more precise one.

2. Statement and proof of Lemmas

We use the notations:

$$M(r) = \max_{0 \leqq \vartheta \leqq 2\pi} |f(re^{i\vartheta})|; \quad M(r, \alpha, \beta) = \max_{\alpha \leqq \vartheta \leqq \beta} |f(re^{i\vartheta})|; \quad \mu(r) = \max_n |c_n| r^n.$$

Lemma 1. *If the condition* (1. 1) *is satisfied, and if*

$$\beta - \alpha > 2\pi\Delta, \tag{2. 1}$$

then we have for entire functions of finite order, that

$$\log M(r, \alpha, \beta) \sim \log M(r). \tag{2. 2}$$

Proof. According to the Wiener—Ingham inequality [4],

$$\int_0^{2\pi} |f(re^{i\vartheta})|^2 \, d\vartheta < K(\alpha, \beta, \Delta) \cdot \int_\alpha^\beta |f(re^{i\vartheta})|^2 \, d\vartheta.$$

Thus

$$\mu^2(r) \leqq \int_0^{2\pi} |f(re^{i\vartheta})|^2 \, d\vartheta < K \int_\alpha^\beta |f(re^{i\vartheta})|^2 \, d\vartheta \leqq K(\beta - \alpha) M^2(r, \alpha, \beta),$$

$$\mu(r) \leqq K' M^2(r, \alpha, \beta), \quad \log \mu(r) \leqq C + \log M(r, \alpha, \beta). \tag{2. 3}$$

On the other hand it is well known [5, p. 34] that for entire functions of finite order:

$$\log \mu(r) \sim \log M(r). \tag{2. 4}$$

(2. 3), (2. 4), and the trivial inequality

$$M(r, \alpha, \beta) \leqq M(r)$$

immediately give (2. 2).

Lemma 2. *Let D_0 be the unit disc slit along the positive real axis, and let $\omega_0(z)$ be the harmonic measure of $0 \leqq x \leqq 1$ in D_0. Then*

$$\cot \left\{ \frac{\pi}{2} \omega_0(re^{i\vartheta}) \right\} = \frac{2\sqrt{r}}{1 - r} \sin \frac{\vartheta}{2}. \tag{2. 5}$$

Lemma 3. *Suppose that $m(R) = \max_{0 \leqq x \leqq R} |f(x)|$ and $\omega_0(z)$ is defined as in the previous lemma. Then we have:*

$$\log |f(re^{i\vartheta})| \leqq \omega_0 \left(\frac{r}{R} e^{i\vartheta} \right) \log m(R) + \left\{ 1 - \omega_0 \left(\frac{r}{R} e^{i\vartheta} \right) \right\} \log M(R). \tag{2. 6}$$

Lemma 4. *If $f(z)$ is of lower order ϱ and $\varrho' > \varrho$, then there is a sequence: $R_n \to \infty$ such that* [2])

$$R_n \left\{ \frac{d}{dr} \log M(r) \right\}_{r=R_n} \leqq \varrho' \log M(R_n). \tag{2.7}$$

Proof. Suppose that for $r \geqq r'$

$$r \left\{ \frac{d}{dr} \log M(r) \right\} > \varrho' \log M(r).$$

Then

$$\log\log M(r) - \log\log M(r') = \int_{x'}^{x} \frac{\frac{d}{dt} \log M(t)}{\log M(t)}\, dt > \varrho' \int_{x'}^{\pi} \frac{dt}{t} = \varrho' (\log r - \log r'),$$

$$\varliminf_{r\to\infty} \frac{\log\log M(r)}{\log r} \geqq \varrho' > \varrho,$$

which is impossible.

3. Proof of Theorem 1

Suppose that $\{R_n\}$ is the sequence defined in Lemma 4 and that $R_n e^{i\alpha}$ is the first intersection of Γ and the circle $|z_n| = R_n$. Without loss of generality we can assume that $\alpha = 0$.

We have assumed that $f(z)$ is bounded on Γ, without loss of generality we can assume that $|f(z)| \leqq 1$ on Γ.

If $\bar{f}(z) = \overline{f(\bar{z})}$ we find that $|f(z)\bar{f}(z)| \leqq M(R)$ on Γ and also on $\bar{\Gamma}$ which is the reflection of Γ into the real axis. The earlier intersections of Γ with the real axis partition $0 \leqq x \leqq R_n$ into a finite number of segments. (If there is no intersection, there is only one segment.) Each segment is the bisector of a domain bounded by an arc of Γ and an arc of $\bar{\Gamma}$. Hence, by the maximum principle we have:

$$|f(x)|^2 = |f(x)\bar{f}(x)| \leqq M(R),$$

$$m(R) = \max_{0 \leqq x \leqq R} |f(x)| \leqq \sqrt{M(R)}. \tag{3.1}$$

Since $r \dfrac{d}{dr} \log M(r) = \dfrac{d}{d \log r} \log M(r)$ is an increasing function of r, the application of Lemma 4 gives for $0 < h \leqq 1$:

$$\frac{\log M(R_n) - \log M(R_n e^{-h})}{h} \leqq R_n \left\{ \frac{d}{dr} \log M(r) \right\}_{x=R_n} \leqq \varrho' \log M(R_n).$$

[2]) The left-hand side of (2. 7) may have isolated discontinuities but this does not affect the argument.

Hence, writing $r_n = R_n e^{-h}$, we have

$$\frac{\log M(r_n)}{\log M(R_n)} \geqq 1 - \varrho' h. \tag{3.2}$$

From (2. 6), (3. 1), and (3. 2) we obtain

$$\log |f(r_n e^{i\vartheta})| \leqq \left\{\frac{1}{2}\omega_0\left(\frac{r_n}{R_n}e^{i\vartheta}\right) + \left(1 - \omega_0\left(\frac{r_n}{R_n}e^{i\vartheta}\right)\right)\right\} \log M(R_n) =$$

$$= \left\{1 - \frac{1}{2}\omega_0(e^{-h}e^{i\vartheta})\right\} \log M(R_n) \leqq \left\{1 - \frac{1}{2}\omega_0(e^{-h}e^{i\vartheta})\right\}(1-\varrho' h)^{-1} \log M(r_n).$$

In view of (2. 5), $\omega_0(e^{-h}e^{i\vartheta})$ is a decreasing function of ϑ for $0 \leqq \vartheta \leqq \pi$, and hence for $0 < \Delta' < 1$:

$$\log M(r_n, -\pi\Delta, +\pi\Delta') \leqq \left\{1 - \frac{1}{2}\omega_0(e^{-h}e^{i\pi\Delta'})\right\}(1-\varrho' h)^{-1} \log M(r_n). \tag{3.3}$$

On the other hand, if $\varrho'' > \varrho'$ and $\Delta' > \Delta$, we obtain from Lemma 1, that for $n \geqq n_0$

$$\log M(r_n) \leqq \frac{1-\varrho' h}{1-\varrho'' h} \log M(r_n, -\pi\Delta', +\pi\Delta'). \tag{3.4}$$

From (3. 3) and (3. 4) we conclude that

$$\log M(r_n) \leqq \frac{1 - \frac{1}{2}\omega_0(e^{-h}e^{i\pi\Delta'})}{1-\varrho'' h} \log M(r_n), \tag{3.5}$$

$$\omega_0(e^{-h}e^{i\pi\Delta'}) \leqq 2\varrho'' h.$$

Substituting the value of ω_0 from (2. 5) we obtain:

$$\cot(\pi\varrho'' h) \leqq \frac{2\cdot e^{-h/2}}{1-e^{-h}} \sin\frac{\pi}{2}\Delta',$$

(3. 6)

$$\sin\frac{\pi}{2}\Delta' \geqq \frac{1}{2}(e^{h/2} - e^{-h/2})\cot(\pi\varrho'' h) \geqq \frac{h}{2}\cot\pi\varrho'' h.$$

Since:

$$\lim_{h\to 0}\frac{h}{2}\cot\pi\varrho'' h = \frac{1}{2\pi\varrho''}$$

we have that for $\varrho''' > \varrho''$ and $h < \varepsilon_0(\varrho'', \varrho''')$:

$$\frac{\pi}{2}\Delta' \geqq \sin\frac{\pi}{2}\Delta' \geqq \frac{1}{2\pi\varrho'''}, \qquad \varrho'''\Delta' \geqq \frac{1}{\pi^2}.$$

This is valid for every $\varrho''' > \varrho$, $\Delta' > \Delta$, hence:

$$\varrho \cdot \Delta \geqq \frac{1}{\pi^2}$$

which proves (1. 3).

References

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