On representing functions by Dàrboux functions

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In 1953 W. SIERPIŃSKI [7, 8] proved that each real-valued function on a connected, separable metric space could be expressed as (1) a sum of two functions each of which maps each closed, connected subset onto the real line R, and (2) a pointwise limit of a sequence of functions each of which maps each closed, connected subset onto R. These results were later generalized by S. MARCUS [6] to apply to a more general situation, where, in particular, the domain space is not topologized. However, when the domain space is a real interval, both of SIERPIŃSKI's results follow immediately from a general theorem of H. FAST [4] namely: If $\mathfrak{F}$ is a family of functions of cardinality $\leqq \mathbf{c}$, then there exists a function f such that $f+g$ is Darboux for each $g \in \mathfrak{F}$. (A function h is Darboux on a real interval if it maps connected sets onto connected sets.)

The main purpose of this article is to extend FAST's result in two directions. In one direction, we will extend FAST's result to the more general setting considered by S. MARCUS (see the paragraphs preceding Theorem 4). And, secondly, we extend FAST's result to apply to Baire functions and measurable functions on a real interval (see Theorems 1 and 2). From this latter extension we will deduce that when $\alpha > 1$ each Baire α (or measurable) function is both the sum of two Baire $\alpha+1$ (resp. measurable) Darboux functions and a pointwise limit of a sequence of Baire $\alpha+1$ (resp. measurable) Darboux functions. We will also show that when $\alpha > 1$ a Baire α (or measurable) function on an interval is the product of two Baire $\alpha+1$ (resp. measurable) functions each of which assumes each non-zero number on each subinterval.

Throughout the sequel unless otherwise specified the domain space is assumed to be a real interval I and measurable means Lebesgue measurable. For the definition of and facts about Baire Functions of class α, Borel sets of class α, etc. see KURATOWSKI [5]. We will consider cardinals to be ordinals which are not equipotent with smaller ordinals. Thus the cardinal $\mathbf{c}$ is the first ordinal equipotent with R. We will say that a set B is $\mathbf{c}$-dense in A if each open interval which intersects A contains $\mathbf{c}$ points of B.

We begin by invoking the following lemma proven in BRUCKNER, CEDER and WEISS [2]. The first part was first proven by BOBOC and MARCUS [1].

Decomposition Lemma. *If A is any $\mathbf{c}$-dense in itself subset of I, then A can be decomposed into $\mathbf{c}$ disjoint, non-void subsets each of which is $\mathbf{c}$-dense in A. Moreover, if A is any $\mathbf{c}$-dense in itself measurable set (or Borel set of class α), then*

A can be decomposed into countably many disjoint, non-void, subsets each of which is **c**-*dense in A and is measurable (resp. a Borel set of class* max $(\alpha, 2)$*).*

Now we prove our main theorems, both of whose proofs are patterned after FAST's proofs.

Theorem 1. *Let $\mathcal{A}$ be any family of measurable functions having cardinality $\leqq \mathbf{c}$. Then there exists a function f such that $f+g$ is measurable and Darboux for each $g \in \mathcal{A}$.*

Proof. By taking a Cantor set of zero-measure in each rational subinterval of I and then taking their union we obtain a **c**-dense subset A of I having zero-measure. According to the Decomposition Lemma we can then decompose A into disjoint, non-void sets $\{A_\alpha\}_{\alpha<\mathbf{c}}$ each of which is **c**-dense in A. Since A has zero-measure, each A_α will be measurable. Next enumerate the rational subintervals of I as $\{I_i\}_{i=1}^{\infty}$ and further decompose each A_α into countably many disjoint **c**-dense, measurable subsets $\{A_{\alpha n}\}_{n=1}^{\infty}$. Now let $h_{\alpha n}$ be any function mapping $I_n \cap A_{\alpha n}$ onto R (the real line). Clearly $h_{\alpha n}$ is measurable. Now define h_α on A_α by putting $h_\alpha(x) = h_{\alpha n}(x)$, if $x \in I_n \cap A_{\alpha n}$ for some n, and $h_\alpha(x) = 0$ elsewhere in A_α. Again h_α is measurable and each h_α maps $A_\alpha \cap J$ onto R for each subinterval J. Next define h on I by $h(x) = h_\alpha(x)$, if $x \in A_\alpha$, and $h(x) = 0$ elsewhere in I. Since A has zero-measure h will be measurable.

Now let $\{y_\alpha\}_{\alpha \in R}$ be an enumeration of R and define the function k on I by $k(x) = y_\alpha$ if $x \in A_\alpha$ and $k(x) = 0$ otherwise. Next represent $\mathcal{A}$ as $\{F(x, y): y \in R\}$ where F is a real-valued function defined on $I \times R$. Put $f(x) = h(x) - F(x, k(x))$. Now choose any $g \in \mathcal{A}$. Then $g(x) = F(x, y_\alpha)$ for some α. Consider the function $G(x) = f(x) + F(x, y_\alpha)$. To complete the proof we must show G is both Darboux and measurable.

To show G is Darboux, we note that $G(x) = h(x) = h_\alpha(x)$ for $x \in k^{-1}(y_\alpha) = A_\alpha$. But h_α clearly maps $A_\alpha \cap J$ onto R for each subinterval J, hence so does G. To show G is measurable, let M be any interval in R. Then $G \urcorner I - A$ is measurable since $G(x) = F(x, y_\alpha) - F(x, 0)$ on $I - A$. Now since A has zero-measure, it follows that $G^{-1}(M)$ is measurable. This finishes the proof.

Theorem 1 may not be true if $\mathcal{A}$ has cardinality $2^{\mathbf{c}}$. For example, let $\mathcal{A}$ be all measurable functions. Then, if there were a function f such that $f+g$ were Darboux and measurable for each $g \in \mathcal{A}$, then by taking $g = 0$ we would have that f is measurable. Then define $h(x) = -f(x)$ if $x \neq 0$ and $h(0) = -f(0) + 1$. Then $h \in \mathcal{A}$ but $f+h$ fails to be Darboux.

Theorem 2. *Let $\mathcal{A}$ be a countable family of Baire α functions. Then there exists a function f such that $f+g$ is a Darboux function of Baire class* max $(\alpha+1, 3)$ *for any $g \in \mathcal{A}$.*

Proof. First, using the Decomposition Lemma, we decompose I into countably many disjoint, non-void subsets $\{A_{mn}\}_{n,m=1}^{\infty}$ each of which is **c**-dense in I and is a Borel set of class 2. Now enumerate the open rational subintervals of I as $\{I_m\}_{m=1}^{\infty}$. Now pick a Baire function h_{nm} to map a subset of $I_m \cap A_{nm}$ onto R as follows: Since $I_m \cap A_{nm}$ is a Borel set of cardinality **c**, we can find a no-where dense perfect subset P_{nm} of it (see KURATOWSKI [3] p. 387). Next we can map P_{nm} continuously onto

$[0, 1]$ by a function Φ. Then Φ maps the F_σ set $P_{nm} - \Phi^{-1}(0) - \Phi^{-1}(1)$ continuously onto $(0, 1)$, which in turn is homeomorphic to R by a function Ψ. Hence, $\Psi \circ \Phi$ maps an F_σ subset, F_{nm}, of $I_m \cap A_{mn}$ onto R continuously. Now put $h_{nm} = \Psi \circ \Phi$.

Next put $A_n = \bigcup_{m=1}^{\infty} A_{nm}$ and define $h_n = \bigcup_{m=1}^{\infty} h_{nm}$. Then clearly h_n maps each $A_n \cap J$ onto R where J is any open subinterval of I. Define h by $h(x) = h_n(x)$, if $x \in A_{nm}$. Now if F is closed in R and contains 0 we have

$$h^{-1}(F) = \left(\bigcup_{n,m=1}^{\infty} (h_{nm} \urcorner F_{nm})^{-1}(F) \right) \cup \left(I - \bigcup_{n,m=1}^{\infty} F_{nm} \right)$$

which is a F_σ union a G_δ. Hence, at worst, $h^{-1}(F)$ for any closed set is an $F_{\sigma\delta}$, so h is a Baire 2 function.

Now let us enumerate $\mathscr{A}$ as $\{g_n\}_{n=1}^{\infty}$ and enumerate the rationals in I as $\{r_n\}_{n=1}^{\infty}$. Define $F(x, y) = g_n(x)$, if $y = r_n$, and $F(x, y) = 0$, if y is irrational. Now define k by $k(x) = r_n$, if $x \in A_n$, and put $f(x) = h(x) - F(x, k(x))$. Now let $g \in \mathscr{A}$. Then $g = g_n$ for some n so that $g(x) = F(x, r_n)$. Put $G(x) = g(x) + f(x)$. To complete the proof we need show that G is both Darboux and Baire of class $\max(\alpha+1, 3)$.

To show G is Darboux we note that $G(x) = h_n(x)$ for x in the **c**-dense subset A_n of I. But h_n maps $A_n \cap J$ onto R for each open interval J, hence, so does G. To show G is Baire of class $\max(\alpha+1, 3)$, we note that

$$(G \urcorner A_m)(x) = F(x, r_n) + f(x) = F(x, r_n) + h(x) - F(x, r_m).$$

Hence $G \urcorner A_m$ is of Baire class $\max(\alpha, 2)$. Hence $G^{-1}(U)$ for any open set U is the countable union of Borel sets belonging to class $\max(\alpha, 2)$. Hence, G is Baire of class $\max(\alpha, 2) + 1 = \max(\alpha+1, 3)$.

It is unkown whether or not we can improve upon the number $\max(\alpha+1, 3)$ in both Theorems 2 and 3. As an aside, we note that the function h in the proof of Theorem 2 is a Baire 2 function which maps each subinterval onto R. Clearly this can not be accomplished by a Baire 1 function.

Now we have the obvious consequence

Corollary 1. *If $\mathscr{A}$ is a countable family of Baire functions, then there exists a function f such that $f+g$ is Darboux and Baire for each $g \in \mathscr{A}$.*

The above corollary may not be valid when $\mathscr{A}$ is uncountable. For example, by a similar example to that following Theorem 1, it cannot be valid when $\mathscr{A}$ is taken to be all Baire functions. We do not know, however, whether or not Theorem 2 itself can be valid for families $\mathscr{A}$ with cardinality **c**.

Corollary 2. *Every measurable (or Baire α) function is the sum of two Darboux, measurable functions (resp. Darboux functions of Baire class* $\max(\alpha+1, 3)$*).*

Proof. Let g be any measurable function. Then put $\mathscr{A} = \{g, 0\}$. Then according to Theorem 1 there exists a function f such that $f+g$ and f are Darboux and measurable. Hence g is the sum of the two Darboux, measurable functions $f+g$ and $-f$. Similarly with the case when g is Baire α.

The above corollary without the refinement of the Baire class has been proved also by Erdős [3].

Corollary 3. *Every measurable (or Baire α) function is the pointwise limit of a sequence of Darboux, measurable functions (resp. Darboux functions of Baire class* $\max(\alpha+1, 3)$*).*

Proof. Let g be any measurable function. Put $\mathscr{A}=\{ng\}_{n=1}^{\infty}$. So by Theorem 1 there will exist f such that $h_n = \left(1-\frac{1}{n}\right)g+\frac{1}{n}f$ is Darboux and measurable for any n. But obviously $\{h_n\}_{n=1}^{\infty}$ approaches g pointwise. Similarly with the case when g is Baire α.

Not every function can be the product of two Darboux functions, as was pointed out by S. Marcus in [4]. For example, it is easily seen that the Baire 1 function f defined by $f(0)=-1$ and $f(x)=+1$ for $x\neq 0$ cannot be the product of two Darboux functions. However, if a function f were always positive (or negative) then it can be factored into two Darboux functions. For then there are two Darboux functions g and h so that $\log f = g+h$. Hence, $f=e^g e^h$ where e^g and e^h are Darboux (and, moreover, are Baire of class $\max(\alpha+1, 3)$ or measurable if f is Baire α or measurable resp.). However, S. Marcus [4] has proven that each function on I is the product of two functions each of which assumes every non-zero number on each subinterval. We shall now give another proof of Marcus' result, but one which can easily be modified so as to apply to Baire α and measurable functions.

Theorem 3. *Each function is the product of two functions each of which assumes every non-zero number on each subinterval. Moreover, if the original function is measurable of Baire α, then the factoring functions can be taken to be measurable or of Baire class* $\max(\alpha+1, 3)$ *respectively.*

Proof. We first prove the result for an arbitrary function f on I and then note the modifications required for the measurable and Baire cases.

Let C be the closed set consisting of all points $x\in I$ such that each neighborhood of x contains $\mathbf{c}$ points of $f^{-1}(0)$. Let $A = C\cap f^{-1}(0)$. Then either A is empty or $\mathbf{c}$-dense in itself. In the latter case we decompose A into two $\mathbf{c}$-dense subsets A^1 and A^2 by the Decomposition Lemma. Then, again by the Decomposition Lemma we decompose A^1 into $\mathbf{c}$-dense subsets $\{A_n^1\}_{n=1}^{\infty}$ and A^2 into $\mathbf{c}$-dense subsets $\{A_n^2\}_{n=1}^{\infty}$. Now enumerate all open rational intervals which hit A as $\{J_n\}_{n=1}^{\infty}$. Let h_n and g_n be functions which map $A_n^1\cap J_n$ and $A_n^2\cap J_n$ respectively onto R. Next define $h_A(x)=h_n(x)$, if $x\in A_n^1\cap J_n$ for some n, and put $h_A(x)=0$ elsewhere in A. Also define $g_A(x)=g_n(x)$, if $x\in A_n^2\cap J_n$ for some n, and $g_A(x)=0$ elsewhere in A. Clearly $h_A(x)g_A(x)=f(x)$ for $x\in A$ and for any interval J hitting A both h_A and g_A map $J\cap A$ onto R.

Now consider $B = I-A$. If B is non-empty it is $\mathbf{c}$-dense in itself. Hence we can decompose B into $\mathbf{c}$-dense subsets B^1 and B^2. Next decompose B^1 and B^2 into c-dense subsets $\{B_n^i\}_{n=1}^{\infty}$ $(i=1, 2, \dots)$ respectively. Let $\{J_n\}_{n=1}^{\infty}$ be all open rational intervals which hit B. Let h_n' and g_n' be functions mapping $B_n^1\cap J_n$ and $B_n^2\cap J_n$ respectively onto $R-\{0\}$. Next define h_B and g_B as follows:

$$\begin{aligned} h_B(x) &= h_n'(x) && \text{if } x\in B_n^1\cap J_n \text{ for some } n,\\ &= 1 && \text{if } x\in B_n^1-J_n \text{ for some } n,\\ &= 0 && \text{elsewhere in } B, \end{aligned}$$

and

$$\begin{aligned} g_B(x) &= g_n'(x) \text{ if } x\in B_n^2\cap J_n \text{ for some } n,\\ &= 1 \quad \text{ if } x\in B_n^2 - J_n \text{ for some } n,\\ &= 0 \quad \text{ elsewhere in } B. \end{aligned}$$

Then $h_B^{-1}(0)=B^2$ and $g_B^{-1}(0)=B^1$ and h_B maps $J\cap B^1$ and g_B maps $J\cap B^2$ onto either R or $R-\{0\}$ for any subinterval J which intersects B.

Now define h and g as follows:

$$\begin{aligned} h(x) &= h_A(x) &&\text{if } x\in A,\\ &= h_B(x) &&\text{if } x\in B^1,\\ &= f(x)/g_B(x) &&\text{if } x\in B^2,\\ &= 1 &&\text{if } x\in \bar{A}-A, \end{aligned}$$

and

$$\begin{aligned} g(x) &= g_A(x) &&\text{if } x\in A,\\ &= g_B(x) &&\text{if } x\in B^2,\\ &= f(x)/h_B(x) &&\text{if } x\in B^1,\\ &= f(x) &&\text{if } x\in \bar{A}-A. \end{aligned}$$

Clearly $f(x)=h(x)\,g(x)$ for each $x\in I$. Now let J be any subinterval of I. If J hits $\bar{A}$, it also hits A. But h_A and g_A map $J\cap A$ onto R; hence so do h and g. On the other hand if $J\subseteq B$, both h_B and g_B map $J\cap B$ onto R or $R-\{0\}$. Hence we have $R-\{0\}\subseteq g(J)\cap h(J)$.

Now suppose f is measurable. Since $A = C\cap f^{-1}(0)$, A must be measurable. From the fact that for any measurable set C of cardinality $\mathbf{c}$ and any set $D\subseteq R$ of cardinality $\mathbf{c}$ there exists a measurable function mapping C onto D, we can clearly make the functions $h_n, g_n, h_A, g_A, \dots, h$ and g measurable.

Now suppose f is Baire α. Since $A = C\cap f^{-1}(0)$, A must be of Borel class α. Hence, the sets A_n^i can be taken to be of class $\max(\alpha, 2)$. Then we choose h_A, g_A, h_B and g_B similar to the h in the proof of Theorem 2, so that h_A, g_A, h_B and g_B are of class $\max(\alpha+1, 3)$. Since quotients of Baire functions of class $\leqq\beta$ are of Baire class $\leqq\beta$, both $f(x)/g_B(x)$ and $f(x)/h_B(x)$ are of Baire class $\max(\alpha+1, 3)$. It follows then that h and g will be of Baire class $\max(\alpha+1, 3)$. This finishes the proof of Theorem 3.

Let X and Y be arbitrary sets and $\mathscr{P}$ be a family of subsets of X. Then a function from X to Y has, according to MARCUS [6], "the Darboux property in the strong sense relative to $\mathscr{P}$ and Y", if $f(\mathscr{P})=Y$ for all $P\in\mathscr{P}$. For brevity let us call such a function $\langle\mathscr{P}, Y\rangle$-Darboux. If $\mathbf{m}$ is an infinite cardinal we will say that $\mathscr{P}$ is an $\mathbf{m}$-family if $\mathscr{P}$ and each member of $\mathscr{P}$ has cardinality $\mathbf{m}$. In [6] S. MARCUS has proved that if $\mathscr{P}$ is an $\mathbf{m}$-family of subsets of some set X, and Y is an additive group of cardinality $\mathbf{m}$, then each function f from X to Y is (1) the sum of two $\langle\mathscr{P}, Y\rangle$-Darboux functions and (2) the pointwise limit of a sequence $\{f_n\}_{n=1}^{\infty}$ of $\langle\mathscr{P}, Y\rangle$-Darboux functions, where for each $x\in X$ $\{f_n(x)\}_{n=1}^{\infty}$ is eventually constant.

If one takes $Y=R$ and $\mathscr{P}$ to be the family of all infinite, closed, connected subsets of a connected, separable metric space X, then one gets the above cited results of SIERPIŃSKI. Another interesting case is when $Y=R$ and $\mathscr{P}$ is the family of all perfect subsets of $X=R^n$. In this case, S. MARCUS [6] also proved that each function is the product of two functions each of which maps each perfect subset onto either R or $R-\{0\}$.

Since each measurable function on I is continuous when restricted to some closed set of positive measure, there are no measurable functions which map each perfect subset of I onto R. Hence we can't extend Theorems 1 and 2 to give $\langle \mathcal{P}, R\rangle$-Darboux functions, where $\mathcal{P}$ is the family of all perfect subsets of I. However, it is clear that we can easily extend Theorems 1 and 2 to apply, for example, to $\langle \mathcal{C}, R\rangle$-Darboux functions where $\mathcal{C}$ is the family of all non-void open subsets of R^n.

Now we extend FAST's Theorem to apply to general $\langle \mathcal{P}, Y\rangle$-Darboux functions. Then, not only FAST's Theorem but also the above result of MARCUS follows immediately (in the same way corollary 2 followed from Theorem 1).

Theorem 4. *Let Y be an additive group of cardinality* $\mathbf{m}$. *Let $\mathcal{P}$ be an* $\mathbf{m}$-*family of sets. Let $\mathfrak{F}$ be a family of functions of cardinality $\leqq \mathbf{m}$ from $X = \bigcup \mathcal{P}$ into Y. Then, there exists a function f from Y to X such that $f+g$ is $\langle \mathcal{P}, Y\rangle$-Darboux for each $g \in \mathfrak{F}$.*

Proof. According to the Lemma of [6] we can decompose X into $\mathbf{m}$ disjoint sets $\{B_\lambda\}_{\lambda<\mathbf{m}}$ each of which meets each member of $\mathcal{P}$. Since $\mathbf{m}\times\mathbf{m}\times\mathbf{m}$ has the same cardinality as $\mathbf{m}$ we can reexpress this family as $\{B_{\alpha\beta\gamma}\}_{\alpha,\beta,\gamma<\mathbf{m}}$. Now put $A_{\alpha\beta} = \bigcup_{\gamma<\mathbf{m}} B_{\alpha\beta\gamma}$ for each $\alpha, \beta<\mathbf{m}$. Then each $A_{\alpha\beta}$ meets each $P\in\mathcal{P}$ in exactly $\mathbf{m}$ points. Now well order $\mathcal{P}$ as $\{P_\beta\}_{\beta<a}$. For $\alpha, \beta<\mathbf{m}$ let $f_{\alpha\beta}$ be a function mapping $A_{\alpha\beta}\cap P_\beta$ onto Y. Define $h(x)=f_{\alpha\beta}(x)$, if $x\in$ some $A_{\alpha\beta}\cap P_\beta$, and $h(x)=0$ otherwise.

Next well order Y as $\{y_\alpha\}_{\alpha<m}$ and put $A_\alpha = \bigcup_{\beta<\mathbf{m}} A_{\alpha\beta}$. Define a function k from X to Y by $k(x)=y_\alpha$ if $x\in A_\alpha$. Now represent $\mathfrak{F}$, which we can assume without loss of generality to have cardinality $\mathbf{m}$, as $\{F(x, y): y\in Y\}$ where F is a function on $X\times Y$ to Y. Put $f(x) = h(x)-F(x, k(x))$. Now suppose $g\in\mathfrak{F}$. Then $g(x)=F(x, y_\alpha)$ for some α. Then $f(x)+g(x) = h(x)-F(x, k(x))+F(x, y_\alpha) = h(x)$ for all $x\in A_\alpha$. But h maps each $P\cap A_\alpha$ for $P\in\mathcal{P}$ onto Y. Hence, $g+f$ maps each member of $\mathcal{P}$ onto Y, which finishes the proof.

Theorem 4 does not imply MARCUS' result (2), but in case Y is, say, a normed linear space of cardinality $\mathbf{m}$, it clearly does imply that each function from $\bigcup\mathcal{P}$ to Y is a pointwise limit of a sequence of $\langle \mathcal{P}, Y\rangle$-Darboux functions, where $\mathcal{P}$ is an $\mathbf{m}$-family of sets.

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