On spectral sets having connected complement

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1. Introduction

Let X be a compact subset of the plane with a connected complement. The present paper is based on the following result, which was discovered independently by FOIAŞ [3], LEBOW [8] and BERGER [1].

Theorem 0. *Every Hilbert space operator having X as a spectral set has a normal dilation whose spectrum is contained in ∂X.*

(We recall briefly a few definitions. If T is an operator [= bounded linear transformation] on a Hilbert space $\mathfrak{H}$, then a compact subset of the plane Y is called a *spectral set* for T if it contains the spectrum of T and satisfies

$$\|\varrho(T)\| \leqq \sup_{z \in Y} |\varrho(z)| \tag{1}$$

for all rational functions ϱ having no poles on Y. If Y has a connected complement then it suffices that (1) hold whenever ϱ is a polynomial. A *dilation* of T is an operator A which acts on a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace and which satisfies $T^n P = P A^n P$ for all positive integers n, where P is the orthogonal projection in $\mathfrak{K}$ with range $\mathfrak{H}$. This dilation is called *minimal* if $\mathfrak{H}$ is contained in no proper reducing subspace of A. The dilation in Theorem 0 becomes unique to within isomorphism if one imposes on it the condition of minimality. A dilation A of T is called a Y-dilation if its spectrum is contained in ∂Y and if $\varrho(A)$ is a dilation of $\varrho(T)$ for every rational function ϱ having no poles on Y. If Y has a connected complement, then every dilation of T with spectrum on ∂Y is automatically a Y-dilation.)

In the present paper we use Theorem 0 to study operators having X as a spectral set. We eventually obtain a characterization of all such operators (Theorem 4). For the case where X is the closed unit disc, this characterization reduces to a well-known theorem (see LANGER [7] and SZ.-NAGY—FOIAŞ [12]) which states that every contraction operator on a Hilbert space has a decomposition into the direct sum of a unitary operator and a completely non-unitary contraction.

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Our arguments depend largely on properties of the algebra $\mathbf{P}(\partial X)$, the space of functions on ∂X that can be uniformly approximated by polynomials. The required results about $\mathbf{P}(\partial X)$ are exposed in Section 2, where we also introduce certain notations which are retained throughout the paper. Section 2 is rather lengthy as it seemed best to give a fairly complete discussion. Section 3 is devoted to three lemmas concerning the invariant subspaces of normal operators with spectra on ∂X. The main results are in Section 4. These relate chiefly to direct sum decompositions of operators having X as a spectral set. We also discuss briefly a functional calculus for a subclass of these operators.

To conclude this section we obtain a useful lemma concerning semi-invariant subspaces. This result will be stated in somewhat greater generality that is needed for our immediate purposes. Suppose that $\mathbf{S}$ is a semi-group of operators on a Hilbert space $\mathfrak{K}$. Then a subspace $\mathfrak{H}$ of $\mathfrak{K}$ is called *semi-invariant* under $\mathbf{S}$ if the orthogonal projection P onto $\mathfrak{H}$ satisfies $PAPBP = PABP$ for all A and B in $\mathbf{S}$. Of interest in the present paper is the case where $\mathbf{S}$ consists of the non-negative integral powers of some fixed operator A; we then call a semi-invariant subspace of $\mathbf{S}$ a semi-invariant subspace of A. This notion bears an obvious relation to that of an operator dilation.

Lemma 0. *Let $\mathbf{S}$ be a semi-group of operators on a Hilbert space $\mathfrak{K}$. Then a subspace $\mathfrak{H}$ of $\mathfrak{K}$ is semi-invariant under $\mathbf{S}$ if and only if it has the form $\mathfrak{H} = \mathfrak{M} \ominus \mathfrak{N}$ where $\mathfrak{M}$ and $\mathfrak{N}$ are invariant subspaces of $\mathbf{S}$ such that $\mathfrak{N} \subset \mathfrak{M}$.*

Proof. The proof that $\mathfrak{H}$ is semi-invariant if it has the described form is straightforward and we therefore omit it. To prove the other half of the lemma, let $\mathfrak{H}$ be a semi-invariant subspace of $\mathbf{S}$ and assume without loss of generality that $\mathbf{S}$ contains the identity operator. Let $\mathfrak{M}$ be the smallest invariant subspace of $\mathbf{S}$ containing $\mathfrak{H}$, and let P and Q be the orthogonal projections onto $\mathfrak{H}$ and $\mathfrak{M}$ respectively. We can complete the proof by showing that the subspace $\mathfrak{N} = \mathfrak{M} \ominus \mathfrak{H}$ is invariant under $\mathbf{S}$, or equivalently, that

$$(Q-P)A(Q-P) = A(Q-P)$$

for all A in $\mathbf{S}$. From now on, let A denote a fixed operator in $\mathbf{S}$. As the equalities $QAQ = AQ$ and $QAP = AP$ clearly hold, it will be enough to show that $PAP = PAQ$. Now if B is in $\mathbf{S}$ and y is in $\mathfrak{H}$, then by semi-invariance,

$$PAPBy = PAPBPy = PABPy = PABy.$$

But by its definition, $\mathfrak{M}$ is the subspace spanned by all the vectors By with y in $\mathfrak{H}$ and B in $\mathbf{S}$. We may conclude that $PAPx = PAx$ for all x in $\mathfrak{M}$, and this implies the desired equality $PAP = PAQ$. The proof is complete.

The author is grateful to Professor Foiaş for pointing out the preceding proof, which is simpler than the one originally submitted. In spite of its elementary character, Lemma 0 seems to have been thus far overlooked.

2. The algebra $\mathbf{P}(\partial X)$

Let $\mathbf{P}(\partial X)$ denote the algebra of all functions on ∂X that can be uniformly approximated by polynomials. We give $\mathbf{P}(\partial X)$ the supremum norm. The set X is (with the usual abuse of language) the maximal ideal space of this algebra. We list below as propositions the needed properties of $\mathbf{P}(\partial X)$ and provide proofs where none exist in the literature. In terminology we follow WERMER [15] and HOFFMAN [6].

Proposition 1. *$\mathbf{P}(\partial X)$ is a Dirichlet algebra, that is, every real continuous function on ∂X can be uniformly approximated by the real parts of polynomials.*

This result is due to WALSH [14].

Let $G_1, G_2, G_3, \dots$ be the components of the interior of X, and for each j choose a point a_j in G_j. Let m_j denote the (unique by Proposition 1) representing measure on ∂X for the functional of evaluation at a_j on $\mathbf{P}(\partial X)$. We denote by $\mathbf{H}^p(m_j)$, $p=1,2$, the closure of $\mathbf{P}(\partial X)$ in $\mathbf{L}^p(m_j)$, and by $\mathbf{H}_0^p(m_j)$ the subspace of $\mathbf{H}^p(m_j)$ consisting of those functions that are annihilated by m_j.

Proposition 2. *The measure m_j is supported by ∂G_j.*

Proof. Let m_j' be a representing measure on ∂G_j for the functional of evaluation at a_j on $\mathbf{P}(\partial G_j)$ (the algebra of functions on ∂G_j that can be uniformly approximated by polynomials). Since $\partial G_j \subset \partial X$, the measure m_j' also represents the functional of evaluation at a_j on $\mathbf{P}(\partial X)$. Therefore $m_j' = m_j$ by the uniqueness of the latter.

Proposition 3. *The non-degenerate Gleason parts of $\mathbf{P}(\partial X)$ are precisely the sets $G_1, G_2, G_3, \dots$.*

Proof. It is easy to show that each point of ∂X constitutes by itself a Gleason part of $\mathbf{P}(\partial X)$. Hence the Gleason part containing a_j is contained in $X - \partial X$. By the Wermer embedding theorem [15], each Gleason part in a Dirichlet algebra is a continuous image of the open unit disc, and therefore is connected. Hence the Gleason part containing a_j is contained in G_j. On the other hand, it follows immediately from HARNACK's inequality that G_j is contained in a single Gleason part [2].

Proposition 4. *If $i \neq j$ then the measures m_i and m_j are mutually singular. The measures m_j contain no atoms.*

For the proof, see [4, Proposition 4].

Proposition 5. *If the finite complex Borel measure μ on ∂X annihilates $\mathbf{P}(\partial X)$, then μ has the form*

$$d\mu = \sum_j h_j \, dm_j,$$

where each h_j is a function in $\mathbf{H}_0^1(m_j)$ and

$$\sum_j \int |h_j| \, dm_j < \infty.$$

This is proved in [4].

For convenience in notation we henceforth let G stand for any one of the domains G_j, and we let a denote the corresponding point a_j and m the corresponding measure m_j. For z in G we let m_z denote the representing measure for the functional of evaluation at z on $\mathbf{P}(\partial X)$.

Proposition 6. *If z is in G then the measure m_z is absolutely continuous with respect to m and dm_z/dm is bounded. As z varies over any compact subset of G the derivatives dm_z/dm remain uniformly bounded.*

For the proof, see [2].

If f is a function in $\mathbf{H}^p(m)$ $(p=1, 2)$ and $\{f_n\}_1^\infty$ is a sequence of functions in $\mathbf{P}(\partial X)$ converging in $\mathbf{L}^p(m)$ to f, then it follows from Proposition 6 that $\{f_n\}$ converges uniformly on every compact subset of G. The limit function is thus analytic in G and clearly depends only on f, not on the approximating sequence $\{f_n\}$. We denote the analytic function associated in this manner with f by f_G; obviously

$$f_G(z) = \int f\, dm_z, \qquad z \in G.$$

Let $\mathbf{H}^\infty(m)$ denote the weak-star closure of $\mathbf{P}(\partial X)$ in $\mathbf{L}^\infty(m)$. An equivalent definition is $\mathbf{H}^\infty(m) = \mathbf{H}^2(m) \cap \mathbf{L}^\infty(m)$. The space $\mathbf{H}^\infty(m)$ is easily seen to be an algebra. We let $\mathbf{H}^\infty(G)$ denote the algebra of all analytic functions f_G with f in $\mathbf{H}^\infty(m)$. A function in $\mathbf{H}^\infty(m)$ is called an *inner function* if it has unit modulus almost everywhere (m).

Proposition 7. *There is an inner function w in $\mathbf{H}^\infty(m)$ with the following properties.*

(i) *If f is in $\mathbf{H}^1(m)$ and $f_G(a)=0$, then f/w is in $\mathbf{H}^1(m)$.*

(ii) *The function w_G is a univalent map of G onto the open unit disk D, with $w_G(a)=0$.*

(iii) *If f is in $\mathbf{H}^2(m)$, then for z in G*

$$f_G(z) = \sum_{n=0}^{\infty} (f, w^n)[w_G(z)]^n.$$

The function w is unique to within a multiplicative constant of unit modulus.

These results are due to Wermer [15]. Actually, Wermer only proves (i) for functions f in $\mathbf{H}^2(m)$, but the result for functions in $\mathbf{H}^1(m)$ follows immediately.

From now on we let w stand for a fixed function with the properties described in the preceding proposition, and we define $\psi = w_G$, $\varphi = \psi^{-1}$. It is easy to see that $\{w^n\}_0^\infty$ is an orthonormal sequence in $\mathbf{H}^2(m)$.

For $0<r<1$ let Γ_r be the image under φ of the circle $C_r = \{z : |z| = r\}$ in the unit disc D, and let m_r be the measure on Γ_r obtained by transplanting normalized Lebesgue measure from C_r.

Proposition 8. $\lim_{r\to 1} m_r = m$ *in the weak-star topology of the dual of* $\mathbf{C}(\bar{G})$.

Proof. If f is in $\mathbf{P}(\partial X)$, then for $0<r<1$ we have

$$\int f\,dm_r = \frac{1}{2\pi}\int_0^{2\pi} (f\circ\varphi)(re^{it})\,dt = f(\varphi(0)) = f(a).$$

Hence every weak-star cluster point m' of the net $\{m_r\}_{0<r<1}$ satisfies

$$\int f\,dm' = f(a)$$

for all f in $\mathbf{P}(\partial X)$. Also every weak-star cluster point of $\{m_r\}$ is clearly a positive measure supported by ∂G. Since the functional of evaluation at a on $\mathbf{P}(\partial X)$ has a unique representing measure, we may conclude that no measure other than m can be a weak-star cluster point of the net $\{m_r\}$. But every subnet of this net has a weak-star cluster point because the closed unit ball in $\mathbf{C}(\bar{G})^*$ is weak-star compact. This proves the proposition.

For f in $\mathbf{H}^p(m)$ $(p=1, 2)$ we define the analytic function f_D in D by $f_D(z) = f_G(\varphi(z))$.

Proposition 9. *For $p=1, 2$ the map $f\to f_D$ is an isometry of $\mathbf{H}^p(m)$ onto $\mathbf{H}^p(D)$*

Proof. If f is a function in $\mathbf{P}(\partial X)$ then

$$\int |f|^p\,dm = \lim_{r\to 1}\int |f|^p\,dm_r$$

by Proposition 8. On the other hand

$$\int |f|^p\,dm_r = \frac{1}{2\pi}\int_0^{2\pi} |f_D(re^{it})|^p\,dt, \qquad 0<r<1,$$

and as $r\to 1$ the right side here goes to the p-th power of the norm of f_D in $\mathbf{H}^p(D)$. Thus our map is an isometry of a dense subset of $\mathbf{H}^p(m)$ into $\mathbf{H}^p(D)$, and so is isometric on all of $\mathbf{H}^p(m)$. From part (iii) of Proposition 7 we see that the image of $\mathbf{H}^2(m)$ under the isometry contains all functions with square-summable Taylor coefficients and thus consists of all of $\mathbf{H}^2(D)$. Since $\mathbf{H}^2(D)$ is dense in $\mathbf{H}^1(D)$, the image of $\mathbf{H}^1(m)$ consists of all of $\mathbf{H}^1(D)$.

Actually Proposition 9 holds for general p, but that is superfluous to our present needs.

Proposition 10. *If the function f in $\mathbf{H}^1(m)$ does not vanish almost everywhere (m), then it is non-zero almost everywhere (m).*

Proof. When $f_G(a)\neq 0$ this follows from [6, Theorem 6. 4]. Suppose on the other hand that $f_G(a)=0$ but that f does not vanish almost everywhere. Then by Proposition 9 the function f_G is not identically zero, and therefore it has a zero of some finite order k at a. By Proposition 7, the function $g=f/w^k$ then belongs to $\mathbf{H}^1(m)$, and furthermore $g_G(a)\neq 0$. Hence g is non-zero almost everywhere, and since $|w|=1$ a. e. this proves the result for f.

Proposition 11. *The following topologies on* $\mathbf{H}^\infty(m)$ *are identical.*

($\mathfrak{W}_1$) *The weak-star topology on* $\mathbf{H}^\infty(m)$ *as a subspace of the dual of* $\mathbf{L}^1(m)$

($\mathfrak{W}_2$) *The weak operator topology on* $\mathbf{H}^\infty(m)$ *as an algebra of multiplication operators on* $\mathbf{L}^2(m)$

($\mathfrak{W}_3$) *The weak operator topology on* $\mathbf{H}^\infty(m)$ *as an algebra of multiplication operators on* $\mathbf{H}^2(m)$.

Proof. That ($\mathfrak{W}_1$) and ($\mathfrak{W}_2$) are identical follows immediately from their definitions. Also it is obvious that ($\mathfrak{W}_3$) is courser than ($\mathfrak{W}_2$). Now a basic neighborhood of the origin for ($\mathfrak{W}_2$) is a finite intersection of sets of the form

$$V(f, g; \varepsilon) = \left\{h \in H^\infty(m)\colon \left|\int hf\bar{g}\, dm\right| < \varepsilon\right\},$$

where f and g are functions in $\mathbf{L}^2(m)$ and ε is a positive real number. It only remains to show that any such $V(f, g; \varepsilon)$ contains a neighborhood of the origin for ($\mathfrak{W}_3$), that is, a finite intersection of sets $V(f', g'; \varepsilon')$ with f' and g' in $\mathbf{H}^2(m)$. Consider first the special case where f and g are positive and bounded from zero. Then by [6, Theorem 5. 9] there is a function f' in $\mathbf{H}^2(m)$ such that $|f'|^2 = fg$ a. e. (m), and we have $V(f', f'; \varepsilon) = V(f, g; \varepsilon)$, as desired. In the general case we can write $f = (f_1 - f_2) + i(f_3 - f_4)$, $g = (g_1 - g_2) + i(g_3 - g_4)$, where f_j and g_j are positive and bounded from zero. We then have

$$\bigcap_{j,k=1}^{4} V(f_j, g_k; \varepsilon/16) \subset V(f, g; \varepsilon),$$

which reduces the general case to the special case just treated. The proof of the proposition is complete.

Proposition 12. (i) *The space* $\mathbf{H}^\infty(G)$ *consists of all bounded analytic functions in* G.

(ii) *The polynomials in* w *are weak-star dense in* $\mathbf{H}^\infty(m)$.

Proof. We consider the isometry of $\mathbf{H}^2(m)$ onto $\mathbf{H}^2(D)$ defined by $f \to f_D$ (see Proposition 9). This transformation at the same time sends $\mathbf{H}^\infty(m)$ onto a certain subalgebra of $\mathbf{H}^\infty(D)$, and in particular sends the function w onto the coordinate function in D (i. e. $w_D(z) \equiv z$). Also it is obvious that the transformation on $\mathbf{H}^\infty(m)$ is a homeomorphism relative to the weak topologies of $\mathbf{H}^\infty(m)$ and its image as algebras of multiplication operators on $\mathbf{H}^2(m)$ and $\mathbf{H}^2(D)$ respectively. Since, as is well-known, the polynomials are weak-star dense in $\mathbf{H}^\infty(D)$, the present proposition now follows from the preceding one.

Proposition 13. *Let* h *be a function in* $\mathbf{H}^1_0(m)$. *Then the measure* $h\,dm$ *annihilates all rational functions having no poles in* $\bar{G}$.

Proof. By the same argument as used in the proof of Proposition 2, the measure m represents evaluation at a on the algebra of rational functions with no poles in G. Now $h = \lim h_n$ where each h_n is a polynomial vanishing at a and the limit is in the norm of $\mathbf{L}^1(m)$. Hence if ϱ is a rational function without poles in $\bar{G}$, then

$$\int \varrho h\, dm = \lim \int \varrho h_n\, dm = \lim \varrho(a) h_n(a) = 0.$$

3. Normal operators with spectra on ∂X

Let $\mathfrak{K}$ be a Hilbert space and let A be a normal operator on $\mathfrak{K}$ whose spectrum is contained in ∂X. Let E be the spectral measure of A, and for each pair of vectors x, y in $\mathfrak{K}$ let (Ex, y) denote the Borel measure on ∂X that assigns to the Borel set S the mass $(E(S)x, y)$. For $j=1, 2, 3, \ldots$ let $\mathfrak{K}_j$ be the set of all x in $\mathfrak{K}$ such that (Ex, x) is absolutely continuous with respect to m_j, and let $\mathfrak{K}_0$ be the set of all x in $\mathfrak{K}$ such that (Ex, x) is singular with respect to every m_j. The sets $\mathfrak{K}_j$ are mutually orthogonal reducing subspaces of A and $\mathfrak{K} = \sum_0^\infty \oplus \mathfrak{K}_j$ [5, § 66] [2]). We denote by A_j the restriction of A to $\mathfrak{K}_j$ and by R_j the orthogonal projection in $\mathfrak{K}$ with range $\mathfrak{K}_j, j=0, 1, 2, \ldots$. For any two vectors x, y in $\mathfrak{K}$, the measure (ER_jx, y), $j>0$, is the absolutely continuous component of (Ex, y) with respect to m_j, while (ER_0x, y) is the singular component of (Ex, y) with respect to the family of measures $\{m_j : j=1, 2, 3, \ldots\}$.

Lemma 1. *Let $\mathfrak{M}$ be an invariant subspace of A. Then*

$$\mathfrak{M} = \sum_{j=0}^{\infty} \oplus R_j \mathfrak{M}. \tag{2}$$

Moreover the subspace $R_0\mathfrak{M}$ reduces A, and for $j>0$ the subspace $R_j\mathfrak{M}$ is invariant under $\varrho(A_j)$ for every rational functions ϱ having no poles in $\bar{G}_j$.

Proof. Let x be any vector in $\mathfrak{M}$. Then for y in $\mathfrak{M}^\perp$ we have

$$0 = (A^n x, y) = \int z^n \, d(E(z)x, y), \qquad n=0, 1, 2, \ldots,$$

and so the measure (Ex, y) annihilates $\mathbf{P}(\partial X)$. It thus follows from Propositions 5 and 13 that

(a) the measure (ER_0x, y) vanishes identically,

(b) the measure (ER_jx, y) annihilates all rational functions having no poles in $\bar{G}_j$, $j=1, 2, 3, \ldots$.

It follows from (a) and (b) that R_jx is orthogonal to $\mathfrak{M}^\perp$, and therefore that R_jx is in $\mathfrak{M}$ $(j=0, 1, 2, \ldots)$. In other words, $\mathfrak{M}$ is invariant under every R_j. Since $\sum_0^\infty R_j = 1$, the decomposition (2) follows immediately.

From (a) it follows that $E(S)R_0x$ is orthogonal to $\mathfrak{M}^\perp$, and is therefore in $\mathfrak{M}$, for every Borel subset S of ∂X. This implies that A^*R_0x is in $\mathfrak{M}$, and hence in $R_0\mathfrak{M}$. Thus $R_0\mathfrak{M}$ reduces A.

Suppose finally that ϱ is a rational function without poles in $\bar{G}_j$ (j fixed, $j>0$). Then from (b) it follows that

$$(\varrho(A_j)R_jx, y) = \int \varrho \, d(ER_jx, y) = 0.$$

Therefore $\varrho(A_j)R_jx$ is in $R_j\mathfrak{M}$, and we may conclude that $R_j\mathfrak{M}$ is invariant under $\varrho(A_j)$.

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[2]) We shall always write ∞ for the upper limit in summations over j, even though these are actually finite summations in cases where the interior of X has only finitely many components.

Lemma 2. *Suppose that none of the measures m_j are absolutely continuous with respect to E. Then every invariant subspace of A is a reducing subspace of A.*

Proof. Let $\mathfrak{M}$ be an invariant subspace of A, let x be any vector in $\mathfrak{M}$, and let y be any vector in $\mathfrak{M}^\perp$. Then the measure (Ex, y) is orthogonal to $\mathbf{P}(\partial X)$. Since none of the measures m_j are absolutely continuous with respect to (Ex, y), it follows from Propositions 5 and 10 that (Ex, y) vanishes identically. This implies that A^*x is orthogonal to $\mathfrak{M}^\perp$ and therefore in $\mathfrak{M}$. We may conclude that $\mathfrak{M}$ reduces A.

If $\mathfrak{H}$ is a subspace of $\mathfrak{K}$, then by the *projection of A onto* $\mathfrak{H}$ we mean the operator T on $\mathfrak{H}$ defined by $Tx = PAx$, where P is the orthogonal projection in $\mathfrak{K}$ with range $\mathfrak{H}$. Thus A is a dilation of its projection onto $\mathfrak{H}$ if and only if $\mathfrak{H}$ is semi-invariant under A.

Lemma 3. *Assume that $\mathfrak{H}$ is a semi-invariant subspace of A such that the projection T of A onto $\mathfrak{H}$ is normal and has its spectrum on ∂X. Then $\mathfrak{H}$ reduces A.*

Proof. Let F be the spectral measure of T. Suppose x is a vector in $\mathfrak{H}$. Then for every non-negative integer n we have

$$\int z^n \, d(F(z)x, x) = (T^n x, x) = (A^n x, x) = \int z^n \, d(E(z)x, x).$$

Since $\mathbf{P}(\partial X)$ is a Dirichlet algebra, and since (Ex, x) and (Fx, x) are real measures, it follows that $(Ex, x) = (Fx, x)$.

Now by Lemma 0 we have $\mathfrak{H} = \mathfrak{M} \ominus \mathfrak{N}$ where $\mathfrak{M}$ and $\mathfrak{N}$ are invariant subspaces of A such that $\mathfrak{N} \subset \mathfrak{M}$. Let $\mathfrak{J}$ be the set of all x in $\mathfrak{H}$ such that none of the measures m_j are absolutely continuous with respect to (Ex, x). It follows from the observation of the preceding paragraph and from well-known properties of normal operators that $\mathfrak{J}$ is dense in $\mathfrak{H}$. Let x be any vector in $\mathfrak{J}$. Then it follows from Lemma 2 that the two subspaces $\bigvee_0^\infty A^n x$, $\bigvee_0^\infty A^{*n} x$ coincide with one another and with the smallest reducing subspace of A containing x. Since obviously

$$\bigvee_0^\infty A^n x \subset \mathfrak{M}, \quad \bigvee_0^\infty A^{*n} x \subset \mathfrak{N}^\perp,$$

we may conclude that Ax and A^*x are in $\mathfrak{H}$. Hence we have shown that $A\mathfrak{J} \subset \mathfrak{H}$ and $A^*\mathfrak{J} \subset \mathfrak{H}$. This together with the density of $\mathfrak{J}$ in $\mathfrak{H}$ implies that $\mathfrak{H}$ reduces A.

4. On operators having X as a spectral set

Theorem 1. *Let the operator T on the Hilbert space $\mathfrak{H}$ have X as a spectral set. Then T has a decomposition*

$$T = \sum_{j=0}^{\infty} \oplus T_j, \tag{3}$$

where

(a) *T_0 is a normal operator whose spectrum is contained in ∂X and whose spectral measure is singular with respect to every m_j,*

(b) *for* $j>0$, T_j *has* $\bar{G}_j$ *as a spectral set,*

(c) *for* $j>0$, *the spectral measure of the minimal normal* $\bar{G}_j$*-dilation of* T_j *is absolutely continuous with respect to* m_j.

Proof. By Theorem 0, there is a minimal normal X-dilation A of T acting on a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$. We carry over the notations introduced in the preceding section. By Lemma 0 we have $\mathfrak{H} = \mathfrak{M} \ominus \mathfrak{N}$ where $\mathfrak{M}$ and $\mathfrak{N}$ are invariant subspaces of A such that $\mathfrak{N} \subset \mathfrak{M}$. By Lemma 1

$$\mathfrak{M} = \sum_{j=0}^{\infty} \oplus R_j \mathfrak{M}, \quad \mathfrak{N} = \sum_{j=0}^{\infty} \oplus R_j \mathfrak{N},$$

and therefore

$$\mathfrak{H} = \sum_{j=0}^{\infty} \oplus \mathfrak{H}_j$$

where $\mathfrak{H}_j = R_j \mathfrak{M} \ominus R_j \mathfrak{N}$. It is evident that each $\mathfrak{H}_j$ reduces T, and therefore (3) holds with $T_j = T|\mathfrak{H}_j$.

Consider a fixed $j>0$. By Lemma 1, if ϱ is a rational function with no poles in $\bar{G}_j$, then $R_j\mathfrak{M}$ and $R_j\mathfrak{N}$ are invariant under the operator $\varrho(A_j)$. As is easily seen, this implies that the projection of $\varrho(A_j)$ onto $\mathfrak{H}_j$ equals $\varrho(T_j)$. (In particular $\varrho(T_j)$ exists.) Property (b) now follows by Proposition 2. Moreover, it is clear that A_j is a minimal normal $\bar{G}_j$-dilation of T_j, and therefore (c) holds.

Finally, (a) follows immediately from the fact that $\mathfrak{H}_0$ reduces A (see Lemma 1).

If T is an operator having X as a spectral set, then we shall say that T is *X-pure* provided there is no invariant subspace $\mathfrak{H}' \neq \{0\}$ of T such that $T|\mathfrak{H}'$ is normal and has its spectrum on ∂X. For the case where X is the closed unit disk, the concept of X-purity reduces to that of complete non-unitarity. If T has X as a spectral set and is X-pure, then the operator T_0 of Theorem 1 must be trivial, and therefore T has the closure of the interior of X as a spectral set. This conclusion also follows from a result of Foiaş (see the last proposition in [3]).

Theorem 2. *Let T be a Hilbert space operator having X as a spectral set. Then T has a unique decomposition as the direct sum of an X-pure operator and a normal operator with spectrum on* ∂X.

Proof. By Theorem 0 there is a normal X-dilation A of T. If $\mathfrak{H}'$ is an invariant subspace of T such that $T|\mathfrak{H}'$ is normal and has its spectrum on ∂X, then $\mathfrak{H}'$ reduces A by Lemma 3. Therefore the span $\mathfrak{L}$ of all such subspace $\mathfrak{H}'$ is a reducing subspace of T such that $T' = T|\mathfrak{L}$ is normal and has its spectrum on ∂X. It follows immediately from the definition of $\mathfrak{L}$ that the operator $T'' = T|\mathfrak{L}^{\perp}$ is X-pure, and thus the decomposition $T = T' \oplus T''$ is of the required form. The uniqueness of this decomposition follows immediately from the definition of $\mathfrak{L}$.

Theorem 3. *Let the operator T on the Hilbert space* $\mathfrak{H}$ *have X as a spectral set, and assume that T is X-pure. Let the subspaces* $\mathfrak{H}_j$ *of* $\mathfrak{H}$, $j=1, 2, \ldots$, *be as defined in the proof of Theorem* 1. *(It follows from the proof of Theorem* 1 *that* $\mathfrak{H}_0$ *is trivial.) Let A be a normal X-dilation of T acting on a Hilbert space* $\mathfrak{K}$ *containing* $\mathfrak{H}$, *and let E be the spectral measure of A. Then for any non-zero vector x in* $\mathfrak{H}_j$, *the measure* (Ex, x) *is mutually absolutely continuous with* m_j.

Proof. By Lemma 0 we have $\mathfrak{H} = \mathfrak{M} \ominus \mathfrak{N}$ where $\mathfrak{M}$ and $\mathfrak{N}$ are invariant subspaces of A such that $\mathfrak{N} \subset \mathfrak{M}$. Let x be a non-zero vector in $\mathfrak{H}_j$. We know from Theorem 1 that (Ex, x) is absolutely continuous with respect to m_j. Suppose that (Ex, x) is strictly absolutely continuous with respect to m_j. Then, letting $\mathfrak{H}'$ denote the smallest reducing subspace of A containing x, we have

$$\mathfrak{H}' = \bigvee_0^\infty A^n x = \bigvee_0^\infty A^{*n} x$$

by Lemma 2. Since obviously $\bigvee_0^\infty A^n x \subset \mathfrak{M}$ and $\bigvee_0^\infty A^{*n} x \subset \mathfrak{N}^\perp$, we have $\mathfrak{H}' \subset \mathfrak{H}$. Therefore $\mathfrak{H}'$ is a non-trivial reducing subspace of T such that $T|\mathfrak{H}'$ is normal and has its spectrum on ∂X. But this contradicts our hypotheses. We may conclude that (Ex, x) is not strictly absolutely continuous with respect to m_j.

Now let G denote a particular one of the domains G_j and let m denote the corresponding measure m_j. Consider an operator T on a Hilbert space $\mathfrak{H}$ which has $\bar{G}$ as a spectral set and which is $\bar{G}$-pure. Let A be a minimal normal $\bar{G}$-dilation of T. If h is a function in $\mathbf{H}^\infty(m)$, then since the spectral measure of A is absolutely continuous with respect to m (Theorem 3), the operator $h(A)$ is defined by the standard functional calculus for normal operators. We thus have a natural map of $\mathbf{H}^\infty(m)$ onto the operator algebra $\mathbf{H}^\infty(A) = \{h(A): h \in \mathbf{H}^\infty(m)\}$. This map is an algebra isomorphism and is a homeomorphism relative to the weak-star topology on $\mathbf{H}^\infty(m)$ and the weak operator topology on $\mathbf{H}^\infty(A)$. The last assertions follow readily from the fact that not only is the spectral measure of A absolutely continuous with respect to m, but also m is absolutely continuous with respect to the spectral measure of A (Theorem 3). Now each operator $h(A)$ in $\mathbf{H}^\infty(A)$ corresponds by projection to an operator on $\mathfrak{H}$, which we denote by $h(T)$. We thus have a natural map from $\mathbf{H}^\infty(m)$ onto the class of operators $\mathbf{H}^\infty(T) = \{h(T): h \in \mathbf{H}^\infty(m)\}$. It is easily seen that $\mathbf{H}^\infty(T)$ is an algebra, and that the map of $\mathbf{H}^\infty(m)$ onto $\mathbf{H}^\infty(T)$ is an algebra homomorphism and is continuous relative to the weak-star topology on $\mathbf{H}^\infty(m)$ and the weak operator topology on $\mathbf{H}^\infty(T)$. If ω is a bounded analytic function in G, then by Proposition 12 we have $\omega = h_G$ for some (unique) h in $\mathbf{H}^\infty(m)$, and we shall write $\omega(T)$ in place of $h(T)$.

Consider in particular the functions w, $\psi = w_G$, and $\varphi = \psi^{-1}$ (see Proposition 7 and the remarks following it). Since w is an inner function the operator $w(A)$ is unitary, and therefore the operator $S = \psi(T)$ is a contraction. The operators A and $w(A)$ have the same invariant subspaces since each is a weak limit of polynomials in the other (Proposition 12). This makes it clear that $w(A)$ is a minimal unitary dilation of S and that S is completely non-unitary. If ω is a bounded analytic function in the unit disc, then we have the composition law $\omega(S) = (\omega \circ \psi)(T)$. Indeed, this is easy to verify if ω is a polynomial, and therefore it holds in general by weak continuity. In particular $\varphi(S) = T$.

By combining the preceding observations with Theorems 1 and 2, we obtain the following characterization of those operators having X as a spectral set.

Theorem 4. *The Hilbert space operator T has X as a spectral set if and only if it has the form*

$$T = T_0 \oplus \sum_{j=1}^{\infty} \oplus\, \varphi_j(S_j), \tag{4}$$

where T_0 is a normal operator with its spectrum on ∂X, the S_j are completely non-unitary contractions, and φ_j is for each j a conformal map of the unit disc onto G_j.

Actually we have only proved half of this theorem, the half which asserts that T has the form (4) if it has X as a spectral set. But the other half follows easily from well-known properties of contraction operators.

In conclusion we mention that if the operator T has X as a spectral set and is X-pure, then the above discussion shows us how to define $\omega(T)$ whenever ω is a bounded analytic function in the interior of X. The functional calculus used above is an extension of and was motivated by the functional calculus for contractions developed by Sz.-Nagy and Foiaş [11], [13], and by Schreiber [9], [10].

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