On maximum theorems for analytic operator functions

By ARLEN BROWN and R. G. DOUGLAS in Ann Arbor (Michigan, U. S. A.)

1. Introduction. Let $\mathfrak{H}$ be a complex Hilbert space and $\mathfrak{L}$ be the ring of bounded linear operators defined on $\mathfrak{H}$. A function $T=T(\xi)$ from an open subset Δ of the complex plane to $\mathfrak{L}$ is an *analytic operator function* if the scalar function $(T(\xi)x, y)$ is analytic on Δ for every pair of vectors x, y in $\mathfrak{H}$. The purpose of the present note is to discuss various maximum modulus theorems for such functions, as well as certain other results obtainable by a more or less direct exploitation of the maximum modulus principle for ordinary scalar functions.

2. Maximum modulus theorems. If $\|T\|$ is used as a measure of the size of T, then the following theorem, which we state for the sake of completeness, is the natural version of the maximum modulus principle for operator functions.

Theorem 1. *If $T(\xi)$ is an analytic operator function on a connected domain Δ, and if $\|T(\xi)\|$ assumes its maximum on Δ, then $\|T(\xi)\|$ is constant on Δ.*

(This result is in the literature; see e. g., [1, 3. 13]. The proof is an obvious modification of the proof of Theorem 2 below.)

Another gauge of the size of an operator T is the *numerical radius:* $w(T)=$ $=\sup\{|(Tx, x)|: \|x\|=1\}$. Using this we obtain another maximum modulus principle.

Theorem 2. *If $T(\xi)$ is an analytic operator function on a connected domain Δ, and if $w(T(\xi))$ assumes its maximum on Δ, then $w(T(\xi))$ is constant on Δ.*

Proof. Choose $\xi_0 \in \Delta$ such that $w(T(\xi)) \leqq w(T(\xi_0))=w_0$ for all $\xi \in \Delta$, and select a sequence $\{x_n\}$ of unit vectors in $\mathfrak{H}$ such that $|(T(\xi_0)x_n, x_n)| \to w_0$. The functions $\varphi_n(\xi)=(T(\xi)x_n, x_n)$ are all analytic on Δ and are uniformly bounded by w_0. Hence there exists a subsequence converging subuniformly on Δ to an analytic limit ψ. Clearly $|\psi(\xi)| \leqq w_0$, $\xi \in \Delta$, and $\psi(\xi_0)=\varepsilon w_0$, $|\varepsilon|=1$, so that, by the maximum modulus principle $\psi \equiv \varepsilon w_0$. In particular, $\{\varphi_n(\xi)\}$ contains a subsequence tending to εw_0 for each $\xi \in \Delta$, and the theorem follows.

Our next maximum modulus theorem is unique in that it employs a set valued function. For any point set S, we denote by S^- its closure.

Theorem 3. *If $T(\xi)$ is an analytic operator function on a connected domain Δ, and if the numerical range $W(T(\xi))$ assumes its maximum on Δ at a ξ_0, in the sense that $W(T(\xi)) \subset W(T(\xi_0))^-$ for every $\xi \in \Delta$, then $W(T(\xi))$ is independent of ξ.*

Note. The *numerical range* $W(T)$ of an operator T is the (numerical) set $\{(Tx, x): \|x\|=1\}$. We employ the well known fact that $W(T)$ is convex.

Proof. Let l be a line of support for the compact convex set $W(T(\xi_0))^-$, and let λ_0 be a point on l. We shall show (i) if $\lambda_0 \in W(T(\xi_0))^-$ then $\lambda_0 \in W(T(\xi))^-$ for all $\xi \in \Delta$, and likewise (ii) if $\lambda_0 \in W(T(\xi_0))$ then $\lambda_0 \in W(T(\xi))$ for all $\xi \in \Delta$. In both cases it is no loss of generality to assume that $\lambda_0 = 0$ and that l is the imaginary axis, since we may replace the given function T by a linear transform $\alpha_0 T - \lambda_0$ without affecting the hypotheses of the theorem in any way. Similarly, we may and do assume that all the sets $W(T(\xi))$ lie in the positive half-plane $\operatorname{Re} \xi \geqq 0$.

Proof of (ii). Choose a unit vector $x_0 \in \mathfrak{H}$ such that $(T(\xi_0)x_0, x_0) = 0$. Then $\varphi(\xi) = (T(\xi)x_0, x_0)$ is an analytic function on Δ with non-negative real part and with a zero at ξ_0. The ordinary maximum modulus principle applied to $e^{-\varphi}$ shows that $\varphi \equiv 0$ and the result follows.

Proof of (i). Choose a sequence $\{x_n\}$ of unit vectors such that $(T(\xi_0)x_n, x_n) \to 0$ and let $\varphi_n(\xi) = (T(\xi)x_n, x_n)$ for all $\xi \in \Delta$. Since $|\varphi_n(\xi)| \leqq w(T(\xi_0))$ the sequence $\{\varphi_n\}$ is uniformly bounded and therefore possesses a subsequence converging subuniformly on Δ to a limit ψ. Clearly $\operatorname{Re} \psi \geqq 0$ and $\psi(\xi_0) = 0$ and it follows, as before, that $\psi \equiv 0$. Thus 0 is a limit point of the sequence $\{\varphi_n(\xi)\}$ for every $\xi \in \Delta$ and (i) is proved.

Now, using (i) only, we see that each line of support for $W(T(\xi_0))^-$ meets all the sets $W(T(\xi))^-$ in the same segment, whence it follows by convexity that the set function $W(T(\xi))^-$ is constant. But then, the hypotheses of the theorem are satisfied with ξ replacing ξ_0 and it follows, by (ii), that any boundary point present in one $W(T(\xi))$ must be present in all.

The following theorem, while not perhaps deserving to be called a maximum modulus principle, is nevertheless closely related to the foregoing in spirit and method of proof. It was suggested to the authors by a result of DE BRANGES and ROVNYAK in an as yet unpublished manuscript.

Theorem 4. *Let $T(\xi)$ be an analytic operator function on a connected domain Δ and suppose that $T(\xi)$ is contraction valued, i. e., that $\|T(\xi)\| \leqq 1$ everywhere on Δ.*

(1) *If for some $x \neq 0$ in $\mathfrak{H}$ and some $\xi_0 \in \Delta$ we have $\|T(\xi_0)x\| = \|x\|$ then $T(\xi)x$ is constant on Δ.*

(2) *If a complex number γ of modulus one is in the spectrum of $T(\xi)$ for any one ξ_0, then it is in the spectrum of $T(\xi)$ for every $\xi \in \Delta$.*

(3) *If $1 + T(\xi)$ is invertible at any one point ξ_0 then it is invertible for every $\xi \in \Delta$.*

Sketch of proof. (1) Apply the maximum modulus principle to the function $\varphi(\xi) = (T(\xi)x, T(\xi_0)x)$.

(2) Since $|\gamma| = 1$ it is well known that γ is an approximate eigen-value of $T(\xi_0)$. Choose a sequence $\{x_n\}$ of unit vectors such that $(T(\xi_0) - \gamma)x_n \to 0$ and apply the usual argument to the sequence of functions $\varphi_n(\xi) = (T(\xi)x_n, x_n)$.

(3) Since $\|T(\xi)\| \leqq 1$, a simple computation shows that $\lambda = -1$ belongs to the spectrum of $T(\xi)$ if and only if it belongs to the closed numerical range $W(T(\xi))^-$. In other words, $1 + T(\xi)$ fails to be invertible when and only when the vertical line $\operatorname{Re} \lambda = -1$ supports $W(T(\xi))^-$ at the point $\lambda = -1$. The result now follows from the proof of Theorem 3 (see (ii) above).

3. Concluding remarks.

1. In the ordinary maximum modulus theorem for scalar valued functions the presence of a *local* maximum on a connected domain implies that the function *itself* is constant, not just its modulus. It is easy to see, but should be officially remarked, that such strong results cannot be obtained in the present setting. For example, if $1, \omega, \overline{\omega}$ denote the cube roots of unity then

$$T(\xi) = \operatorname{diag}(1, \omega, \overline{\omega}, \xi)$$

(on $\mathfrak{H}$ of dimension 4) has norm and numerical range both constant on the disc $|\xi| < \frac{1}{2}$. Likewise, $\|T\|$, $W(T)$ and $w(T)$ all have a local maximum at $\xi = 0$ on the disc $|\xi| < 1$. (It is, of course, impossible for any one of our three gauges of size of $T(\xi)$ to possess a strict maximum at an interior point.)

2. On the other hand, certain vestiges of the stronger theorem do survive. A case in point is part (1) of Theorem 4, along with the following two remarks, both of which are its immediate consequences.

Corollary to Theorem 4. *Let $T(\xi)$ be a contraction valued analytic operator function on a connected domain Δ. Then*

(a) *the null space of $1 + T(\xi)$ is constant on Δ,*

(b) *if for some $\xi_0 \in \Delta$ the operator $T(\xi_0)$ is an isometry then $T(\xi)$ is constant on Δ.*

3. Is there a maximum modulus theorem for the spectral radius of $T(\xi)$? Or, perhaps, for the spectrum itself? These appear to be open questions. In any event, it does not seem to be possible to answer them with the elementary techniques of the present note.

Reference

[1] E. HILLE and R. S. PHILLIPS, *Functional Analysis and Semigroups* (Providence, 1957).

(Received January 12, 1965)