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Generalized Bernstein Inequalities

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Dedicated to my friend, colleague and teacher Professor Gábor Szegő on his 70th birthday

§ 1. Introduction

Two celebrated theorems due to BERNSTEIN and MARKOFF describe extremal characterizations of the Tchebycheff polynomials. The origin of these developments stems from interest in the following problems.

Problem 1. Determine the polynomial of degree $m-1$ which maximizes

$$\max_{-1\leqq x\leqq 1} |P_{m-1}(x)| \tag{1}$$

among all polynomials of degree $m-1$ satisfying the conditions

$$(1-x^2)P_{m-1}^2(x) \leqq 1 \qquad (-1\leqq x\leqq 1). \tag{2}$$

Problem 2. Let $P_m(x)$ denote any polynomial of degree m obeying the restriction

$$\max_{-1\leqq x\leqq 1} |P_m(x)| \leqq 1.$$

Find an upper bound for

$$\max_{-1\leqq x\leqq 1} |P_m'(x)|.$$

The extremal polynomial in each case turns out to be a classical Tchebycheff polynomial. The solutions to Problems 1 and 2 lead to what is known as the Markoff—Bernstein inequalities.

The usual method of analyzing Problem 2 is to reduce it to Problem 1. It is customary to first formulate a trigonometric version of Problem 2 which is easily solved. It is then possible to combine the result of the trigonometric case with the conclusion of Problem 1 and thereby uncover the solution of Problem 2 (for the details of this method see PÓLYA—SZEGŐ [14, page 90]).

The extremal characterization and the uniqueness proof for Problem 1 depend on the existence of a polynomial which exhibits maximum oscillation under the constraint (2). In the classical case this polynomial is $U_{m-1}(t)$, the Tchebycheff polynomial of the second kind. The essential fact duly exploited in the proof is that any other polynomial $P_{m-1}(t)$ meeting the constraint (2) cannot provide a larger value in (1) or otherwise the difference $P_{m-1}^2(t)-U_{m-1}^2(t)$ has too many zeros.

Extensions, refinements and elaborations of Problems 1 and 2 have moved in several directions. In most instances only Problem 2 has been generalized. For example:

1. The result of Problem 2 has been suitably extended to certain classes of entire functions of order ϱ, e. g., BOAS [6, Chap. 8], ACHIESER [1, p. 140], BERNSTEIN [2].

2. An extensive accounting of aspects of Problem 2 when the domain of definition of $P_n(z)$ is enlarged so that z ranges over a region of the complex plane is available (see SZEGŐ [20] and BERNSTEIN [5] and DOCEV [7]).

3. A multivariate generalization operates in terms of harmonic polynomials where the derivative function of Problem 2 is replaced by a gradient expression (see e. g., SZEGŐ [15]). In HÖRMANDER [8] these considerations are related to certain concepts of hyperbolic cones.

We propose a different kind of generalization with Problem 1 as the point of departure. The novelty of these generalizations is to replace polynomials $P_n(t)=\sum_{i=0}^{n} a_i t^i$ formed from the system of functions $\{t^i\}_{i=0}^{n}$ by u-polynomials generated as linear combinations of a Tchebycheff system of functions $\{u_i\}_0^n$.

Let $u_0(t), \ldots, u_n(t)$ be continuous functions on a finite interval $[a, b]$. These functions are called a Tchebycheffian system or T-system provided all the determinants

$$U\begin{bmatrix} t_0, \ldots, t_n \\ 0, \ldots, n \end{bmatrix} = \begin{vmatrix} u_0(t_0) & u_1(t_0) & \ldots & u_n(t_0) \\ u_0(t_1) & u_1(t_1) & \ldots & u_n(t_1) \\ \vdots & \vdots & & \vdots \\ u_0(t_n) & u_1(t_n) & \ldots & u_n(t_n) \end{vmatrix} \tag{3}$$

for arbitrary choices of $\{t_i\}_{i=0}^{n}$ satisfying

$$a \leqq t_0 < t_1 < \ldots < t_n \leqq b$$

maintain a single strict sign. Without restricting generality (multiply $u_n(t)$ suitably by $+1$ or -1), we may assume the determinants in (3) are positive.

In the particular case $u_i(t)=t^i$ $(i=0, \ldots, n)$, (3) reduces to the familiar Vandermonde determinant.

Tchebycheff systems occur naturally in various domains of mathematics. For example, GANTMACHER and KREIN [9] establish that for regular Sturm—Liouville eigenvalue problems with discrete positive spectrum the first $n+1$ eigenfunctions $\varphi_0, \varphi_1, \ldots, \varphi_n$ constitute a T-system. More generally the first $n+1$ eigenfunctions associated with an integral transformation

$$T\varphi = \int_a^b K(x, y)\varphi(y)\, d\sigma(y) \qquad (d\sigma \geqq 0),$$

where $[a, b]$ is finite and K has an iterate which is strictly totally positive, i. e., satisfies certain determinantal inequalities, form a T-system.

T-systems play a role in interpolation problems, moment theory, the study of oscillation properties of polynomials and in other branches of analysis. For a geometrical study of T-systems, the reader may consult KREIN [12] or a forthcoming book [11] by the author and W. STUDDEN.

If the functions $u_i(t)$ $(i=0, 1, \ldots, n)$ are sufficiently differentiable we sometimes extend the definition of

$$U\begin{bmatrix} t_0, \ldots, t_n \\ 0, \ldots, n \end{bmatrix}$$

given in (3) to allow for equalities amongst the t_i. Thus, if $a \leqq t_0 \leqq t_1 \leqq \ldots \leqq t_n \leqq b$ then

$$U^*\begin{bmatrix} t_0, \ldots, t_n \\ 0, \ldots, n \end{bmatrix}$$

is defined as the determinant of (3) where for each set of equal t_i we replace successive rows by their successive derivatives. For example if $a \leqq t_0 = t_1 = \ldots = t_{k-1} < t_k < \ldots$ $\ldots < t_{n-2} < t_{n-1} = t_n \leqq b$, then

$$U^*\begin{bmatrix} t_0, \ldots, t_n \\ 0, \ldots, n \end{bmatrix} = \begin{vmatrix} u_0(t_0) & \cdots & u_n(t_0) \\ u_0^{(1)}(t_0) & \cdots & u_n^{(1)}(t_0) \\ \vdots & & \vdots \\ u_0^{(k-1)}(t_0) & \cdots & u_n^{(k-1)}(t_0) \\ u_0(t_k) & \cdots & u_n(t_k) \\ \vdots & & \vdots \\ u_0(t_{n-2}) & \cdots & u_n(t_{n-2}) \\ u_0(t_n) & \cdots & u_n(t) \\ u_0'(t_n) & \cdots & u_n'(t_n) \end{vmatrix}.$$

The system $\{u_i(t)\}_0^n$ will be called *extended Techebycheffian* of order r (abbreviated ET_r) provided $u_i(t)$ are of class C^{r-1} and

$$U^*\begin{bmatrix} t_0, \ldots, t_n \\ 0, \ldots, n \end{bmatrix} > 0$$

for all $t_0, \ldots, t_n$ satisfying $a \leqq t_0 \leqq t_1 \leqq \ldots \leqq t_n \leqq b$, where equalities are permitted in groups consisting of at most r successive t values.

In the following, the term *polynomial* will refer to a function of the form $u(t) = \sum_{i=0}^{n} a_i u_i(t)$, where the a_i $(i=0, \ldots, n)$ are real constants and the functions $u_i(t)$ $(i=0, 1, \ldots, n)$ constitute either a T-system or an ET-system on a closed interval $[a, b]$. By the *index* of a set $\{t_1, \ldots, t_k\}$ for $t_i \in [a, b]$ $(i=1, \ldots, k)$, we shall mean the number of distinct points in this set under the special convention that the endpoints a and b are counted as one-half while interior points are given a count of one. For example the set $\{a, (a+b)/2\}$ has index 3/2 and the set $\{a, (a+b)/2, b\}$ has index 2.

Before introducing our main theorems we present one lemma due essentially to KREIN [12] which provides information concerning the structure of polynomials and the nature of their zeros. For any polynomial u, t_0 is said to be a non-nodal zero if $u(t_0)=0$ and $u(t) \leqq 0$ or $u(t) \geqq 0$ for t in some open neighborhood of t_0. All other zeros including a and b are called nodal. The symbol $Z(u)$ denotes the number of zeros of u where nodal zeros are counted once and non-nodal zeros twice. The maximum number of sign changes in the sequence $\{a_i\}_1^m$, where zero terms can be counted as either plus or minus, is denoted by $V(a_1, a_2, \ldots, a_m)$.

We have

Lemma 1. (a) *If* $\{u_i(t)\}_0^n$ *is a T-system on* $[a, b]$ *and* $u(t)=\sum_{i=0}^{n} a_i u_i(t)\not\equiv 0$ *is a polynomial, then*

(i) $V(u(t_0), \ldots, u(t_{n+1}))\leqq n$ *for all* t_i *satisfying* $a\leqq t_0<\ldots<t_{n+1}\leqq b$,

(ii) $Z(u)\leqq n$.

(b) *If* $\{u_i(t)\}_0^n$ *is an* ET_r*-system then any polynomial possesses at most n zeros (counting multiplicities up to order r). The number of zeros of a polynomial u by this counting procedure is denoted by* $Z_r^*(u)$.

The proof of Lemma 1 is simple; a formal argument appears in [11].

The key tool of this paper is the representation theorem expressing a positive polynomial as a unique combination of two other polynomials exhibiting special oscillation properties. Various extremal problems are solved by suitably invoking the representation theorem. In this category we include a number of cases of best minimax approximation, the Markoff—Bernstein inequalities for polynomials and extremal problems of a type introduced by BERNSTEIN in [4].

We state the key fact established in [10] which underlies most of the developments of this paper.

Theorem A. *If* $\{u_i(t)\}_0^n$ *is a T-system and* $p(t)$ *and* $q(t)$ *are continuous functions on* $[a, b]$ *such that there exists a polynomial* $v(t)$ *with* $p(t)>v(t)\geqq q(t)$ *then there exists exactly two polynomials* $\underline{u}(t)$ *and* $\bar{u}(t)$ *satisfying the following properties:*

(i) $p(t)\geqq u(t)\geqq q(t)$ *and* $v-u$ *vanishes at n interior points,*

(ii) *there exists* $n+1$ *points* $s_1<s_2<\ldots<s_{n+1}$ *which interlace the zeros of* $v-u$ *such that for* $n=2m$

$$\bar{u}(s_i)=\begin{Bmatrix} p(s_i), & i \text{ odd} \\ q(s_i), & i \text{ even}\end{Bmatrix}, \quad \underline{u}(s_i)=\begin{Bmatrix} p(s_i), & i \text{ even} \\ q(s_i), & i \text{ odd}\end{Bmatrix}$$

and for $n=2m+1$

$$\underline{u}(s_i)=\begin{Bmatrix} p(s_i), & i \text{ odd} \\ q(s_i), & i \text{ even}\end{Bmatrix}, \quad \bar{u}(s_i)=\begin{Bmatrix} p(s_i), & i \text{ even} \\ q(s_i), & i \text{ odd}\end{Bmatrix}.$$

When p and q are polynomials then $\bar{u}$ *can be distinguished from* $\underline{u}$ *in that* $\bar{u}(b)=0$ *while* $\underline{u}(b)\neq 0$.

Corollary. *If p is a polynomial then* $p-\underline{u}$ *and* $p-\bar{u}$ *vanish on a set of index* $n/2$.

There are many extremal problems in the theory of approximation of functions by polynomials whose solutions are intimately connected with the special polynomials $\underline{u}$ and $\bar{u}$ for appropriate choices of p and q. In several natural examples discussed later we will find that the extremal polynomial often coincides with $\underline{u}$ or $\bar{u}$. The validity for this result rests on a simple counting principle which exploits the special oscillation properties of $\bar{u}$ and $\underline{u}$. Actually, since the polynomials $\underline{u}$ and $\bar{u}$ cover the distance between $p(t)$ and $q(t)$ at least n times as t traverses $[a, b]$ it is clear that if $u(t)$ is an arbitrary polynomial lying between p and q then $\bar{u}-u$ and $\underline{u}-u$ exhibit at least n

zeros under the convention that non-nodal zeros are counted twice. But Lemma 1 tells us that $\bar{u}-u$ and similarly $\underline{u}-u$ cannot possess $n+1$ zeros without vanishing identically. This requires that $\bar{u}-u$ or $\underline{u}-u$ obey certain inequalities. Such inequalities can be interpreted as extremal characterizations of $\bar{u}$ and $\underline{u}$ corresponding to certain variational problems.

§ 2. Extremal Problems

Unless stated otherwise we assume throughout this section that $\{u_i(t)\}_{i=0}^{n}$ constitutes an ET_2-system.

Let $p(t)$ be a continuous function on $[a, b]$ and $q(t)$ a fixed *polynomial* satisfying $p(t)>q(t)$ for all $t\in[a, b]$. Consider the class of polynomials,

$$\mathcal{U} = \{u|q(t)\leqq u(t)\leqq p(t),\ t\in[a, b];\ u(a) = q(a)\}. \tag{4}$$

Theorem A (see [10]) and its corollary affirm the existence of a unique polynomial $v_*\in\mathcal{U}$ characterized by

Property A: v_*-q vanishes on a set of index $n/2$ and $p-v_*$ vanishes at least once between each pair of zeros of v_*-q and between the largest interior zero and the endpoint b. The special polynomial v_* is, in fact, the polynomial $\bar{u}$ when n is even and $\underline{u}$ when n is odd. Moreover, v_* enjoys several remarkable extremal properties as attested to by Theorems 1 and 2 which follow.

Theorem 1. *Let $\mathcal{U}$ be defined as in* (4) *and let v_* be the unique polynomial characterized by Property* A. *Then*

$$\max_{u\in\mathcal{U}} u'(a) \tag{5}$$

is attained uniquely in $\mathcal{U}$ by the polynomial v_*.

Proof. Since the polynomials in $\mathcal{U}$ are uniformly bounded, we easily infer that $\mathcal{U}$ is a compact family of polynomials. Hence the maximum in (5) is attained. If a polynomial w attains this maximum then $w'(a)\geqq v_*'(a)$.

Let s_1 be the first zero of $p(t)-v_*(t)$. Clearly $w(t)-v_*(t)$ possesses at least $n-1$ zeros on $[s_1, b]$ with the convention that non-nodal zeros are counted twice. If $w'(a)>v_*'(a)$ then $w(t)-v_*(t)$ has a zero in $(a, s_1]$ which together with the endpoint a and the $n-1$ zeros in $[s_1, b]$ provide a total of $n+1$ zeros. If $w'(a)=v_*'(a)$ then $w(t)-v_*(t)$ exhibits a zero at $t=a$ of multiplicity at least two. In either case $w(t)-v_*'(t)$ has $n+1$ zeros where multiple zeros are counted twice. It follows from Lemma 1 that $w(t)\equiv v_*(t)$.

We next introduce a class of polynomials slightly more restricted than (4), namely

$$\mathcal{U}_0 = \begin{cases} \mathcal{U} & n \text{ odd,} \\ \{u|u\in\mathcal{U},\ u(b) = q(b)\} & n \text{ even.} \end{cases} \tag{6}$$

Assume p is of class C^1 and that

(i) for n even the function

$$f(t) = \frac{p(t)-q(t)}{(b-t)(t-a)}$$

is strictly decreasing on (a, t_0) and strictly increasing on (t_0, b) for some t_0 in (a, b),

(ii) for n odd the function

$$g(t) = \frac{p(t)-q(t)}{t-a}$$

is strictly decreasing on $[a, b]$.

The functions $f(t)$ and $g(t)$ as well as the expressions in (7) and (8) below are extended by continuity where the denominators vanish.

Theorem 2 (Generalized Bernstein—Markoff Inequality). *Let the assumptions stated in* (i) *and* (ii) *prevail. The polynomial* $v_* \in \mathcal{U}_0$, *explicitly characterized by Property* A, *uniquely attains*

$$\max_{u \in \mathcal{U}_0} \max_{t \in [a,b]} \frac{u(t)-q(t)}{(b-t)(t-a)} \tag{7}$$

if n is even and

$$\max_{u \in \mathcal{U}_0} \max_{t \in [a,b]} \frac{u(t)-q(t)}{(t-a)} \tag{8}$$

if n is odd.

Remark. The relation of this theorem and Problem 1 is made explicit in Section 5 where other applications are also indicated.

Proof. We deal only with (7). Note that the special polynomial v_* obeys the property that

$$\frac{v_*(t)-q(t)}{(b-t)(t-a)} \tag{9}$$

oscillates between 0 and $f(t)$ $(a<t<b)$ in a manner that it equals $f(t)$ at m $(n=2m)$ points $s_1, \ldots, s_m$ and equals zero at $m-1$ points $t_1, \ldots, t_{m-1}$ which together satisfy $a<s_1<t_1<s_2<t_2<\ldots<t_{m-1}<s_m<b$. From the monotonicity properties of $f(t)$ ((i) and (ii) above) it follows that the maximum in (7) is achieved for some t in $[a, s_1] \cup [s_m, b]$. Now if $w \in \mathcal{U}_0$ attains (7) and $w \neq v_*$ we infer on the basis of Theorem 1 that

$$\frac{w'(a)-q'(a)}{b-a} < \frac{v_*'(a)-q'(a)}{b-a}.$$

By reversing the interval $[a, b]$ Theorem 1 also implies that

$$\frac{w'(b)-q'(b)}{b-a} > \frac{v_*'(b)-q'(b)}{b-a},$$

so that
$$\frac{w(x_0)-q(x_0)}{(b-x_0)(x_0-a)} \geqq \frac{v_*(x_0)-q(x_0)}{(b-x_0)(x_0-a)}$$

for some $x_0 \in (a, s_1] \cup [s_m, b)$. However, in this case $w-v_*$ has $n-1$ zeros on (a, b)

counting non-nodal zeros twice so that $w-v_*$ possesses $n+1$ zeros on $[a, b]$ and hence $w \equiv v_*$.

The demonstration that v_* is the only polynomial attaining (8) is accomplished by analogous reasoning. The proof of the theorem is complete.

In Theorem 1 it was shown that the polynomial v_* characterized by Property A possesses the maximum derivative at the end point a. This same property of v_* actually holds for an arbitrary $z \in [a, b]$ in a sense that we now describe.

For n odd, say $n = 2m+1$, there exist points $a = \tilde{t}_0 < \tilde{s}_1 < \tilde{t}_1 < \tilde{s}_2 < \ldots < \tilde{t}_m < < \tilde{s}_{m+1} \leqq b$ at which the polynomial v_* satisfies the relations

$$v_*(\tilde{t}_k) = q(\tilde{t}_k) \qquad (k=0, 1, \ldots, m)$$

and

$$v_*(\tilde{s}_k) = p(\tilde{s}_k) \qquad (k=1, 2, \ldots, m+1).$$

For $n = 2m+2$ the only modification is that $\tilde{s}_{m+1} < b$ and $v_*(b) = q(b)$. As t traverses an interval of the type $(\tilde{t}_i, \tilde{s}_{i+1})$ the polynomial v_* extends from the value $q(\tilde{t}_i)$ to the value $p(\tilde{s}_{i+1})$ while on an interval of the type $(\tilde{s}_i, \tilde{t}_i)$ the values of $v_*(t)$ vary from $p(\tilde{s}_i)$ to $q(\tilde{t}_i)$. Generally v_* is increasing on $(\tilde{t}_i, \tilde{s}_{i+1})$ and decreasing on $(\tilde{s}_i, \tilde{t}_i)$.

Bearing this in mind we define the sets A and B as follows.

$$A = (\tilde{t}_0, \tilde{s}_1) \cup (\tilde{t}_1, \tilde{s}_2) \cup \ldots \cup (\tilde{t}_m, \tilde{s}_{m+1}),$$

$$B = \begin{cases} (\tilde{s}_1, \tilde{t}_1 \cup (\tilde{s}_2, \tilde{t}_2) \cup \ldots \cup (\tilde{s}_m, \tilde{t}_m) & (n = 2m+1), \\ (\tilde{s}_1, \tilde{t}_1) \cup (\tilde{s}_2, \tilde{t}_2) \cup \ldots \cup (\tilde{s}_m, \tilde{t}_m) \cup (\tilde{s}_{m+1}, b) & (n = 2m+2). \end{cases}$$

For each fixed $z \in [a, b]$ let $\mathcal{U}(z)$ be the class of polynomials

$$\mathcal{U}(z) = \{u \mid q(t) \leqq u(t) \leqq p(t),\ t \in [a, b],\ u(z) = v_*(z)\}.$$

Theorem 3. *The polynomial v_* uniquely attains*

$$\max_{u \in \mathcal{U}(z)} u'(z) \quad \textit{if} \quad z \in A, \tag{10}$$

$$\min_{u \in \mathcal{U}(z)} u'(z) \quad \textit{if} \quad z \in B. \tag{11}$$

The proof is similar to that of Theorem 2 and is therefore omitted.

Remark. We emphasize again that if $\{u_i\}_0^n$ is not an ET-system of order 2 but simply a T-system, the preceding theorems remain in force except for the uniqueness assertion.

§ 3. Generalized Bernstein—Markoff Inequalities for Infinite Intervals

The result of Theorem 2 may easily be extended to the semi-infinite interval $[0, \infty)$. We assume that $\{u_i\}_0^n$ is an ET-system of order 2 on $[0, \infty)$ and in addition we impose the following requirements:

(i) $u_n(t) > 0$, $t \geqq \hat{t}$ for some $\hat{t} > 0$,

(ii) $\lim_{t \to \infty} u_i(t)/u_n(t) = 0$, $i = 0, \ldots, n-1$,

(iii) $\{u_i\}_0^{n-1}$ is a T-system on $[0, \infty)$.

Consider a continuously differentiable function $f(t)>0$ on $[0, \infty)$ satisfying the condition that

$$\lim_{t\to\infty} \frac{f(t)}{u_n(t)} \tag{12}$$

exists where its value is positive or plus infinity. Notice that these assumptions include the case where $u_i(t)=t^i$ $(i=0, 1, \ldots, n)$ and $f(t)=e^t$.

We wish to determine the polynomial $\tilde{u}$ which attains

$$\max_{u\in A} \max_{t\in[0,\infty]} \frac{u(t)}{tf(t)} \tag{13}$$

where A is the class of polynomials defined by the conditions

$$A = \{u \mid 0 \leqq u(t) \leqq f(t),\ u(0) = 0\}$$

and the value $t^{-1}u(t)$ is defined to be $u'(0)$ for $t=0$.

To solve this problem we first determine that polynomial (as in Theorem 1) which attains

$$\max_{u\in A} u'(0). \tag{14}$$

As in the case of the finite interval the extremal polynomials which yield the maximums in (13) and (14) agree.

Let $w(t)$ be a strictly positive function on $[0, \infty)$ such that $w(t)=u_n(t)$, $t\geqq\hat{t}$ and set

$$v_k(x) = \begin{cases} \dfrac{u_k(\tan x)}{w(\tan x)} & (x\in[0, \pi/2)), \\ \delta_{kn} & (x = \pi/2), \end{cases}$$

$$\bar{f}(x) = \frac{f(\tan x)}{w(\tan x)} \qquad (x\in[0, \pi/2]).$$

The system $\{v_k\}_0^n$ is a T-system on $[0, \pi/2]$ and $\bar{f}(x)>0$ on $[0, \pi/2)$ and its value at $\pi/2$ is positive or possibly infinite because of (12).

Suppose first that $\bar{f}(\pi/2)$ is finite. Let v_* be the polynomial satisfying Property A. If $\bar{f}(\pi/2)$ is infinite we construct v_* first for

$$\bar{f}_N(x) = \begin{cases} \bar{f}(x) & (f(x)\leqq N), \\ N & (f(x)>N) \end{cases}$$

observing that the same polynomial v_* occurs for all N sufficiently large.

We now transform the polynomial v_* according to

$$u_*(t) = w(t)v_*(\tan^{-1} t) \qquad (t\in[0, \infty)). \tag{15}$$

For $n=2m$ the polynomial $u_*(t)$ vanishes at the points

$$0 = \bar{t}_0 < \bar{t}_1 < \ldots < \bar{t}_{m-1}$$

and agrees with $f(t)$ at the points $\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_m$. Also, these values interlace in the manner

$$0 = \tilde{t}_0 < \tilde{s}_1 < \tilde{t}_1 < \ldots < \tilde{t}_{m-1} < \tilde{s}_m < \infty$$

The corresponding points for the case $n = 2m+1$ have the form

$$0 = \tilde{t}_0 < \tilde{s}_1 < \tilde{t}_1 < \tilde{s}_2 < \ldots < \tilde{s}_m < \tilde{t}_m < \tilde{s}_{m+1} < \infty.$$

Note that in the even case we have $2m-1$ zeros counting non-nodal zeros twice. An additional zero occurs at plus infinity and necessarily $v_*(t) = \alpha_0 v_0(t) + \ldots + \alpha_{n-1} v_{n-1}(t)$, the coefficient of $v_n(t)$ being zero since v_* vanishes at $\pi/2$.

Theorem 4. *The polynomial u_* defined by* (15) *uniquely attains*

$$\max_{u \in \Lambda} u'(0)$$

and

(16)
$$\max_{u \in \Lambda} \max_{t \in [0, \infty]} \frac{u(t)}{tf(t)}$$

where
$$\Lambda = \{u | 0 \leqq u(t) \leqq f(t),\ u(0) = 0\}.$$

The proof of this theorem parallels the preceding analysis. Note that $u(t)/f(t) \leqq 1$ and that the function $1/t$ is decreasing and so the maximum in (16) is achieved in the interval $(0, \tilde{s}_1)$. From here on the argument proceeds as in Theorem 2.

§ 4. Generalized Bernstein—Markoff Inequalities for Periodic Functions

In this section we develop some periodic versions of the generalized Bernstein—Markoff type inequalities.

As is natural to the circle case we restrict n to be even, say $n = 2m$, and assume that $\{u_i\}_0^{2m}$ constitutes an *ET* system of order 2 consisting of periodic functions (see [11]). An interval of periodicity is assumed to be of length $b-a$. We sometimes require that $\{u_i\}_0^{2m}$ be *ET* of order 3. The extended Tchebycheffian assumptions of order 2 and 3 are imposed in order to assure unique solutions to various extremal problems.

The following basic theorem proved in [10] is the counterpart of Theorem A.

Theorem B. *Let $p(t)$ be a positive, periodic and continuous function of period length $b-a$. For a fixed $t_0 \in [a, b)$, let $v(t; t_0)$ represent the unique polynomial constructed in Theorem* 6 *of* [10] *possessing the properties:*

(i) $p(t) \geqq v(t; t_0) \geqq 0$,

(ii) *$v(t; t_0)$ has m distinct zeros one of which is t_0,*

(iii) *$p(t) - v(t; t_0)$ vanishes at least once between each pair of zeros of $v(t; t_0)$ (viewed in the periodic sense).*

Consider the class of polynomials

$$\mathcal{W}(t_0) = \{u | 0 \leqq u(t) \leqq p(t),\ t \in [a, b),\ u(t_0) = 0\}.$$

An extremal characterization of $v(t; t_0)$ is embodied in the following result. (Compare with Theorem 1.)

Theorem 5. *If* $\{u_i\}_0^{2m}$ *is an ET-system of order* 3 *then*

$$\max_{u\in\mathscr{W}(t_0)} u''(t_0)$$

is uniquely attained by the polynomial $v(t;t_0)$.

Proof. Let w be any polynomial in $\mathscr{W}(t_0)$ for which

$$w''(t_0) = \max_{u\in\mathscr{W}(t_0)} u''(t_0).$$

Then $w''(t_0)\geqq v''(t_0;t_0)$. If $w''(t_0)>v''(t_0;t_0)$ the function $w(t)-v(t;t_0)$ has at least $2m+1$ zeros counting multiple zeros twice. If $w''(t_0)=v''(t_0;t_0)$ then $w(t)-$ $-v(t;t_0)$ has at least $2m+1$ zeros where the zero at t_0 is of order 3. In both contingencies we contradict the assumptions on the system $\{u_i\}_0^{2m}$ unless $w(t)\equiv v(t;t_0)$.

Our next objective concerns the formulation of the analog of Theorem 3. Let $p(t)$ and $q(t)$ be two continuous periodic functions on $[a,b)$ for which $p(t)>q(t)$ and suppose there exists a polynomial $\tilde{u}$ satisfying $p(t)>\tilde{u}(t)>q(t)$. For each $t_0\in[a,b)$, Theorem 7 of [10] affirms the existence of a unique polynomial $u(t;t_0)$ possessing the properties

(i) $p(t)\geqq u(t;t_0)\geqq q(t)$,

(ii) $\tilde{u}(t_0)=u(t_0;t_0)$ and there exists n points $\{s_i\}_1^n$, $s_1<t_0<s_2<\ldots<s_n<s_1+b-a$ such that and $u(t;t_0)$ equals $q(t)$ and $p(t)$ alternately at $s_1, s_2, \ldots, s_n$.

An example of the class of polynomials $u(t;t_0)$ is given later in the case of trigonometric polynomials where $p(t)=-q(t)=a$ positive polynomial $h(t)$ of order at most m. The oscillation properties of $u(t;t_0)$ are basic to the solution of certain extremal problems as described in Theorem 6 below. In order to prepare for Theorem 6, we note some preliminaries.

Suppose we specify a point $z_0\in[a,b)$ and a value c satisfying $q(z_0)<c<p(z_0)$. Since the coefficients of $u(t;t_0)$ are continuous functions of the parameter t_0 we deduce the existence of two polynomials $\bar{u}$ and $\underline{u}$ each equal to c at the point z_0 satisfying conditions (i) and (ii) except that $\bar{u}$ and $\underline{u}$ alternate in the opposite direction. Specifically,

$$\underline{u}(s_1)=p(s_1) \quad\text{and}\quad \underline{u}(s_2)=q(s_2), \quad\text{etc.}$$

while

$$\bar{u}(s_1)=q(s_1) \quad\text{and}\quad \bar{u}(s_2)=p(s_2), \quad\text{etc.}$$

The sets of points $\{s_i\}$ associated with the two polynomials $\bar{u}$ and $\underline{u}$ will in general differ.

Consider the class of polynomials $\mathscr{V}$ defined by

$$\mathscr{V}(z_0)=\{u\,|\,q(t)\leqq u(t)\leqq p(t),\ t\in[a,b),\ u(z_0)=c,\ q(z_0)<c<p(z_0)\}.$$

Lemma 2. *If* $\{u_i\}_0^{2m}$ *is a periodic ET-system of order* 2 *then*

$$\max_{u\in\mathscr{V}(z_0)} u'(z_0) = \bar{u}'(z_0)$$

and

$$\min_{u\in\mathscr{V}(z_0)} u'(z_0) = \underline{u}'(z_0).$$

In each case the extremal polynomial is unique.

Proof. The proof again is accomplished by appropriately counting zeros and using the fact that $\bar{u}$ and $\underline{u}$ oscillate a maximum number of times between $p(t)$ and $q(t)$. We omit the details.

We are now in possession of the ingredients necessary to prove the principal theorem of this section. The admissible class of polynomials consists of

$$\mathscr{V}_h = \{u \mid |u(t)| \leqq h(t)\}$$

where $h(t)$ is a strictly positive continuous periodic function on $[a, b)$.

Theorem 7. *Let $\{u_i\}_0^{2m}$ be a periodic ET-system of order 2. The value*

$$\max_{u \in \mathscr{V}_h} \max_{t \in [a,b)} |u'(t)|$$

is attained by a polynomial $u(t; t_0)$ for some t_0. In other words, in computing the maximum it is enough to restrict attention to the one parameter family of polynomials $u(t; t_0)$.

Proof. Consider any member $u \in \mathscr{V}_h$ and fix a point $t^* \in [a, b)$. We distinguish two cases according as $|u(t^*)| < h(t^*)$ or $|u(t^*)| = h(t^*)$. If the first possibility prevails then we appeal to Lemma 2 which affirms the existence of a polynomial $u(t; t_0) \in \mathscr{V}_h$ for some t_0 with the properties

$$u(t^*; t_0) = u(t^*) \quad \text{and} \quad |u'(t^*, t_0)| \geqq |u'(t^*)|. \tag{17}$$

In the second case when $|u(t^*)| = h(t^*)$ we increase the function h slightly to h_ε creating the situation $|u(t^*)| < h_\varepsilon(t^*)$. The polynomials $u_\varepsilon(t; t_0)$ and their derivatives $u'_\varepsilon(t, t_0)$ vary continuously with ε uniformly in t. (This is so since the coefficients are continuous functions of ε and t_0.) We can now appeal to the preceding case and deduce again the validity of (17) with $u(t^*; t_0)$ replaced by $u_\varepsilon(t^*; t_0)$ (here t_0 may also depend on ε). Invoking the standard limiting process on ε we infer in all circumstances the validity of (17).

The second relation of (17) can be expressed in the form

$$\sup_{t_0} |u'(t^*; t_0)| \geqq \sup_{u \in \mathscr{V}_h} |u'(t^*)|.$$

Since t^* is arbitrary in $[a, b)$, the assertion of the theorem is established.

§ 5. Examples

We begin with two examples of Theorem 7.

Example 1. Consider the T-system of trigonometric functions

$$1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \ldots, \cos m\theta, \sin m\theta. \tag{18}$$

The special polynomials $u(\theta; \theta_0)$ oscillating between $p(\theta) \equiv 1$ and $q(\theta) \equiv -1$ asserted in Theorem B are $\sin m(\theta + \theta_0)$. Application of Theorem 7 and Lemma 2 gives the following classical result. If $g(\theta)$ is a trigonometric polynomial of degree m and

$$|g(\theta)| \leqq 1$$

then

$$|g'(\theta)| \leqq m,$$

with equality if and only if $g(\theta) = \sin m(\theta + \theta_0)$.

Example 2. The T-system under consideration is again (18). Let $h(\theta)$ be a fixed positive trigonometric polynomial of degree $l \leqq m$ and let

$$h(\theta) = |g(z)|^2$$

where $z = e^{i\theta}$ and

$$g(z) = \gamma \prod_{\nu=1}^{l} (z - z_\nu) \qquad (\gamma > 0, |z_\nu| < 1, \nu = 1, \ldots, l).$$

The special polynomials of Theorem B lying between $h(\theta)$ and $-h(\theta)$ of maximum oscillation can be explicitly determined. In fact the family of polynomials

$$u(\theta;\, \theta_0) = \left(\Re e^{i\psi(\theta_0)} z^{m-2l}[g(z)]^2\right), \quad z = e^{i\theta}, \quad 0 \leqq \theta < 2\pi,$$

where ψ is a real parameter depending on θ_0 ($\Re$ = real part), fulfills the requirements

$$-h(\theta) \leqq u(\theta;\, \theta_0) \leqq h(\theta)$$

and $u(\theta; \theta_0)$ touches $h(\theta)$ and $-h(\theta)$ alternately m times. The formal proof of this fact appears in [11], see also SZEGŐ [18]. These polynomials coincide with the class $u(\theta; \theta_0)$ described in Theorem B.

We apply Theorem 7 as follows.

Suppose $P(\theta)$ is a trigonometric polynomial of order at most m satisfying

$$|P(\theta)| \leqq \gamma^{-2} h(\theta).$$

Then the value

$$\max_P \max_\theta |P'(\theta)| \tag{19}$$

is achieved by a polynomial of the form

$$u(\theta;\, \varphi) = \gamma^{-2} \Re[e^{i\varphi} z^{m-2l} (g(z))^2] \quad (z = e^{i\theta}),$$

where φ is a parameter.

In the special case where $h(\theta) = 1 - 2r \cos\theta + r^2$ ($|r| < 1$) a few elementary calculations show that the value of (19) is $(1 + |r|)^2 (m + |r|(m-2))^2$.

We next turn to

Example 3. All the results of this example are classical, however, it is instructive to fit them into the framework of the previous sections.

We start with the representation

$$1 = T_m^2(t) + (1-t)^2 U_{m-1}^2(t) \qquad (-1 \leqq t \leqq 1), \tag{20}$$

where

$$T_m(t) = \cos m\theta \qquad (t = \cos\theta),$$

$$U_m(t) = \frac{1}{m+1} T'_{m+1}(t) = \frac{\sin(m+1)\theta}{\sin\theta}$$

are the Tchebycheff polynomials of the first and second kind respectively. In the notation of Theorem A, equation (20) expresses the polynomial $u(t)\equiv 1$ as a sum of the extreme polynomials $\underline{u}(t)=T_m^2(t)$ and $\bar{u}(t)=(1-t^2)U_{m-1}^2(t)$.

(i) Let $q(t)\equiv 0$ and $p(t)=1$ in Theorem 1. We then recognize the class of polynomials $\mathscr{U}$ defined in (4) for $n=2m$ as those polynomials $P_{2m}(t)=\sum_{i=0}^{n} a_i t^i$ satisfying

(21) $$0\leqq P_{2m}(t)\leqq 1, \quad -1\leqq t\leqq 1 \quad \text{and} \quad P_{2m}(-1)=0.$$

Invoking Theorem 1 we conclude that

$$P_{2m}'(-1)\leqq \frac{d}{dt}(1-t^2)U_{m-1}^2(t)|_{t=1}$$

with equality prevailing only if $P_{2m}(t)=v_*(t)=(1-t^2)U_{m-1}^2(t)$.

(ii) Problem 1 posed at the start of the paper was concerned with the task of calculating the maximum of

$$\max_{-1\leqq t\leqq 1}|P_{m-1}(t)|$$

over the set of polynomials of degree $m-1$ satisfying the condition

$$(1-t^2)P_{m-1}^2(t)\leqq 1 \qquad (-1\leqq t\leqq 1).$$

The solution to this problem is contained in the following slightly more general result. We consider the class $\mathscr{U}_0$ (see (6)) of polynomials satisfying (21) and the further condition that $P_{2m}(+1)=0$. Consulting Theorem 2 we may conclude that for $P_{2m}\in\mathscr{U}_0$ we have

(22) $$\max_{-1\leqq t\leqq 1}\frac{P_{2m}(t)}{1-t^2}\leqq \max_{-1\leqq t\leqq 1}U_{m-1}^2(t)=m^2$$

with equality present only when $P_{2m}(t)=(1-t^2)U_{m-1}^2(t)$.

Furthermore an induction argument using the relation

$$U_{m-1}(t)=\frac{\sin m\theta}{\sin\theta}=\cos(m-1)\theta+\cos\theta\,\frac{\sin(m-1)\theta}{\sin\theta} \qquad (m\geqq 1)$$

shows that equality on the right side of (22) is attained exclusively at $t=\pm 1$.

The solution of Problem 1 which we have just described may be cast in the following form: if $P_{m-1}(t)$ is a polynomial of degree $m-1$ on $[-1,1]$ and

$$\sqrt{1-t^2}\,|P_{m-1}(t)|\leqq 1$$

then

$$|P_{m-1}(t)|\leqq m$$

with equality only if $P_{m-1}(t)=\gamma U_{m-1}(t)$, ($|\gamma|=1$) and $t=\pm 1$.

(iii) In continuing to apply the results of Section 2 we consider Theorem 3. The polynomial $(1-t^2)U_{m-1}^2(t)$ vanishes at

$$t_k=-\cos\frac{k\pi}{m} \qquad (k=0,1,\ldots,m)$$

and equals one at the zeros of $T_m(t)$, i.e., at

$$s_k = \cos\frac{2k-1}{2m}\pi, \qquad (k = 1, 2, \dots, m).$$

Theorem 3 asserts that for $z \in (-1, s_1) \cup (t_1, s_2) \cup \dots \cup (t_{m-1}, s_m)$ the polynomial $(1-t^2)U_{m-1}^2(t)$ has the maximum derivative at the point z among all polynomials P_{2m} obeying the contraints $0 \leqq P_{2m}(t) \leqq 1$ and $P_{2m}(z) = (1-z^2)U_{m-1}^2(z)$.

Example 4. The applications of the preceding paragraph involved the specific functions $p(t) \equiv 1$ and $q(t) \equiv 0$. Other specifications lead to new inequalities. For example we suppose again that $q(t) \equiv 0$ and now define $p(t)$ by

$$(23) \quad m(m+\alpha+\beta+1)p(t) = m(m+\alpha+\beta+1)\{P_m^{(\alpha,\beta)}(t)\}^2 + (1-t^2)\left\{\frac{d}{dt}P_m^{(\alpha,\beta)}(t)\right\}^2$$

where $P_m^{(\alpha,\beta)}(t)$ are the Jacobi polynomials, orthogonal on $[-1, 1]$ with respect to the weight function $w(t) = (1-t)^\alpha(1+t)^\beta$ $(\beta > -1, \alpha > -1)$ and normalized by the condition

$$P_m^{(\alpha,\beta)}(1) = \binom{m+\alpha}{m}.$$

In order to ascertain the monotonicity properties of $p(t)$ required in Theorem 2 we proceed as follows. It is familiar that $y = P_m^{(\alpha,\beta)}(t)$ satisfies the differential equation

$$(1-t^2)y'' + [\beta-\alpha-(\alpha+\beta+2)t]y' + m(m+\alpha+\beta+1)y = 0$$

and therefore

$$(24) \qquad m(m+\alpha+\beta+1)p'(t) = 2(\alpha-\beta+(\alpha+\beta+1)t)\left\{\frac{d}{dt}P_m^{(\alpha,\beta)}(t)\right\}^2$$

(see Szegő [17, Chap. 7]) from which it is clear that $p'(t)$ changes sign once at $t_0 = (\beta-\alpha)/(\alpha+\beta+1)$. Note that $-1 < t_0 < 1$ if and only if $(\alpha+1/2)(\beta+1/2) > 0$.

If $\alpha = \beta = \lambda - 1/2$, the ultraspherical polynomials $P_m^{(\lambda)}(t)$ are defined by

$$P_m^{(\lambda)}(t) = \frac{\Gamma(\lambda+1/2)\Gamma(m+2\lambda)}{\Gamma(2\lambda)\Gamma(m+\lambda+1/2)} P_m^{(\lambda-1/2,\lambda-1/2)}(t), \qquad \lambda > -\frac{1}{2}.$$

In this case the calculation in (24) reduces to

$$m(m+2\lambda)p'(t) = 2\lambda t\left\{\frac{d}{dx}P_m^{(\alpha,\beta)}(t)\right\}^2.$$

If $\lambda > 0$ then $p'(t) > 0$ for $0 < t < 1$ and $p'(t) < 0$ for $-1 < t < 0$. Therefore

$$f(t) = \frac{p(t)}{1-t^2}$$

satisfies the monotonicity conditions stipulated in Theorem 2.

By applying Theorem 2 with the function

$$m(m+2\lambda)p(t) = m(m+2\lambda)\{P_m^{(\lambda)}(t)\}^2 + (1-t^2)\left\{\frac{d}{dt}P_m^{(\lambda)}(t)\right\}^2$$

we obtain the result that any polynomial $Q_{2m}(t)$ of degree $\leqq 2m$ obeying the restrictions $0 \leqq Q_{2m}(t) \leqq m(m+2\lambda)p(t)$ and $Q_{2m}(\pm 1)=0$ also fulfills the inequality

$$\max_{-1 \leqq t \leqq 1} \frac{Q_{2m}(t)}{1-t^2} \leqq \max_{-1 \leqq t \leqq 1} \left\{\frac{d}{dt} P_m^{(\lambda)}(t)\right\}^2 \tag{25}$$

and equality occurs only if $Q_{2m}(t) = (1-t^2)\left\{\frac{d}{dt} P_m^{(\lambda)}(t)\right\}^2$.

For $p(t)$ defined by (23) Theorems 1 and 3 apply for all values $\alpha > -1$ and $\beta > -1$.

These examples are typical expressions of Theorems $1-7$. We refer the reader to [11] for other applications and refinements of these ideas.

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