On certain classes of power-bounded operators in Hilbert space

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Dedicated to P. R. Halmos on his 50th birthday

1. Let $\mathscr{C}_\varrho\,(\varrho>0)$ denote the class of those (bounded, linear) operators T in Hilbert space $\mathfrak{H}$, whose powers T^n $(n=1,2,\ldots)$ admit a representation

$$T^n=\varrho\cdot \mathrm{pr}\, U^n \qquad (n=1,2,\ldots), \tag{1}$$

where U is a unitary operator in some Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace.

It is known that the class $\mathscr{C}_1$ consists precisely of the contraction operators T, i.e. for which

$$\|T\|\leqq 1, \tag{2}$$

cf. [1], and that $\mathscr{C}_2$ consists precisely of those T for which

$$w(T)\leqq 1. \tag{3}$$

The latter fact was discovered by C. A. BERGER (not yet published); simplified proofs appear in [2] and [3]. *Norm* $\|T\|$ and *numerical radius* $w(T)$ of an operator are defined by

$$T=\sup\frac{\|Th\|}{\|h\|},\quad w(T)=\sup\frac{|(Th,h)|}{\|h\|^2}\qquad (h\in\mathfrak{H},\ h\neq 0).$$

Clearly, every operator T of class $\mathscr{C}_\varrho$ is power-bouded, indeed we have $\|T^n\|\leqq\varrho$, but the converse is not true. We shall give an example of a power-bounded operator which is not contained in any of the classes $\mathscr{C}_\varrho(\varrho>0)$.

2. First we give a characterization of the classes $\mathscr{C}_\varrho$.

Theorem. *In order that the operator T in $\mathfrak{H}$ belong to the class $\mathscr{C}_\varrho$ it is necessary and sufficient that the following conditions be satisfied:*

$$\|h\|^2-2\left(1-\frac{1}{\varrho}\right)\mathrm{Re}\,(zTh,h)+\left(1-\frac{2}{\varrho}\right)\|zTh\|^2\geqq 0 \quad \textit{for}\quad h\in\mathfrak{H}\quad \textit{and}\quad |z|\leqq 1, \tag{I$_\varrho$}$$

(II) *the spectrum of T lies in the closed unit disk.*

For $\varrho\leqq 2$ condition (I$_\varrho$) *implies* (II).

Proof. Suppose that (1) holds. Since U is unitary, the series $I_{\mathfrak{K}}+2zU+\cdots+ +2z^nU^n+\cdots$ converges in the norm for every z, $|z|<1$, its sum being equal to $(I_{\mathfrak{K}}+zU)(I_{\mathfrak{K}}-zU)^{-1}$. By virtue of (1), the series $I_{\mathfrak{H}}+\frac{2}{\varrho}zT+\cdots+\frac{2}{\varrho}z^nT^n+\cdots$ also will converge in the norm, its sum being then equal to $\left(1-\frac{2}{\varrho}\right)I_{\mathfrak{H}}+\frac{2}{\varrho}(I_{\mathfrak{H}}-zT)^{-1}$. Hence we infer first that $(\mu I_{\mathfrak{H}}-T)^{-1}$ exists as a bounded operator in $\mathfrak{H}$ for $|\mu|>1$, i.e. condition (II) is fulfilled. Moreover, we have

$$(4)\qquad \left(1-\frac{2}{\varrho}\right)I+\frac{2}{\varrho}(I-zT)^{-1} = \mathrm{pr}\,(I+zU)(I-zU)^{-1} \qquad (|z|<1).$$

Since

$$\mathrm{Re}\big((I+zU)k,\ (I-zU)k\big) = \|k\|^2-|z|^2\|Uk\|^2 = (1-|z|^2)\|k\|^2 \geqq 0 \text{ for } k\in\mathfrak{K},\ |z|<1,$$

we have

$$\mathrm{Re}\big((I+zU)(I-zU)^{-1}k,\ k\big)\geqq 0 \quad\text{for}\quad k\in\mathfrak{K},\ |z|<1,$$

thus, by (4),

$$(5)\qquad \mathrm{Re}\left[\left(1-\frac{2}{\varrho}\right)(l,l)+\frac{2}{\varrho}\big((I-zT)^{-1}l,\ l\big)\right] \geqq 0 \quad\text{for}\quad l\in\mathfrak{H},\ |z|<1.$$

Set $l = l_z = (I-zT)h$, where h is an arbitrary element of $\mathfrak{H}$. Then (5) yields

$$(6)\qquad \left(1-\frac{2}{\varrho}\right)\|(I-zT)h\|^2+\frac{2}{\varrho}\,\mathrm{Re}\big(h,\ (I-zT)h\big) \geqq 0 \quad\text{for}\quad h\in\mathfrak{H},\ |z|<1,$$

whence (I_ϱ) follows by a simple rearrangement, at least for $|z|<1$. The limit case $|z|=1$ can be included by continuity.

Suppose now, conversely, that (I_ϱ) and (II) hold for T. By (II), $(I-zT)^{-1}$ exists as a bounded operator in $\mathfrak{H}$, for $|z|<1$. From (I_ϱ) we obtain (6) by the inverse of the above mentioned rearrangement. Setting $h = h_z = (I-zT)^{-1}l$ in (6), where l is an arbitrary element of $\mathfrak{H}$, we get (5). This means that the operator valued function

$$(7)\qquad F(z) = \left(1-\frac{2}{\varrho}\right)I+\frac{2}{\varrho}(I-zT)^{-1}$$

satisfies the condition

$$(8)\qquad \mathrm{Re}\, F(z)\geqq O.$$

Since, moreover, $F(z)$ is holomorphic in the unit disc ($|z|<1$), and $F(0)=I$, it follows from a theorem of F. Riesz, generalized to operator valued functions, that there exists a unitary operator U in some space $\mathfrak{K}(\supseteqq\mathfrak{H})$, such that

$$(9)\qquad F(z)=\mathrm{pr}\,(I+zU)(I-zU)^{-1} \qquad (|z|<1),$$

cf. e.g. [1]. Since

$$(I+zU)(I-zU)^{-1} = I+2zU+\cdots+2z^nU^n+\cdots \quad\text{for}\quad |z|<1,$$

and, by (7),

$$F(z) = I+\frac{2}{\varrho}zT+\cdots+\frac{2}{\varrho}z^nT^n+\cdots \quad\text{at least for}\quad |z|\,\|T\|<1,$$

it results from (9) by comparing coefficients that

$$\frac{1}{\varrho} T^n = \mathrm{pr}\, U^n \qquad (n = 1, 2, \ldots),$$

i.e. $T \in \mathscr{C}_\varrho$.

Thus we have proved that (I_ϱ) and (II) characterize the operators of class $\mathscr{C}_\varrho$.

We have still to prove the last statement of the theorem. We start with relation (6) which is an equivalent form of (I_ϱ). If $\varrho \leqq 2$ we have $1-2/\varrho \leqq 0$ so that (6) implies

$$\mathrm{Re}\big(h, (I-zT)h\big) \geqq 0 \qquad (h \in \mathfrak{H},\ |z| \leqq 1);$$

choosing an adequate value for z we obtain hence

(10) $$\|h\|^2 \geqq |(Th, h)| \qquad (h \in \mathfrak{H}).$$

Consider the self-adjoint operator $R_z = \mathrm{Re}\,(I-zT)$. Since

$$(R_z h, h) = \mathrm{Re}\big((I-zT)h, h\big) = \|h\|^2 - \mathrm{Re}\, z(Th, h) \geqq (1-|z|)\|h\|^2,$$

we have $R_z \geqq (1-|z|)I$, thus if $|z|<1$ then $Q_z = R_z^{1/2} \geqq (1-|z|)^{1/2} I$, Q_z^{-1} exists as a bounded, everywhere defined operator, $\|Q_z^{-1}\| \leqq (1-|z|)^{-1/2}$. We have for $|z|<1$

$$I - zT = R_z + i\,\mathrm{Im}\,(I-zT) = R_z - i\,\mathrm{Im}\,(zT) = Q_z[I - iQ_z^{-1}\,\mathrm{Im}\,(zT) Q_z^{-1}]Q_z.$$

Since $Q_z^{-1}\,\mathrm{Im}\,(zT)\, Q_z^{-1}$ is selfadjoint, the operator in [ ] has an inverse, everywhere defined and bounded by 1. Thus $I-zT$ also has a bounded and everywhere defined inverse; indeed,

$$\|(I-zT)^{-1}\| \leqq (1-|z|)^{-1} \qquad (|z|<1).$$

This implies (II), moreover the inequality

(10′) $$\|(\mu I - T)^{-1}\| \leqq \frac{1}{|\mu|-1} \quad \text{for} \quad 1 < |\mu| < \infty.$$

This concludes the proof of the theorem.

It is clear that for $\varrho = 1$ and $\varrho = 2$, (I_ϱ) reduces to condition (2) and (3), respectively. Thus our theorem generalizes the characterizations of the classes $\mathscr{C}_1$ and $\mathscr{C}_2$ mentioned in § 1.

We may complete this remark with the following ones:

Remark 1. *If* $0<\varrho<2$, $\varrho \neq 1$, (I_ϱ) *reduces to the condition*

(I'_ϱ) $$\|(\mu I - T)h\| \leqq \frac{|\mu|}{|\varrho-1|}\|h\| \quad \textit{for} \quad \left|\frac{\varrho-1}{\varrho-2}\right| \leqq |\mu| < \infty, \quad h \in \mathfrak{H},$$

while if $2<\varrho<\infty$, (I_ϱ) *reduces to the condition*

(I''_ϱ) $$\|(\mu I - T)h\| \geqq \frac{|\mu|}{\varrho-1}\|h\| \quad \textit{for} \quad \frac{\varrho-1}{\varrho-2} \leqq |\mu| < \infty, \quad h \in \mathfrak{H}.$$

Proof. If $0<\varrho<2$, $\varrho\neq 1$, multiplication by the negative factor $\varrho/(\varrho-2)$, and an easy rearrangement transforms (I_ϱ) into the equivalent form

$$(11)\qquad \left\|\left(\frac{\varrho-1}{\varrho-2}I-zT\right)h\right\|^2-\frac{1}{(\varrho-2)^2}\|h\|^2\leqq 0\qquad (h\in\mathfrak{H},\ |z|\leqq 1).$$

Setting $z=\dfrac{\varrho-1}{\varrho-2}\dfrac{1}{\mu}$, (11) can be expressed in the equivalent form (I'_ϱ).

If $2<\varrho<\infty$, multiplication by the positive factor $\varrho/(\varrho-2)$ and the same easy rearrangement transforms (I_ϱ) into the equivalent form

$$(12)\qquad \left\|\left(\frac{\varrho-1}{\varrho-2}I-zT\right)h\right\|^2-\frac{1}{(\varrho-2)^2}\|h\|^2\geqq 0\qquad (h\in\mathfrak{H},\ |z|\leqq 1).$$

Setting, as above, $z=\dfrac{\varrho-1}{\varrho-2}\dfrac{1}{\mu}$, (12) transforms into the equivalent form (I''_ϱ).

Remark 2. *In order that T be of class $\mathscr{C}_\varrho$ with $1<\varrho<2$, it is necessary and sufficient that the condition*

$$(\mathrm{III}'_\varrho)\qquad \|\mu I-T\|\leqq|\mu|+1\quad \textit{for}\quad \frac{\varrho-1}{2-\varrho}\leqq|\mu|<\infty$$

hold.

Proof. (III'_ϱ) implies (I'_ϱ) since

$$|\mu|+1\leqq\frac{|\mu|}{\varrho-1}$$

for $|\mu|\geqq(\varrho-1)(2-\varrho)^{-1}$. On the other hand, if $|\mu|\geqq(\varrho-1)(2-\varrho)^{-1}$ and $\mu=\varepsilon|\mu|$, (I'_ϱ) gives

$$\|\mu I-T\|\leqq\left|\mu-\varepsilon\frac{\varrho-1}{2-\varrho}\right|+\left\|\varepsilon\frac{\varrho-1}{2-\varrho}I-T\right\|\leqq|\mu|-\frac{\varrho-1}{2-\varrho}+\frac{1}{2-\varrho}=|\mu|+1,$$

thus (I'_ϱ) implies (III'_ϱ).

Remark 3. *In order that T be of class $\mathscr{C}_\varrho$ with $2\leqq\varrho<\infty$, it is necessary and sufficient that T verify the conditions* (II) *and*

$$(\mathrm{III}''_\varrho)\qquad \|(\mu I-T)^{-1}\|\leqq\frac{1}{|\mu|-1}\begin{cases}\textit{for}\ \ 1<|\mu|<\infty & \textit{if}\ \ \varrho=2,\\ \textit{for}\ \ 1<|\mu|\leqq r_\varrho=\dfrac{\varrho-1}{\varrho-2} & \textit{if}\ \ \varrho>2.\end{cases}$$

Proof. *Case* $\varrho=2$. We know already that (I_2) implies (10′), i.e. (III''_2). Suppose, conversely, that (III''_2) holds. Then $\|(\mu I-T)h\|\geqq(|\mu|-1)\|h\|$ for $1<|\mu|<\infty$ hence $\|(I-\varepsilon rT)h\|\geqq(1-r)\|h\|$ for $0<r<1$, $|\varepsilon|=1$. This gives

$$0\leqq\|h-\varepsilon rTh\|^2-(1-r)^2\|h\|^2=2r\|h\|^2-2\,\mathrm{Re}\,\varepsilon r(Th,h)+r^2\|Th\|^2-r^2\|h\|^2.$$

Dividing by $2r$ and letting $r\to 0$ it results $\|h\|^2-\mathrm{Re}\,\varepsilon(Th,h)\geqq 0$. Since this holds for arbitrary ε, $|\varepsilon|=1$, we get $|(Th,h)|\leqq\|h\|^2$ for any $h\in\mathfrak{H}$, i.e. $w(T)\leqq 1$. Thus (III''_2) implies (3), i.e. (I_2).

Case $\varrho > 2$. Suppose first that (I_ϱ), i.e. (I''_ϱ) holds. Then we have $\|(\mu I - T)h\| \geqq \geqq \frac{|\mu|}{\varrho - 1}\|h\|$ for $|\mu| \geqq r_\varrho$, in particular

$$\|(\varepsilon r_\varrho I - T)h\| \geqq \frac{r_\varrho}{\varrho - 1}\|h\| = \frac{1}{\varrho - 2}\|h\| \qquad (|\varepsilon| = 1, h \in \mathfrak{H}).$$

If $1 < |\mu| < r_\varrho$ and $\varepsilon = \mu/|\mu|$, we obtain hence

$$\|(\mu I - T)h\| \geqq \|(\varepsilon r_\varrho I - T)h\| - \|(\varepsilon r_\varrho - \mu)h\| \geqq \left[\frac{1}{\varrho - 2} - (r_\varrho - |\mu|)\right]\|h\|,$$

i. e.

$$\|(\mu I - T)h\| \geqq (|\mu| - 1)\|h\| \quad \text{for} \quad 1 < \mu < r_\varrho.$$

This implies (III''_ϱ).

Suppose, conversely, that (III''_ϱ) holds. Then we have in particular

$$\text{(13)} \quad \|(I - \zeta T)^{-1}\| = \frac{1}{|\zeta|}\left\|\left(\frac{1}{\zeta}I - T\right)^{-1}\right\| \leqq r_\varrho \frac{1}{r_\varrho - 1} = \varrho - 1 \quad \text{for} \quad |\zeta| = \frac{1}{r_\varrho}.$$

Since, by (II), $(I - zT)^{-1}$ is a holomorphic function of z for $|z| < 1$, we conclude from (13) by the maximum principle that

$$\|(I - zT)^{-1}\| \leqq \varrho - 1 \quad \text{for} \quad |z| \leqq \frac{1}{r_\varrho}.$$

Thus, if $|\mu| \geqq r_\varrho$, we have

$$|\mu| \cdot \|(\mu I - T)^{-1}\| = \left\|\left(I - \frac{1}{\mu}T\right)^{-1}\right\| \leqq \varrho - 1,$$

i.e. (I''_ϱ). This finishes the proof.

3. Let $\|T\| \leqq 1$. Then for every complex μ we have $\|\mu I - T\| \leqq |\mu| + \|T\| \leqq |\mu| + 1$; thus in virtue of Remark 2 we have

$$\text{(14)} \qquad \mathscr{C}_1 \subset \mathscr{C}_\varrho \quad \text{for } 1 \leqq \varrho < 2.$$

Let now $T \in \mathscr{C}_{\varrho_1}$ with $0 < \varrho_1 < \infty$, and let ϱ_2 be such that $\varrho_1 \leqq \varrho_2 < 2\varrho_1$. Since $T \in \mathscr{C}_{\varrho_1}$, there exists a unitary operator U in some Hilbert space $\mathfrak{K} \supseteq \mathfrak{H}$ such that

$$\text{(15)} \qquad T^n = \varrho_1 \cdot \text{pr } U^n \qquad (n = 1, 2, \ldots).$$

Since $U \in \mathscr{C}_1$ and $1 \leqq \varrho_2/\varrho_1 < 2$, we have $U \in \mathscr{C}_{\varrho_2/\varrho_1}$, by (14). Thus there exists a unitary operator V in a Hilbert space $\mathfrak{L} \supseteq \mathfrak{H}$ such that

$$\text{(15')} \qquad U^n = \frac{\varrho_2}{\varrho_1} \text{pr } V^n \qquad (n = 1, 2, \ldots).$$

Comparing (15) with (15') we obtain $T^n = \varrho_2 \text{ pr } V^n$ $(n = 1, 2, \ldots,)$, i.e. $T \in \mathscr{C}_{\varrho_2}$. From this remark it follows readily the following

Proposition 1. *The classes* $\mathscr{C}_\varrho (0 < \varrho < \infty)$ *form a non-decreasing scale, i.e.*

$$\text{(16)} \qquad \mathscr{C}_{\varrho_1} \subset \mathscr{C}_{\varrho_2} \quad \text{if} \quad 0 < \varrho_1 < \varrho_2 < \infty.$$

In order to complete this result let us consider a simple example. Let T_s $(s>0)$ be the operator in complex Euclidean 2-space with the matrix $\begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$. Obviously, $\|T_s\|=s$, $T_s^2=O$, and

$$(\mu I-T_s)^{-1} = \frac{1}{\mu^2}(\mu I+T_s).$$

Hence the spectrum of T_s consists of the single point 0, thus condition (II) is fulfilled. Moreover, we have

$$\|(\mu I-T_s)^{-1}\| \leqq \frac{|\mu|+s}{|\mu|^2} = \left(1+\frac{s}{|\mu|}\right)\frac{|\mu|-1}{|\mu|}\frac{1}{|\mu|-1} \leqq \frac{1}{|\mu|-1}$$

if $1<|\mu|\leqq\frac{s}{s-1}$. Thus, if $s\geqq 1$, condition (III_ϱ'') also is fulfilled, with $\varrho=s+1$. Thus

$$T_s\in\mathscr{C}_{s+1} \quad \text{if} \quad s\geqq 1,$$

but, since $\|T_s\|=s$, T_s does not belong to any of the classes $\mathscr{C}_\varrho$ with $\varrho<s$.

This shows that the increasing scale of the classes $\mathscr{C}_\varrho$ does not attain a maximum (indeed, $\mathscr{C}_\varrho$ is properly contained in $\mathscr{C}_{\varrho'}$, if $1\leqq\varrho<\varrho'-1$).

Now, let $0<s<1$. Then, putting $\varrho=\frac{2s}{1+s}$, we have

$$\|\mu I-T_s\| \leqq |\mu|+s \leqq \frac{|\mu|}{1-\varrho} \quad \text{for} \quad \frac{1-\varrho}{2-\varrho}\leqq|\mu|<\infty,$$

i.e. (I_ϱ') is verified. Hence $T_s\in\mathscr{C}_\varrho$. Since $s=\frac{\varrho}{2-\varrho}$, this result also can be expressed in the form

$$T_{\frac{\varrho}{2-\varrho}}\in\mathscr{C}_\varrho \quad \text{if} \quad 0<\varrho<1.$$

But $\|T_s\|=s$ again implies that $T_{\frac{\varrho}{2-\varrho}}$ does not belong to any of the classes $\mathscr{C}_{\varrho'}$ with $\varrho'<\frac{\varrho}{2-\varrho}$.

This shows that none of the classes $\mathscr{C}_\varrho$ is void and, moreover, that the scale of the classes $\mathscr{C}_\varrho$ does not attain a minimum $\left(\text{indeed, } \mathscr{C}_\varrho \text{ properly contains } \mathscr{C}_{\varrho'} \text{ if } 1>\varrho>2\frac{\varrho'}{\varrho'+1}\right)$.

Thus none of the classes $\mathscr{C}_\varrho(\varrho>0)$ is void, and the scale of the classes $\mathscr{C}_\varrho$ neither attains a minimum nor a maximum.

4. There exist power-bouded operators which do not belong to any of the classes $\mathscr{C}_\varrho$. More precisely, *we shall give an example of an operator T such that $\|T^n\|\leqq 2$ for every integer n, and which is not contained in any of the classes $\mathscr{C}_\varrho(\varrho>0)$.*

Consider, to this effect, the space $L^2(-1, 1)$, and the operators V and A defined in this space by

$$(Vf)(x) = f(-x) \quad \text{and} \quad (Af)(x) = a(x)f(x),$$

where $a(x) = 2^{1/2}$ if $-1<x<0$, and $=2^{-1/2}$ if $0<x<1$.

Set $T=AVA^{-1}$, i.e.

$$(Tf)(x) = \frac{a(x)}{a(-x)} f(-x). \tag{17}$$

Since $V^2=I$, we have $T^2=I$, i.e. $T^n=\begin{cases} T \text{ for } n \text{ odd,} \\ I \text{ for } n \text{ even.}\end{cases}$ Let us show that $\|T\|=2$. Indeed, we have

$$\|Tf\|^2 = \int_{-1}^{+1} \left| \frac{a(x)}{a(-x)} f(-x) \right|^2 dx \leqq 4\|f\|^2 \quad \text{for every} \quad f \in L^2,$$

because

$$\frac{a(x)}{a(-x)} = \begin{cases} 2 & \text{if} \quad -1<x<0, \\ 1/2 & \text{if} \quad 0<x<1,\end{cases}$$

and

$$\|Tf\|^2=4\|f\|^2 \quad \text{if} \quad f(x)=0 \quad \text{for} \quad -1<x<0.$$

We assert that T belongs to none of the classes $\mathscr{C}_\varrho$.

Since $\|T\|=2$, the values $\varrho<2$ are *a priori* impossible. We shall show, using the condition (III''_ϱ), that the values $\varrho \geqq 2$ also are impossible.

To this end, observe first that, since $T^2=I$, we have

$$(\mu I-T)^{-1} = \frac{1}{\mu^2-1}(\mu I+T) \quad \text{for} \quad \mu \neq \pm 1. \tag{18}$$

Choose the function $f_0(x)=\begin{cases} 0 \text{ for } -1<x<0 \\ 1 \text{ for } \quad 0<x<1\end{cases}$. Then

$$\|(\mu I+T)f_0\|^2 = \int_{-1}^{1} \left| \mu f_0(x) + \frac{a(x)}{a(-x)} f_0(-x) \right|^2 dx =$$

$$= |\mu|^2+4 = (|\mu|^2+4)\|f_0\|^2,$$

whence

$$\|\mu I+T\| \geqq (|\mu|^2+4)^{1/2}$$

and by (18),

$$\|(\mu I-T)^{-1}\| \geqq \frac{1}{|\mu^2-1|}(|\mu|^2+4)^{1/2}.$$

Now, if μ is real, $\mu>1$, we have

$$\frac{1}{\mu^2-1}(\mu^2+4)^{1/2} > \frac{1}{\mu-1}$$

for μ sufficiently close to 1, namely for $1<\mu<1.5$, and this shows that condition (III$''_\varrho$) is not satisfied for any $\varrho \geqq 2$.

5. Denote by A the class of the functions

$$u(z) = \sum_0^\infty a_n z^n \quad \text{with} \quad \sum_0^\infty |a_n| < \infty.$$

From (1) it follows for every $T \in \mathscr{C}_\varrho$ and $u \in A$:

$$u(T) = \mathrm{pr}\,[\varrho \cdot u(U) + (1-\varrho)\cdot u(0) I_{\mathfrak{K}}]. \tag{19}$$

This relation implies, by virtue of the spectral theory for unitary operators, the following

Proposition 2. *For $T \in \mathscr{C}_\varrho$ and $u \in A$ we have*

$$\|u(T)\| \leqq \max_{|z| \leqq 1} |u_\varrho(z)| \tag{20}$$

and

$$\big[\min_{|z|\leqq 1} \mathrm{Re}\, u_\varrho(z)\big] I_{\mathfrak{H}} \leqq \mathrm{Re}\, u(T) \leqq \big[\max_{|z|\leqq 1} \mathrm{Re}\, u_\varrho(z)\big] I_{\mathfrak{H}} \tag{21}$$

where

$$u_\varrho(z) = \varrho \cdot u(z) + (1-\varrho)\cdot u(0).$$

Obviously, (20) and (21) generalize, for the classes $\mathscr{C}_\varrho$, the inequalities of von Neumann and Heinz, respectively, on contractions, i.e. for the class $\mathscr{C}_1$. *Cf.* [4], p. 431.

It is clear that if $\varrho = 1$, (20) implies $\|T\| \leqq 1$: one has only to set $u(z) = z$. In the case $\varrho \neq 1$, (20) does not seem to imply that $T \in \mathscr{C}_\varrho$. But (21) does: in fact we shall prove the following

Proposition 3. *Suppose T is a power-bounded operator which satisfies* (21) *for every function $u \in A$. Then $T \in \mathscr{C}_\varrho$.*

Proof. Since T is power-bounded, its spectrum is contained in the unit disc, i.e. T satisfies (II). Moreover, power-boundedness assures that $u(T) = a_0 I + a_1 T + \cdots + a_n T^n + \cdots$ converges in norm. Concerning (I$_\varrho$) it suffices to verify (5), or, equivalently, (8). To this effect, choose

$$u(z) = u(\zeta; z) = 1 - \frac{2}{\varrho} + \frac{2}{\varrho}\frac{1}{1-\zeta z} = 1 + \frac{2}{\varrho}\zeta z + \frac{2}{\varrho}\zeta^2 z^2 + \cdots \qquad (|\zeta|<1,\ |z|\leqq 1).$$

Then
$$u_\varrho(z) = 1 + 2\zeta z + 2\zeta^2 z^2 + \cdots = \frac{1+\zeta z}{1-\zeta z},$$

hence $\mathrm{Re}\, u_\varrho(z) \geqq 0$ for $|z| \leqq 1$. Thus, by (21),

$$O \leqq \mathrm{Re}\, u(T) = \left(1 - \frac{2}{\varrho}\right) I_{\mathfrak{H}} + \frac{2}{\varrho}(I - \zeta T)^{-1},$$

and this result coincides with (8).

Finally, we make the following

Proposition 4. *Let* $u(z)\in A$, *with* $|u(z)|\leqq 1$ *for* $|z|\leqq 1$, *and* $u(0)=0$. *Then* $T\in\mathscr{C}_\varrho$ *implies* $u(T)\in\mathscr{C}_\varrho$.

Proof. Since $u_n(z)=[u(z)]^n$ also belongs to A for every $n=1,2,\ldots$, and since $u_n(0)=0$, we have for $T\in\mathscr{C}_\varrho$ by (19):

$$u(T)^n=u_n(T)=\mathrm{pr}\ \varrho\cdot u_n(U)=\varrho\cdot\mathrm{pr}\ u(U)^n. \tag{22}$$

Now, since $|u(z)|\leqq 1$ for $|z|\leqq 1$, $u(U)$ is a contraction. Thus there exists a unitary operator V such that $u(U)^n=\mathrm{pr}\ V^n$ $(n=0,1,\ldots)$. Comparing this with (22) we conclude that

$$u(T)^n=\varrho\cdot\mathrm{pr}\ V^n \qquad (n=1,2,\ldots),$$

i.e. $u(T)\in\mathscr{C}_\varrho$.

For $\varrho=2$, Proposition 4 reduces to a result obtained by STAMPFLI, *cf.* [2].

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