On $|C, 1|_k$ summability factors of infinite series

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1. Let Σa_n be a given infinite series with partial sums s_n, and let $t_n = t_n^0 = na_n$. By σ_n^α and t_n^α we denote the n-th Cesàro means of order α $(\alpha > -1)$ of the sequences $\{s_n\}$ and $\{t_n\}$, respectively. The series Σa_n is said to be absolutely summable (C, α) with index k, or simply summable $|C, \alpha|_k$ $(k \geqq 1)$, if

(1.1) $$\sum n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty \quad ([1]).$$

Summability $|C, \alpha|_1$ is the same as summability $|C, \alpha|$.
Since

$$t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha),$$

condition (1.1) can also be written as

(1.2) $$\sum \frac{|t_n^\alpha|^k}{n} < \infty.$$

If

(1.3) $$\sum_1^n \frac{|s_\nu|}{\nu} = O(\log n),$$

as $n \to \infty$, then Σa_n is said to be strongly bounded by logarithmic means with index 1, or bounded $[R, \log n, 1]$.

2. Recently PATI [2] proved the following theorem concerning summability $|C, 1|$ of a factored infinite series.

Let $\{\lambda_n\}$ be a convex sequence such that $\Sigma \frac{\lambda_n}{n}$ is convergent (then, necessarily, $\lambda_n \geqq 0$). *If Σa_n is bounded $[R, \log n, 1]$, then $\Sigma a_n \lambda_n$ is summable $|C, 1|$.*

The object of this note is to generalize this result by obtaining a theorem for summability $|C, 1|_k$.

3. In what follows we shall establish the following theorem.

Theorem. *If $\{\lambda_n\}$ is a convex sequence such that $\Sigma \frac{\lambda_n}{n} < \infty$, and*

(3.1) $$\sum_1^n |s_\nu|^k/\nu = O(\log n) \qquad (k \geqq 1),$$

then $\Sigma a_n \lambda_n$ is summable $|C, 1|_k$.

It is clear that in the special case $k=1$ our theorem includes the above theorem of PATI. For $k>1$ $\left(\frac{1}{k}+\frac{1}{k'}=1\right)$, we observe that

$$\sum_{1}^{n}\frac{|s_\nu|}{\nu} \leqq \left(\sum_{1}^{n}\frac{|s_\nu|^k}{\nu}\right)^{1/k}\left(\sum_{1}^{n}\frac{1}{\nu}\right)^{1/k'} = O\{(\log n)^{1/k}(\log n)^{1/k'}\} = O(\log n).$$

Thus condition (3. 1) implies condition (1. 3). However the results of FLETT [1] show that summability $|C, 1|_k$ and summability $|C, 1|$ in general are independent of each other.

4. The following lemmas will be required for the proof of this theorem.

Lemma 1. [2] *If $\{\lambda_n\}$ is a convex sequence such that $\sum\frac{\lambda_n}{n}<\infty$, then*

$$\Sigma \log(n+1)\Delta\lambda_n<\infty$$

and

$$m\log(m+1)\Delta\lambda_m=O(1),$$

as $m\to\infty$.

Lemma 2. [2] *Under the condition of Lemma* 1, *we have*

$$\sum_{1}^{m} n\log(n+1)\Delta^2\lambda_n = O(1), \quad as \quad m\to\infty.$$

5. Proof of the Theorem. Let T_n denote the n-th Cesàro mean of order 1 of the sequence $\{na_n\lambda_n\}$. Then we have to show that

$$\sum_{1}^{\infty} n^{-1}|T_n|^k<\infty. \tag{5. 1}$$

Now,

$$T_n=\frac{1}{n+1}\sum_{\nu=1}^{n}\nu a_\nu\lambda_\nu=\frac{1}{n+1}\sum_{1}^{n-1}\Delta(\nu\lambda_\nu)s_\nu+\frac{ns_n\lambda_n}{n+1}-\frac{a_0\lambda_1}{n+1}=$$

$$=\frac{1}{n+1}\sum_{\nu=1}^{n}\Delta(\nu\lambda_\nu)s_\nu-\frac{s_n}{n+1}(n\lambda_n-(n+1)\lambda_{n+1})+\frac{ns_n\lambda_n}{n+1}-\frac{a_0\lambda_1}{n+1}=$$

$$=\frac{1}{n+1}\sum_{1}^{n}\Delta(\nu\lambda_\nu)s_\nu+s_n\lambda_{n+1}-\frac{a_0\lambda_1}{n+1}=L_1^{(n)}+L_2^{(n)}+L_3^{(n)}.$$

By MINKOWSKI's inequality it is therefore sufficient to prove that

$$\sum\frac{|L_1^{(n)}|^k}{n}<\infty, \tag{5. 2}$$

$$\sum\frac{|L_2^{(n)}|^k}{n}<\infty, \tag{5. 3}$$

$$\sum\frac{|L_3^{(n)}|^k}{n}<\infty. \tag{5. 4}$$

Proof of (5. 2). In the sequel C_1, C_2 denote positive constants. We have

$$\sum_1^\infty \frac{|L_1^{(n)}|^k}{n} = \sum_1^\infty \frac{1}{n(n+1)^k}\left|\sum_1^n (\Delta\nu\lambda_\nu)s_\nu\right|^k \leqq \sum_1^\infty \frac{1}{n^{k+1}}\left(\sum_1^n |\Delta\nu\lambda_\nu|\,|s_\nu|\right)^k \leqq$$

$$\leqq C_1 \sum_1^\infty \frac{1}{n^{k+1}}\left(\sum_1^n \nu\Delta\lambda_\nu |s_\nu|\right)^k + C_1 \sum_1^\infty \frac{1}{n^{k+1}}\left(\sum_1^n \lambda_{\nu+1}|s_\nu|\right)^k \leqq$$

$$\leqq C_1 \sum_1^\infty \frac{1}{n^{k+1}} \sum_1^n \nu\Delta\lambda_\nu |s_\nu|^k \left(\sum_1^n \nu\Delta\lambda_\nu\right)^{k/k'} + C_1 \sum_1^\infty \frac{1}{n^{k+1}} \sum_{\nu=1}^n \lambda_{\nu+1}|s_\nu|^k \left(\sum_{\nu=1}^n \lambda_{\nu+1}\right)^{k/k'} =$$

$$= O\left(\sum_1^\infty \frac{1}{n^2} \sum_{\nu=1}^n \nu\Delta\lambda_\nu |s_\nu|^k\right) + O\left(\sum_{n=1}^\infty \frac{1}{n^2} \sum_{\nu=1}^n \lambda_\nu |s_\nu|^k \left(\sum_1^\infty \frac{\lambda_\nu}{\nu}\right)^{k/k'}\right) =$$

$$= O\left(\sum_{\nu=1}^\infty \nu\Delta\lambda_\nu |s_\nu|^k \sum_{n=\nu}^\infty \frac{1}{n^2}\right) + O\left(\sum_{\nu=1}^\infty \lambda_\nu |s_\nu|^k \sum_{n=\nu}^\infty \frac{1}{n^2}\right) =$$

$$= O\left(\sum_{\nu=1}^\infty \Delta\lambda_\nu |s_\nu|^k\right) + O\left(\sum_{\nu=1}^\infty \frac{\lambda_\nu}{\nu}|s_\nu|^k\right).$$

Now

$$\sum_{\nu=1}^m \Delta\lambda_\nu |s_\nu|^k = \sum_1^m \nu\Delta\lambda_\nu \frac{|s_\nu|^k}{\nu} =$$

$$= \sum_1^{m-1} \Delta(\nu\Delta\lambda_\nu) \sum_{\mu=1}^\nu \frac{|s_\mu|^k}{\mu} + m\Delta\lambda_m \sum_{\mu=1}^m \frac{|s_\mu|^k}{\mu} =$$

$$= O\left(\sum_1^{m-1} \nu\Delta^2\lambda_\nu \log(\nu+1)\right) + O\left(\sum_1^{m-1} \Delta\lambda_\nu \log(\nu+1)\right) + O(m\Delta\lambda_m \log(m+1)) = O(1),$$

by virtue of Lemmas 1 and 2.

Also applying Lemma 1 we have

$$\sum_1^m \frac{\lambda_\nu}{\nu}|s_\nu|^k = \sum_1^{m-1} \Delta\lambda_\nu \sum_{\mu=1}^\nu \frac{|s_\mu|^k}{\mu} + \lambda_m \sum_{\mu=1}^m \frac{|s_\mu|^k}{\mu} =$$

$$= O\left(\sum_1^{m-1} \Delta\lambda_\nu \log(\nu+1)\right) + O(\lambda_m \log(m+1)) = O(1).$$

Hence

$$\sum_1^\infty \frac{|L_1^{(n)}|^k}{n} = O(1).$$

Proof of (5. 3). We have by virtue of the hypothesis

$$\sum_1^m \frac{|L_2^{(n)}|^k}{n} = \sum_1^m \frac{1}{n}|s_n\lambda_{n+1}|^k \leqq \sum_1^m \lambda_n^k \frac{|s_n|^k}{n} = \sum_1^{m-1} \Delta\lambda_n^k \sum_{\mu=1}^n \frac{|s_\mu|^k}{\mu} + \lambda_m^k \sum_1^m \frac{|s_\mu|^k}{\mu} =$$

$$= \sum_1^{m-1} \Delta\lambda_n^k O(\log n) + O(\lambda_m^k \log m) = O\left(\sum_1^m \Delta\lambda_n^k \log n\right) + O(1) = O(1).$$

Finally, it is clear that

$$\sum \frac{|L_3^{(n)}|^k}{n} \leqq C_2 \sum_1^\infty \frac{1}{n^{k+1}} < \infty.$$

This completes the proof of the theorem.

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References

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