On the strong summability of orthogonal series

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Let $\{\varphi_\nu(x)\}$ $(\nu=0, 1, 2, \ldots)$ be a normalized orthogonal system in $[a, b]$ and

(1) $$\sum_{\nu=0}^{\infty} c_\nu \varphi_\nu(x)$$

be an orthogonal series which satisfies

(2) $$\sum_{\nu=0}^{\infty} c_\nu^2 < \infty.$$

We write

$$s_n(x) = \sum_{\nu=0}^{n} c_\nu \varphi_\nu(x)$$

and $$\sigma_n^\alpha(x) = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu(x) \qquad \left(A_n^\alpha = \binom{n+\alpha}{n}\right).$$

Concerning the strong summability of orthogonal series, TANDORI proved the following theorems.

Theorem A. (TANDORI [3]) *If the orthogonal series* (1) *with* (2) *is* $(C, 1)$*-summable to* $f(x)$ *almost everywhere in* $[a, b]$, *then*

$$\lim_{n\to\infty} \sigma_{2^n}^\alpha([s_\nu - f]^2; x) = 0 \qquad (0<\alpha<1)$$

almost everywhere in $[a, b]$, *where*

$$\sigma_n^\alpha([s_\nu - f]^2; x) = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} [s_\nu(x) - f(x)]^2.$$

Theorem B. (TANDORI [4]) *If*

$$\sum_{\nu=2}^{\infty} c_\nu^2 (\log\log \nu)^2 < \infty$$

then there exists a square-integrable function $f(x)$ *such that*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} [s_{n_\nu}(x) - f(x)]^2 = 0$$

almost everywhere for any increasing sequence n_ν.

In the present note, the author intends to generalize these results to the following form.

Theorem 1. *If the orthogonal series* (1) *with* (2) *is* $(C, 1)$*-summable to* $f(x)$ *almost everywhere in* $[a, b]$, *then*

$$\lim_{n\to\infty} \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_\nu(x)-f(x)|^k = 0$$

almost everywhere in $[a, b]$ *for any* $\alpha>0$ *and* $k>0$.

Theorem 2. *If*

$$\sum_{\nu=2}^{\infty} c_\nu^2 (\log\log \nu)^2 < \infty,$$

then there exists a square-integrable function $f(x)$ *such that*

$$\lim_{n\to\infty} \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_{n_\nu}(x)-f(x)|^k = 0$$

for any $\alpha>0$ *and* $k>0$, *almost everywhere in* $[a, b]$ *for any increasing sequence* n_ν.

In the sequel, we use $A, B, \ldots$ to denote positive constants, not necessarily the same on any two occurences.

Lemma 1. *If* $\sum_{\nu=0}^{\infty} c_\nu^2 < \infty$, *then*

$$\int_a^b \left\{ \sum_{n=1}^{\infty} n^{-1} |\sigma_n^{\alpha-1}(x) - \sigma_n^\alpha(x)|^2 \right\} dx \leqq A \sum_{n=0}^{\infty} c_n^2 \qquad (\alpha > 1/2).$$

Proof. This is well known. In fact, it can be proved by application of Bessel's inequality.

For any scalar series

$$\sum_{\nu=0}^{\infty} a_\nu, \tag{3}$$

we write

$$s_n = \sigma_n^0 = \sum_{\nu=0}^{n} a_\nu,$$

$$s_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu, \quad \sigma_n^\alpha = s_n^\alpha / A_n^\alpha, \quad \sigma_n = \sigma_n^1.$$

Then we have the identity

$$n(\sigma_n^\alpha - \sigma_{n-1}^\alpha) = -\alpha(\sigma_n^\alpha - \sigma_n^{\alpha-1}) \qquad (\alpha>0)$$

and we write this τ_n^α. If

$$\sum_{n=1}^{\infty} n^{-1} |\tau_n^\alpha|^k < \infty,$$

then FLETT [1] says that the series (3) is summable $|C, \alpha|_k$.

Lemma 2. *If* $\Sigma c_n^2 < \infty$, *then*

$$\int_a^b \{\sum_{n=1}^{\infty} n^{-1} |s_n(x) - \sigma_n(x)|^r\}^{2/r} \, dx \leqq B \sum_{n=0}^{\infty} c_n^2 \qquad (r \geqq 2),$$

and (1) *is* $|C, 1|_r$*-summable almost everywhere for any* $r \geqq 2$.

Proof. FLETT [1, p. 115] proved that

$$\left(\sum_{n=1}^{\infty} n^{-1} |\tau_n^\beta|^r\right)^{1/r} \leqq A(k, r, \alpha, \beta) \left(\sum_{n=1}^{\infty} n^{-1} |\tau_n^\alpha|^k\right)^{1/k}$$

whenever

$$r \geqq k > 1, \ \alpha > -1, \ \beta \geqq \alpha + 1/k - 1/r.$$

In particular, setting $k = 2$, $\alpha = 1/2 + \varepsilon$ $(\varepsilon > 0)$, we have

$$\beta \geqq 1 + \varepsilon - 1/r.$$

For any given $r \geqq 2$, we can select $\varepsilon < r^{-1}$. Hence we have

$$\left(\sum_{n=1}^{\infty} n^{-1} |\tau_n^1|^r\right)^{1/r} \leqq A \left(\sum_{n=1}^{\infty} n^{-1} |\tau_n^\alpha|^2\right)^{1/2} \qquad (\alpha = 1/2 + \varepsilon).$$

By Lemma 1, for $\alpha > 1/2$, $r \geqq 2$,

$$\int_a^b \left\{\sum_{n=1}^{\infty} n^{-1} |s_n(x) - \sigma_n(x)|^r\right\}^{2/r} dx \leqq A \int_a^b \left\{\sum_{n=1}^{\infty} n^{-1} |\sigma_n^{\alpha-1}(x) - \sigma_n^\alpha(x)|^2\right\} dx \leqq B \sum_{n=0}^{\infty} c_n^2.$$

Hence

$$\sum_{n=1}^{\infty} n^{-1} |s_n(x) - \sigma_n(x)|^r$$

converges almost everywhere.

Lemma 3. *If* $\Sigma c_n^2 < \infty$, *then*

$$\int_a^b \left\{ \sup_{1 \leqq n < \infty} \left(\frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_\nu(x) - \sigma_\nu(x)|^k \right)^{1/k} \right\}^2 dx \leqq A \sum_{n=0}^{\infty} c_n^2.$$

Proof. For any given $\alpha>0$, we select s near to 1 such as $\alpha>1-s^{-1}(s>1)$ and set $r^{-1}+s^{-1}=1$. Then, by HÖLDER's inequality,

$$\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-1}|s_\nu(x)-\sigma_\nu(x)|^k \leqq \frac{1}{A_n^\alpha}\left\{\sum_{\nu=1}^{n}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right\}^{1/r}\left\{\sum_{\nu=1}^{n}\nu^{s/r}(A_{n-\nu}^{\alpha-1})^s\right\}^{1/s} \leqq$$

$$\leqq \frac{B}{A_n^\alpha}\left\{\sum_{\nu=1}^{n}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right\}^{1/r}\{n^{s/r}n^{(\alpha-1)s+1}\}^{1/s} \leqq$$

$$\leqq \frac{Bn^\alpha}{A_n^\alpha}\left\{\sum_{\nu=1}^{n}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right\}^{1/r} \leqq C\left\{\sum_{\nu=1}^{n}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right\}^{1/r}.$$

Hence
$$\int_a^b\left\{\sup_{1\leqq n<\infty}\left(\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-1}|s_\nu(x)-\sigma_\nu(x)|^k\right)^{1/k}\right\}^2 dx \leqq$$

$$\leqq C\int_a^b\left(\sum_{\nu=1}^{\infty}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right)^{2/rk} dx.$$

For a given k, we take s sufficiently near to 1, then rk is greater than 2 because $r^{-1}+ +s^{-1}=1$. Hence, by Lemma 2, we get the required result.

The method of proof of Lemma 3 has been given in SUNOUCHI and YANO [2].

Proof of Theorem 1. For any positive $\varepsilon>0$, we take

$$\sum_{\nu=N+1}^{\infty}c_\nu^2<\varepsilon^3$$

and split Σc_ν^2 into

$$\sum_{\nu=1}^{N}c_\nu^2 \quad\text{and}\quad \sum_{\nu=N+1}^{\infty}c_\nu^2.$$

Consider the orthogonal series

(4) $$\sum_{n=1}^{N}c_n\varphi_n(x)$$

and

(5) $$\sum_{n=N+1}^{\infty}c_n\varphi_n(x).$$

For the series (4), the conclusion is valid evidently and for the second series (5),

$$\int_a^b\left\{\sup_{1\leqq n<\infty}\left(\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-1}|t_\nu(x)-T_\nu(x)|^k\right)^{1/k}\right\}^2 dx \leqq A\sum_{\nu=N+1}^{\infty}c_\nu^2<A\varepsilon^3,$$

where $t_n(x)$ and $T_n(x)$ is the partial sums and the arithmetic means of (5). Hence

$$\text{meas}\left\{x \mid \limsup \left(\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}|t_\nu(x)-T_\nu(x)|^k\right)^{1/k} > \varepsilon\right\} \leqq A\varepsilon.$$

That is,
$$\lim_{n\to\infty} \frac{1}{A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}|t_\nu(x)-T_\nu(x)|^k = 0$$

almost everywhere for the series (5). Combining the two results for series (4) and (5) we conclude that if

$$\Sigma c_n^2 < \infty,$$

then

$$\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|s_\nu(x)-\sigma_\nu(x)|^k \qquad (\alpha>0,\ k>0)$$

converges to zero almost everywhere. By the hypothesis, $\sigma_n(x)$ converges to $f(x)$ a.e. and so

$$\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|s_\nu(x)-f(x)|^k \qquad (\alpha>0,\ k>0)$$

converges to zero almost everywhere.

Proof of Theorem 2. Let us set

$$\psi_\nu(x) = \frac{1}{\gamma_n}[c_{n_\nu+1}\varphi_{n_\nu+1}(x)+\dots+c_{n_{\nu+1}}\varphi_{n_{\nu+1}}(x)]$$

where

$$\gamma_\nu = (c_{n_\nu+1}^2+\dots+c_{n_{\nu+1}}^2)^{1/2}.$$

Then $\{\psi_\nu(x)\}$ $(\nu=0, 1, 2, \dots)$ also is a normalized orthogonal system and the orthogonal series

(6) $$\sum_{\nu=0}^{\infty}\gamma_\nu\psi_\nu(x)$$

satisfies

$$\Sigma\gamma_\nu^2 < \infty.$$

Since n_ν is an increasing sequence and $n_\nu \geqq \nu$,

$$\sum_{\nu=2}^{\infty}\gamma_\nu^2(\log\log\nu)^2 = \sum_{\nu=2}^{\infty}(c_{n_\nu+1}^2+\dots+c_{n_{\nu+1}}^2)(\log\log\nu)^2 \leqq$$

$$\leqq \sum_{\nu=2}^{\infty}(c_{n_\nu+1}^2(\log\log(n_\nu+1))^2+\dots+c_{n_{\nu+1}}^2(\log\log n_{\nu+1})^2) \leqq \sum_{n=2}^{\infty}c_n^2(\log\log n)^2 < \infty,$$

by the hypothesis.

By the well-known theorem, the sequence of the $(C, 1)$-means of the series (6) tends to $f(x)$ almost everywhere. Moreover,

$$\sum_{\nu=0}^{m}\gamma_\nu\psi_\nu(x) = \sum_{n=0}^{n_m}c_n\varphi_n(x)$$

and so the m-th ordinary partial sum of (6) is identical with the n_m-th partial sum of (1). Applying Theorem 1, we get the required.

Remark. Actually our argument proves the following maximal theorem:

$$\int_a^b \left\{ \sup_{1 \leqq n < \infty} \left(\frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_{n_\nu}(x) - f(x)|^k \right)^{1/k} \right\}^2 dx \leqq A \sum_{n=2}^{\infty} c_n^2 (\log\log n)^2,$$

for any $\alpha > 0$ and $k > 0$.

Literature

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