# On the characterization of Stone lattices

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## 1. Introduction

M. H. STONE proposed the following problem:

"What is the most general pseudo-complemented distributive lattice in which $a^* \vee a^{**} = 1$ identically?". See [1], p. 149, problem 70.

The first solution to this problem was given by G. GRÄTZER and E. T. SCHMIDT in [3]:

"Let $L$ be a distributive pseudo-complemented lattice with unit element. Then $L$ is a Stone lattice if and only if the lattice-theoretical join of any two distinct minimal prime ideals of $L$ is $L$."

In this note, we give a short proof of an equivalent form of this theorem and of the corresponding one for relative Stone lattices.*)

The reader is referred to [3] and [4] for the notions and notations. We only replace the words "dual ideal" by "filter".

## 2. Stone lattices

Theorem I. *A distributive pseudo-complemented lattice is a Stone lattice if and only if every prime filter is contained in only one proper maximal filter.*

Proof. 1°) *if*: Let us suppose that $L$ is a distributive pseudo-complemented lattice which satisfies the above condition but which is not a Stone lattice. Then there exists an element $a$ such that $a^* \vee a^{**} = b < 1$. By STONE's theorem (cf. [3], lemma I), there exists a prime filter $P$, containing 1 and disjoint from the principal ideal $(b]$.

Let us consider the filter $F = P \vee [a^*)$. $F$ cannot contain $a^{**}$; otherwise $0 = a^* \wedge a^{**}$ would belong to $F$ and there would exist an element $x \in P$ such that $x \wedge a^* = 0$. But this last equality implies $x \leqq a^{**}$, hence $x \leqq b$, contradicting that $P$ is disjoint from $(b]$.

The family of filters containing $F$ and disjoint from $(a^{**}]$ has a maximal element $F_1$ and this filter $F_1$ is a maximal proper filter of $L$, for any filter containing properly $F_1$ would contain $a^{**}$, and consequently 0 (because it also contains $a^*$).

*) The author wishes to express his indebtedness to Professor G. GRÄTZER for his encouragement and advice in the preparation of this paper.

In the same way, we can show that there exists a maximal proper filter $G_1$ containing the filter $G = P \vee [a^{**})$ but not containing $a^*$.

Thus the prime filter $P$ would be contained in two distinct maximal filters, contradicting the hypothesis.

2°) *only if*: Let us suppose that there exists in the Stone lattice $L$ a prime filter $P$ contained in two distinct maximal proper filters $M_1$ and $M_2$. Let $a$ be an element belonging to $M_1$ but not to $M_2$. Since $a \wedge a^* = 0$, therefore $a^* \notin M_1$ and $a^* \notin P$. Since $M_2$ is maximal, for any element not belonging to it, there exists in $M_2$ an element which is disjoint from the first. Thus there exists $b \in M_2$ such that $b \wedge a = 0$. Since $a^* \geqq b$, we have: $a^* \in M_2$ and $a^{**} \notin M_2$, $a^{**} \notin P$. Then $a^* \vee a^{**} = 1 \in P$, and $a^*, a^{**} \notin P$, contradicting that $P$ is a prime filter.

## 3. Equivalent propositions

By dualizing the statements, it is easy to show that the condition

(A) *any prime filter is contained in only one maximal filter*

is equivalent, in a distributive lattice with elements 0 and 1, to the condition

(A′) *any prime ideal contains only one minimal prime ideal.*

Then we can verify the equivalence, in any distributive lattice, of the condition (A′) to the condition cited above:

(B) *the lattice-theoretical join of any two distinct minimal prime ideals of $L$ is $L$.*

Proof. (A′)$\Rightarrow$ (B). Let us consider, in the distributive lattice $L$, the two minimal prime ideals $P$ and $Q$ such that $P \vee Q \neq L$. There exists a prime filter $F$ disjoint from $P \vee Q$. But then $L - F$ is a prime ideal containing $P$ and $Q$, contradicting (A′).

(B) $\Rightarrow$ (A′). Let us suppose that, in the distributive lattice $L$, the prime ideal $P$ contains two minimal prime ideals $Q$ and $R$. We would have $Q \vee R \subseteqq P$ and (B) would be contradicted.

## 4. Relative Stone lattices

Theorem 2. *A distributive lattice in which every closed interval (as a sublattice) is a pseudo-complemented lattice, is a relative Stone lattice if and only if every proper filter which contains a prime filter is prime.*

Proof. 1°) *if*: Let us suppose that $L$ is a distributive lattice satisfying the conditions of the hypothesis which is not a relative Stone lattice. There exists in $L$ an interval $[k, l]$ in which a prime filter $F'$ is contained in two distinct maximal filters $G'$ and $H'$. Consider the mapping $x \to x' = (x \vee k) \wedge l$ of $L$ onto $[k, l]$. Since $L$ is distributive, this mapping is an endomorphism. Let $F, G$ and $H$ be the inverse images of $F', G'$ and $H'$ respectively. Obviously $F, G$ and $H$ are filters. Moreover, $G \supset F$, $H \supset F$, $G$ and $H$ are non-comparable. By the lemma III of [4], $F$ is a prime filter. Thus we come to a contradiction since the prime filter $F$ is contained in a proper filter $G \wedge H$, which is not prime*).

*) In a distributive lattice, a filter is prime if and only if it is $\wedge$-irreducible.

2) *only if:* Let us suppose that in the relative Stone lattice $L$ there exists a prime filter $F$ contained in a non-prime proper filter $G$. $G$ being non-prime, there would be in $L-G$ two elements $a$ and $b$ such that $a \vee b = d \in G$. More precisely, $d$ belongs to $G-F$ since $F$ is prime. Let $e \in F$, $e > d$. Let us put $a \wedge b = c$. By hypothesis, the interval $[c, e]$ is a Stone lattice. We have: $a^* \vee a^{**} = e$ and $b \leqq a^*$ (where * denotes the pseudo-complement in $[c, e]$). Since $d \wedge a^* = (a \vee b) \wedge a^* = b \wedge a^* = b$ and $d \in G$, $b \notin G$, we conclude: $a^* \notin G$ (and $a^* \notin F$). A similar argument shows that $a^{**} \notin F$. Since $F$ is a prime filter, this is a contradiction.

## 5. Equivalent propositions

The condition

(C) *any proper filter which contains a prime filter is prime*

is equivalent, in a distributive lattice, to the condition of G. GRÄTZER and E. T. SCHMIDT (cf. [3], theorem 3):

(D) *for any pair of prime ideals $P$ and $Q$, neither of which contains the other, $P \vee Q$ is the whole lattice.*

Proof. (C) $\Rightarrow$ (D). Let us suppose that, in the distributive lattice $L$, there exist two non-comparable prime ideals $A$ and $B$ such that $A \vee B \neq L$. By Stone's theorem, there exists a prime filter $F$ disjoint from the ideal $A \vee B$. $L-A$ and $L-B$ are non-comparable prime filters the intersection of which is a filter $G$ containing $F$. By assumption (C), this filter $G$ is prime, which is impossible since that would imply that $G$ is $\wedge$-irreducible.

(D) $\Rightarrow$ (C). Again, let us demonstrate this implication in an indirect way. Let us assume the existence, in the distributive lattice $L$, of a prime filter $F$ and a non-prime proper filter $G$ containing $F$. Thus $G$ is $\wedge$-reducible: there exist non-comparable filters $A$ and $B$ such that $A \wedge B = G$. Thus we can find two elements $a$ and $b$ such that $a \in A$, $a \notin B$, $b \in B$, $b \notin A$. There exists a prime filter $A_1$ containing $A$ and disjoint from $(b]$ and a prime filter $B_1$ containing $B$ and disjoint from $(a]$. $A_1$ and $B_1$ contain $G$ and are non-comparable.

The non-comparable prime ideals $L-A_1$ and $L-B_1$ should have, by (D), the lattice-theoretical join $L$. This would imply that any element of $F$ is the join of two elements not belonging to $F$, hence a contradiction.

In conclusion, let us recall that A. MONTEIRO gives in [6] two conditions, equivalent to (C) in a distributive lattice:

(C′) *the family of all filters including a prime filter is linearly ordered;*

(C″) *the family of all prime filters including a prime filter is linearly ordered.*

The equivalence of conditions (C), (C′) and (C″) can be easily proved.

## References

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