Semigroups having left or right zeroid elements

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An element μ of a semigroup S is called a left zeroid of S if for each $b \in S$ there exists $x \in S$ such that $xb = \mu$ [1]. A right zeroid of S is defined similarly. An element of S is a zeroid of S if it is both a left and right zeroid of S. We will assume that the reader is familiar with the notions of "simple semigroup" [2, p. 5] and "regular semigroup" [2, p. 26]. In this note we determine some properties of semigroups which are assumed to have one-sided zeroid elements. Semigroups having zeroid elements have been studied extensively by CLIFFORD and MILLER [1]. We begin with a simple but useful lemma, and agree that theorems and lemmas involving "left" and "right" cases will be stated and proved for the "left" case only, unless otherwise indicated.

Lemma 1. *If S is a semigroup with left zeroid μ and idempotent e, then $\mu e = \mu$.*

Proof. There exists $x \in S$ such that $xe = \mu$. Thus $\mu e = (xe)e = x(ee) = xe = \mu$.

Lemma 2. *If e is a left zeroid idempotent of a semigroup S, then eS is a group and Se is regular.*

Proof. Clearly e is a left identity of eS. Suppose $eb \in eS$. There exists $c \in S$ such that $c(eb) = e$. Thus $(ec)(eb) = ee = e$. Therefore e is a left zeroid of eS. Hence eS is a group.

It is obvious that e is a right identity of Se. Suppose $ze \in Se$. There exists $g \in S$ such that $g(ze) = e$. There exists $h \in S$ such that $hg = e$ and there exists $k \in S$ such that $kh = e$. Thus $eg = ke$. We have then $e = ee = (eg)(ze) = (ke)(ze)$. Therefore e is a left zeroid of Se. Suppose $x \in Se$. There exists $y \in Se$ such that $yx = e$. Thus $xyx = xe = x$, and Se is regular.

Theorem 1. *A simple semigroup with a left zeroid is regular if it contains an idempotent.*

Proof. Suppose S is a simple semigroup with left zeroid μ and idempotent e. Suppose $b \in S$. There exists $c \in S$ such that $cb = \mu$, and there exists $d \in S$ such that $dc = \mu$. Thus $\mu b = d\mu$. Therefore $(S\mu)b = (Sd)\mu \subseteqq S\mu$ and consequently $S\mu$ is a right ideal. Clearly then $S\mu$ is an ideal, and so $S = S\mu$ since S is simple. Thus every element of S is a left zeroid of S. Hence e is a left zeroid idempotent of S, and by Lemma 2, Se is regular. But by Lemma 1 we have $Se = (S\mu)e = S(\mu e) = S\mu = S$. This completes the proof.

CLIFFORD and MILLER [1] proved the next theorem and the results mentioned in the remark following. We will give a proof here because of the directness and simplicity of our proof.

Theorem 2. (CLIFFORD and MILLER) *If a semigroup S has both a left zeroid and a right zeroid, then every left or right zeroid of S is a zeroid of S.*

Proof. Suppose μ is a left zeroid of S and μ' is a right zeroid of S. Let $e=s\mu$, where $s(\mu\mu)=\mu$. Then e is a left zeroid of S such that $e\mu=\mu$. Similarly, if $f=\mu' t$, where $(\mu'\mu')t=\mu'$, then f is a right zeroid of S and $\mu' f=\mu'$. Let $g\mu'=e$ and $\mu h=f$. Then $e=g\mu'=g(\mu' f)=(g\mu')f=ef=e(\mu h)=(e\mu)h=\mu h=f$. Thus if $c\in S$, then $c(y\mu)=\mu$ if $cy=f=e$. Therefore μ is a right zeroid of S. Similarly μ' is a left zeroid of S.

Remark. We note from the proof above that e is a zeroid idempotent of S and if U is the set of all zeroids of S, then $U=eS=Se$. From this it follows easily that U is a group with identity e. We further note that if $x\in S$, then $ex=e(ex)=(ex)e=e(xe)=xe$. (Thus S is a homogroup [3]). It is obvious that the mapping $x\to ex$ is a homomorphism of S onto eS.

Theorem 3. *Suppose S is a semigroup with left zeroid μ and $L=\{x\in S|x\mu=\mu\}$. Then L is a subsemigroup of S and each of the following conditions on L is sufficient for S to contain an idempotent e such that eS is a group and Se is regular:*

(1) *L has a left zeroid idempotent,*
(2) *L is degenerate,*
(3) *L has a right zeroid,*
(4) *L is regular,*
(5) *L is simple and contains an idempotent.*

In each case the mapping of S onto eS defined by $x\to ex$ is a homomorphism of S onto eS.

Proof. Clearly L is a semigroup. Suppose (1) holds. Let $f=t\mu$, where $t(\mu\mu)=\mu$. Then f is a left zeroid of S and $f\mu=\mu$. Let e be a left zeroid idempotent of L. If $b\in S$, there exists $c\in S$ such that $cb=f$. There exists $k\in L$ such that $kf=e$. Thus $(kc)b=kf=e$, and e is a left zeroid of S. By Lemma 2, eS is a group and Se is regular.

Suppose (2) holds. Then there exists a unique $e\in S$ such that $e\mu=\mu$. Thus e is an idempotent since $ee\mu=e\mu=\mu$. Since in the proof of the sufficiency of (1) we showed that L contains a left zeroid of S, then e is a left zeroid of S. Thus by Lemma 2, eS is a group and Se is regular.

Suppose (3) holds. Consider the element f in the proof of the sufficiency of (1). Since f is a left zeroid of S, if $x\in L$, there exists $y\in S$ such that $yx=f$. Thus $yx\mu=f\mu$, and so $y\mu=\mu$. Therefore $y\in L$ and f is a left zeroid of L. Now since L has a left zeroid and a right zeroid, then by Theorem 2, L has a zeroid, and by the remark following Theorem 2, L has a zeroid idempotent. Hence the conclusion follows from the sufficiency of (1).

Suppose (4) holds. Again we consider the element $f=t\mu$, where $t(\mu\mu)=\mu$. Since L is regular, there exists $z\in L$ such that $f=fzf$. It is obvious that zf is an idempotent. Clearly zf is a left zeroid of S since f is a left zeroid of S. Let $e=zf$. Again by Lemma 2, eS is a group and Se is regular.

Suppose (5) holds. Since we have shown that L contains a left zeroid, then the conclusion follows from Theorem 1 and the sufficiency of (4).

In each case the mapping $x \to ex$ of S onto eS is a homomorphism of S onto eS because e is the identity of the group eS. If $b \in S$ and $c \in S$, then $bc \to e(bc) = [(eb)e]c = (eb)(ec)$.

T. TAMURA [4] showed that if a semigroup S contains exactly one idempotent e, then e is a left zeroid of S if and only if e is a right zeroid of S. We need to prove the following variation of TAMURA's theorem in order to prove Theorem 5.

Theorem 4. *If S is a semigroup which contains among its idempotents one and only one left zeroid e, then e is a zeroid of S.*

Proof. By Lemma 2, Se is regular. Thus if $b \in Se$, there exists $x \in Se$ such that $b=bxb$. But every element of Se is a left zeroid of S. Thus the only idempotent of Se is e. Hence $bx=e$ since bx is an idempotent in Se, and so each element of Se has a right inverse in Se with respect to e. Clearly e is a right identity of Se. Therefore Se is a group and e is its identity. Hence if $c \in S$, there exists $de \in Se$ such that $(ce)(de)=e$. But $(ce)(de)=c[e(de)]=c(de)$. Therefore e is a right zeroid of S. Thus e is a zeroid of S.

K. ISEKI [3] defined a relation "$\leqq$" on the nonempty set of idempotents of a semigroup as follows: $e \leqq f$ provided $ef=e$. He showed that a homogroup always has a unique least idempotent. We can now prove the following theorem concerning ISEKI's relation.

Theorem 5. *Suppose S is a semigroup with a unique least idempotent e. If e is a left or right zeroid of S, then e is a zeroid of S.*

Proof. We will first prove the theorem for the case that e is a left zeroid of S. Suppose S contains an idempotent f which is a left zeroid of S. By Lemma 1, $fe=f$, and so $f \leqq e$. Hence because "$\leqq$" is transitive and e is the unique least idempotent of S, $f=e$. Therefore e is the only idempotent of S which is a left zeroid of S. By Theorem 4, e is a zeroid of S.

Next suppose that e is a right zeroid of S. Suppose S contains an idempotent f which is a right zeroid of S. By the dual of Lemma 1, $ef=f$. But $ef=e$ since $e \leqq f$. Hence $e=f$, and e is the only idempotent of S which is a right zeroid of S. By the dual of Theorem 4, e is a zeroid of S. This completes the proof.

We close with a theorem which gives a simple necessary and sufficient condition for a semigroup with a left zeroid to have a left zeroid idempotent.

Theorem 6. *A semigroup S with a left zeroid μ contains a left zeroid idempotent if and only if the equation $\mu=(\mu\mu)x$ has a solution $x \in S$.*

Proof. Suppose there exists $x \in S$ such that $\mu=(\mu\mu)x$. Let $f=\mu x$. Let $e=t\mu$, where $t(\mu\mu)=\mu$. Then e is a left zeroid of S and $e\mu=\mu$. We have $ef=e\mu x=\mu x$. Hence $ef=f$. But $ef=t\mu f=t\mu$. Thus $ef=e$. Therefore $e=f$ and f is a left zeroid idempotent of S.

Now we prove the "only if" part of the theorem. Suppose e is a left zeroid idempotent of S. We wish to show that e is a left zeroid of $S\mu$. Suppose $b\mu \in S\mu$. There exists $c \in S$ such that $c(b\mu)=e$. There exists $d \in S$ such that $dc=e$ and there exists $g \in S$ such that $gd=e$. Hence $ec=ge$. We have then $e=(ec)(b\mu)=(ge)(b\mu)=$ $=(gcb\mu)(b\mu)$. Therefore e is a left zeroid of $S\mu$. By Lemma 2, $(S\mu)e$ is regular. But $(S\mu)e=S(\mu e)=S\mu$, by Lemma 1. Therefore $S\mu$ is regular. Clearly $\mu \in S\mu$ since μ is a left zeroid of S. Hence there exists $x \in S\mu$ such that $\mu=\mu x\mu$. But μx is an idempotent and by Lemma 1, $\mu=\mu(\mu x)$. This completes the proof.

References

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